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Existence and attractivity results for ψ-Hilfer hybrid fractional differential equations

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ABSTRACT

In this work, we present some results on the existence of attractive solutions of fractional differential equations of the ψ-Hilfer hybrid type. The results on the existence of solutions are a consequence of the Schauder fixed point theorem. Next, we prove that all solutions are uniformly locally attractive.

RESUMEN

En este trabajo, presentamos algunos resultados sobre la existencia de soluciones atractivas de ecuaciones diferenciales fraccionarias de tipo ψ-Hilfer híbridas. Los resultados de existencia de soluciones son consecuencia del teorema de punto fijo de Schauder. A continuación, probamos que todas las soluciones son uniformemente localmente atractivas.

Keywords and Phrases: ψ-Hilfer fractional derivative; Schauder fixed-point Theorem; uniformly locally attractive.

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1 Introduction

The theory of derivatives and integrals to a real or complex order is none other than the fractional theory which began in 1695 between G.A. de L'Hospital and G.W. Leibniz. The fractional integration and differentiation go back to Leibniz, Riemann, Liouville, Abel, Weyl, and Riesz [27]. Many monographs to which the reader can refer such as Abbas *et al.* [1, 5, 6], Diethelm [13], Kilbas *et al.* [17], Oldham *et al.* [22], Podlubny [23], Samko *et al.* [24], Zhou [32, 33], Zhou *et al.* [34] and the works by Abbas and Benchohra [2], Lakshmikantham *et al.* [19, 20, 21]. Recently several works have been done concerning hybrid fractional differential equations see [9, 12, 14, 26, 31], and the references therein.

Functional $\psi-$ fractional differential equations received a great importance in applied mathematics and other sciences, see [8, 16, 18, 25, 28, 29, 30], and the references therein.

Some interesting results on existence and attractivity have been obtained in [3, 4, 7]. In this work, we are interested in the existence and attractivity of solutions for the following problem

$$\begin{cases} D_{0^+}^{\lambda,\sigma;\psi} \frac{u(t)}{v(t,u(t))} = w\left(t, u(t)\right); \ a.e. \quad t \in \mathbb{R}_+, \\ (\psi(t)-\psi(0))^{1-\varsigma} u(t) \mid_{t=0} = u_0; \qquad u_0 \in \mathbb{R}, \end{cases} \tag{1.1}$$

where $\mathbb{R}_+ := [0, +\infty),\ 0 < \lambda < 1,\ 0 \le \sigma \le 1,\ \varsigma = \lambda + \sigma(1-\lambda),\ {}^H D_{0^+}^{\lambda,\sigma;\psi}$ is the ψ -Hilfer fractional derivative of order λ and type σ, $v : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}^*$ and $w : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$, are given functions.

Special cases:

• For $\sigma = 0, \psi(t) = t, u_0 = 0$, we will get nonlinear hybrid FDEs of the form

$$\begin{cases} {}^{RL}D_{0^+}^{\lambda} \left[\frac{u(t)}{v(t,u(t))}\right] = w(t, u(t)),\ \text{a.e. } t \in \mathbb{R}_+, \\ u(0) = 0. \end{cases}$$

• For $\lambda = 1, \sigma = 1, \psi(t) = t$, we will get nonlinear integer order hybrid differential equations of the form

$$\begin{cases} \frac{d}{dt} \left[\frac{u(t)}{v(t,u(t))}\right] = w(t, y(t)),\ \text{a.e. } t \in \mathbb{R}_+, \\ u(0) = u_0 \in \mathbb{R}. \end{cases}$$

For $v = 1$, we will get nonlinear ψ-Hilfer FDEs of the form

$$\begin{cases} {}^H D_{0^+}^{\lambda,\sigma;\psi} u(t) = w(t, y(t)),\ \text{a.e. } t \in \mathbb{R}_+, \\ (\psi(t)-\psi(0))^{1-\varsigma} u(t)\big|_{t=0} = u_0 \in \mathbb{R}. \end{cases}$$

• For $v = 1, \sigma = 0$ (in this case $\varsigma = \lambda$), $\psi(t) = t$, we will get nonlinear FDEs involving Riemann-Liouville fractional derivative

$${}^{RL}D_{0^+}^{\lambda} u(t) = w(t, y(t)),\ \text{a.e. } t \in \mathbb{R}_+.$$

2 Preliminaries

Let $\psi : [a, b] \to \mathbb{R}$ be an increasing differentiable function such that $\psi'(t) \neq 0$, for all $t \in [a, b]$, $(-\infty \leq a < b \leq +\infty)$. Define on $[a, b]$, $(0 < a < b < \infty)$ the weighted space

$$C_{\varsigma\psi[a,b]} = \{\tau : (a, b] \to \mathbb{R} : (\psi(t) - \psi(a))^{\varsigma}\tau(t) \in C[a, b]\}, \quad 0 \leq \varsigma < 1,$$

with the norm

$$\|\tau\|_{C_{\varsigma;\psi}[a,b]} = \|(\psi(t) - \psi(a))^{\varsigma}\tau(t)\|_{C[a,b]} = \max\left\{|(\psi(t) - \psi(a))^{\varsigma}\tau(t)| : t \in [a, b]\right\},$$

where $C([a, b])$ denotes the Banach space of all real continuous functions on $[a, b]$.

Let $BC := BC(\mathbb{R}_+)$ be the Banach space of all bounded and continuous functions from $\mathbb{R}_+$ into $\mathbb{R}$. By $BC_{\varsigma} := BC_{\varsigma}(\mathbb{R}_+)$, we denote the weighted space of all bounded and continuous functions defined by $BC_{\varsigma} = \{\phi : \mathbb{R}_+ \to \mathbb{R} : (\psi(t) - \psi(0))^{1-\varsigma}\phi(t) \in BC\}$, with the norm

$$\|\phi\|_{BC_{\varsigma}} := \sup_{t\in\mathbb{R}_+} \left|(\psi(t) - \psi(0))^{1-\varsigma}\phi(t)\right|.$$

Let us recall some definitions and properties of fractional calculus.

Definition 2.1. *[17] The left-sided ψ-Riemann-Liouville fractional integral and fractional derivative of order λ, $(n - 1 < \lambda < n)$ for an integrable function $\phi : [a, b] \to \mathbb{R}$ with respect to another function $\psi : [a, b] \to \mathbb{R}$, that is an increasing differentiable function such that $\psi'(t) \neq 0$, for all $t \in [a, b], (-\infty \leq a < b \leq +\infty)$, are respectively defined as follows:*

$$I_{a^+}^{\lambda;\psi}\phi(t) = \frac{1}{\Gamma(\lambda)} \int_a^t \psi'(s)(\psi(t) - \psi(s))^{\lambda-1}\phi(s)ds,$$

and

$$D_{a+}^{\lambda;\psi}\phi(t) = \left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n I_{a^+}^{n-\alpha;\psi}\phi(t) = \frac{1}{\Gamma(n-\lambda)}\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n \int_a^t \psi'(s)(\psi(t) - \psi(s))^{n-\lambda-1}\phi(s)ds,$$

where $\Gamma(\cdot)$ is the Euler gamma function defined by

$$\Gamma(\delta) = \int_0^{\infty} e^{-t}t^{\delta-1}dt, \quad \delta > 0.$$

Definition 2.2. *[10] The left-sided ψ -Caputo fractional derivative of function $\chi \in C^n[a, b]$, $(n - 1 < \lambda < n)$ $n = [\alpha] + 1$ with respect to another function ψ is defined by*

$$^cD_{a^+}^{\lambda;\psi}\phi(t) = I_{a^+}^{n-\lambda;\psi}\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n \phi(t) = \frac{1}{\Gamma(n-\lambda)} \int_a^t \psi'(s)(\psi(t) - \psi(s))^{n-\lambda-1}\phi_{\psi}^{[n]}(s)ds,$$

where $\phi_{\psi}^{[n]}(t) = \left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n \phi(t)$ and ψ defined as in Definition $\mathbb{Q}$. Moreover, the $\psi-$ Caputo fractional derivative of function $\phi \in AC^n[a, b]$ is determined as

$$^cD_{a^+}^{\lambda;\psi}\phi(t) = D_{a^+}^{\lambda;\psi}\left[\phi(t) - \sum_{k=0}^{n-1} \frac{\left[\frac{1}{\psi'(t)}\frac{d}{dt}\right]^k \phi(a)}{k!}(\psi(t) - \psi(a))^k\right].$$

Definition 2.3. *[29] Let $n-1<\lambda<n, n\in\mathbb{N}$, with $[a,b], -\infty\le a<b\le+\infty$, and $\psi\in C^n([a,b],\mathbb{R})$ a function such that $\psi(t)$ is increasing and $\psi'(t)\neq 0$, for all $t\in[a,b]$. The ψ -Hilfer fractional derivative (left-sided) of function $\phi\in C^n([a,b],\mathbb{R})$ of order λ and type $\sigma\in[0,1]$ is determined as*

$$D_{a^+}^{\lambda,\sigma;\psi}\phi(t)=I_{a^+}^{\sigma(n-\lambda);\psi}\left[\frac{1}{\psi'(t)}\frac{d}{dt}\right]^n I_{a^+}^{(1-\sigma)(n-\lambda);\psi}\phi(t), t>a.$$

In other way

$$D_{a^+}^{\lambda,\sigma;\psi}\phi(t)=I_{a^+}^{\sigma(n-\lambda);\psi}D_{a^+}^{\gamma;\psi}\phi(t), t>a,$$

where

$$D_{a^+}^{\gamma;\psi}\phi(t)=\left[\frac{1}{\psi'(t)}\frac{d}{dt}\right]^n I_{a^+}^{(1-\sigma)(n-\lambda);\psi}\phi(t).$$

In particular, the ψ -Hilfer fractional derivative of order $\lambda\in(0,1)$ and type $\sigma\in[0,1]$, can be written in the following form

$$\begin{aligned}D_{a^+}^{\lambda,\sigma;\psi}\phi(t)&=\frac{1}{\Gamma(\varsigma-\lambda)}\int_a^t(\psi(t)-\psi(s))^{\varsigma-\lambda-1}D_{a^+}^{\gamma;\psi}\phi(s)ds\\&=I_{a^+}^{\varsigma-\lambda;\psi}D_{a^+}^{\varsigma;\psi}\phi(t),\end{aligned}$$

where $\varsigma=\lambda+\sigma-\lambda\sigma$, and $D_{a^+}^{\varsigma;\psi}\phi(t)=\left[\frac{1}{\psi'(t)}\frac{d}{dt}\right]I_{a^+}^{1-\varsigma;\psi}\phi(t)$.

Lemma 2.4. *[29] Let $\lambda>0, 0\le\varsigma<1$ and $\phi\in L^1(a,b)$. Then*

$$I_{a^+}^{\lambda;\psi}I_{a^+}^{\sigma;\psi}\phi(t)=I_{a^+}^{\lambda+\sigma;\psi}\phi(t),\ \ a.e.\ t\in[a,b].$$

In particular (i) if $\phi\in C_{\varsigma;\psi}[a,b]$, then $I_{a+}^{\lambda;\psi}I_{a+}^{\sigma;\psi}\phi(t)=I_{a+}^{\lambda+\sigma;\psi}\phi(t), t\in(a,b]$.
(ii) If $\phi\in C[a,b]$, then $I_{a^+}^{\lambda;\psi}I_{a^+}^{\sigma;\psi}\phi(t)=I_{a^+}^{\lambda+\sigma;\psi}\phi(t), t\in[a,b]$.

Lemma 2.5. *[29] Let $\lambda>0$, $0\le\sigma\le1$ and $0\le\varsigma<1$. If $h\in C_{\varsigma;\psi}[a,b]$ then*

$$D_{a^+}^{\lambda,\sigma;\psi}I_{a^+}^{\lambda;\psi}\phi(t)=\phi(t),\ \ t\in(a,b].$$

If $\phi\in C^1[a,b]$ then

$$D_{a+}^{\lambda,\sigma;\psi}I_{a+}^{\alpha;\psi}\phi(t)=\phi(t),\ \ t\in[a,b].$$

Lemma 2.6. *Let $\lambda>0, 0\le\varsigma<1$ and $\phi\in C_{\varsigma;\psi}[a,b]$. If $\lambda>\varsigma$, then $I_{a^+}^{\lambda;\psi}\phi\in C[a,b]$ and*

$$I_{a^+}^{\lambda;\psi}\phi(a)=\lim_{t\to a^+}I_{a^+}^{\lambda;\psi}\phi(t)=0.$$

Lemma 2.7. *[29] Let $\phi\in C^n[a,b], n-1<\lambda<n, 0\le\sigma\le1$, and $\varsigma=\lambda+\sigma-\lambda\sigma$. Then for all $t\in(a,b]$*

$$I_{a^+}^{\lambda;\psi}D_{a^+}^{\lambda,\sigma;\psi}\phi(t)=\phi(t)-\sum_{k=1}^n\frac{[\psi(t)-\psi(a)]^{\varsigma-k}}{\Gamma(\varsigma-k+1)}\phi_\psi^{(n-k)}I_{a^+}^{(1-\sigma)(n-\lambda);\psi}\phi(a).$$

In particular, if $0<\lambda<1$, we have

$$I_{a^+}^{\lambda;\psi}D_{a^+}^{\lambda,\sigma;\psi}\phi(t)=\phi(t)-\frac{[\psi(t)-\psi(a)]^{\varsigma-1}}{\Gamma(\varsigma)}I_{a^+}^{(1-\sigma)(1-\lambda);\psi}\phi(a).$$

Moreover, if $\phi \in C_{1-\varsigma;\psi}[a,b]$ *and* $I_{a^+}^{1-\varsigma;\psi}\phi \in C^1_{1-\varsigma;\psi}[a,b]$ *such that* $0<\varsigma<1$. *Then for all* $t \in (a,b]$

$$I_{a^+}^{\varsigma;\psi} D_{a^+}^{\varsigma;\psi}\phi(t) = \phi(t) - \frac{[\psi(t)-\psi(a)]^{\gamma-1}}{\Gamma(\varsigma)} I_{a^+}^{1-\varsigma;\psi}\phi(a).$$

We deduce from the above lemma the following lemmas:

Lemma 2.8. *[18] Let* $v \in C(\Upsilon \times \mathbb{R}, \mathbb{R}^*)$; $\Upsilon := [0,d]$, $d>0$, $\kappa \in C_{1-\zeta,\psi}(\Upsilon)$. *Then the problem*

$$\begin{cases} D_{0^+}^{\lambda,\sigma;\psi} \frac{u(t)}{v(t,u(t))} = \kappa(t), a.e. \quad t \in \Upsilon. \\ (\psi(t)-\psi(0))^{1-\varsigma} u(t) \mid_{t=0} = u_0, \qquad u_0 \in \mathbb{R}, \end{cases}$$

has a unique solution given by

$$u(t) = v(t,u(t)) \left\{ \frac{u_0}{v(0,u(0))} (\psi(t)-\psi(0))^{\varsigma-1} + I_{0^+}^{\lambda;\psi}\kappa(t) \right\}.$$

Lemma 2.9. *Let* $v \in C(\Upsilon \times \mathbb{R}, \mathbb{R}^*)$, $w : \Upsilon \times \mathbb{R} \to \mathbb{R}$ *be a function such that* $w(\cdot, u(\cdot)) \in BC_\varsigma$ *for any* $u \in BC_\varsigma$. *Then the problem (1.1) is equivalent to the problem of obtaining the solutions of the integral equation*

$$u(t) = v(t,u(t)) \left\{ \frac{u_0}{v(0,u(0))} (\psi(t)-\psi(0))^{\varsigma-1} + I_{0^+}^{\lambda;\psi} w(\cdot, u(\cdot))(t) \right\}.$$

Let $\varnothing \neq \Lambda \subset BC$ and let $K : \Lambda \to \Lambda$. We consider the solution of the equation

$$(Ku)(t) = u(t). \tag{2.1}$$

We introduce the concept of attractivity of solutions for equation (2.1).

Definition 2.10. *Solutions of equation (2.1) are locally attractive if there exists a ball* $B(u_0,\mu)$ *in the space* BC *such that, for any solutions* $\tau=\tau(t)$ *and* $\xi=\xi(t)$ *of equations (2.1) that belong to* $B(u_0,\mu) \cap \Lambda$, *we can write*

$$\lim_{t\to\infty} (\tau(t)-\xi(t)) = 0. \tag{2.2}$$

If the limit (2.2) is uniform with respect to $B(u_0,\mu)\cap\Lambda$, *then the solutions of equation (2.1) are said to be uniformly locally attractive (or, equivalently, that the solutions of (2.1) are locally asymptotically stable).*

Lemma 2.11. *[11] Let* $M \subset BC$. *Then* M *is relatively compact in* BC *if the following conditions are satisfied:*

(a) *M is uniformly bounded in BC;*

(b) *the functions belonging to M are almost equicontinuous in $\mathbb{R}_+$, i.e., equicontinuous on every compact set in $\mathbb{R}_+$;*

(c) *the functions from M are equiconvergent, i.e., given $\varepsilon > 0$, there exists $L(\varepsilon) > 0$ such that*

$$\left| u(t) - \lim_{t\to\infty} u(t) \right| < \varepsilon,$$

for any $t \geq L(\varepsilon)$ and $u \in M$.

Theorem 2.12. *(Schauder Fixed-Point Theorem [15]). Let F be a Banach space, let U be a nonempty bounded convex and closed subset of F, and let $K : U \to U$ be a compact and continuous map. Then K has at least one fixed point in U.*

3 Existence and Attractivity Results

Definition 3.1. *A measurable function $u \in BC_\varsigma$ is a solution of problem (1.1) if it verifies the initial condition $(\psi(t)-\psi(0))^{1-\varsigma}u(t)\mid_{t=0}= u_0$ and the equation $D_{0^+}^{\lambda,\sigma;\psi}\frac{u(t)}{v(t,u(t))} = w\,(t, u(t))$ on $\mathbb{R}_+$.*

We will give the following hypotheses:

(H_1) The function $t \mapsto w(t, u)$ is measurable on $\mathbb{R}_+$ for each $u \in BC_\varsigma$, the function $u \mapsto w(t, u)$ is continuous on BC_ς for a.e. $t \in \mathbb{R}_+$, and the function v is bounded such that $u \mapsto v(t, u)$ is continuous.

(H_2) There exists a continuous function $T : \mathbb{R}_+ \to \mathbb{R}_+$ such that for a.e. $t \in \mathbb{R}_+$ and each $u \in \mathbb{R}$,

$$|w(t, u)| \leq \frac{T(t)}{1 + |u|},$$

and

$$\lim_{t\to\infty} (\psi(t) - \psi(0))^{1-\varsigma} \left(I_{0^+}^{\lambda;\psi} T \right)(t) = 0.$$

Set

$$T^* = \sup_{t\in\mathbb{R}_+} (\psi(t) - \psi(0))^{1-\varsigma} \left(I_{0^+}^{\lambda;\psi} T \right)(t) < \infty.$$

Now we present a theorem on the existence and attractivity of solutions of the problem (1.1).

Theorem 3.2. *Assume that the hypotheses (H_1) and (H_2) hold. Then the problem (1.1) has at least one solution defined on $\mathbb{R}_+$ and the solutions of problem (1.1) are uniformly locally attractive.*

Proof. Consider the operator K such that, for any $u \in BC_\varsigma$,

$$(Ku)(t) = v(t, u(t)) \left\{ \frac{u_0}{v(0, u(0))} (\psi(t) - \psi(0))^{\varsigma-1} + \frac{1}{\Gamma(\lambda)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\lambda-1} w(s, u(s)) ds \right\}.$$

Let L be a bound of the function v. For any $u \in BC_\varsigma$, and for each $t \in \mathbb{R}_+$, we have

$$\begin{aligned}
&\left|(\psi(t)-\psi(0))^{1-\varsigma}(Ku)(t)\right| \\
\leq&|v(t,u(t))|\left\{\left|\frac{u_0}{v(0,u(0))}\right| + \frac{(\psi(t)-\psi(0))^{1-\varsigma}}{\Gamma(\lambda)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\lambda-1}|w(s,u(s))|ds\right\} \\
\leq&|v(t,u(t))|\left\{\left|\frac{u_0}{v(0,u(0))}\right| + \frac{(\psi(t)-\psi(0))^{1-\varsigma}}{\Gamma(\lambda)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\lambda-1}T(s)ds\right\} \\
\leq&L\left\{\left|\frac{u_0}{v(0,u(0))}\right| + T^*\right\} \\
:=&R_*.
\end{aligned}$$

So

$$|K(u)\|_{BC} \leq R_*. \tag{3.1}$$

Therefore, $K(u) \in BC_\varsigma$. Since, the map $K(u)$ is continuous on $\mathbb{R}_+$; for any $u \in BC_\varsigma$, and $K(BC_\varsigma) \subset BC_\varsigma$, then the operator K maps BC_ς into itself. Furthermore, equation (3.1) implies that the operator K transforms the ball

$$B_{R_*} := B(0,R_*) = \{v \in BC_\varsigma : \|v\|_{BC_\varsigma} \leq R_*\}$$

into itself. From Lemma 2.9 the solution of problem (1.1) is reduced to finding the solutions of the operator equation $K(u) = u$. We show that the operator $K : BC_\varsigma \to BC_\varsigma$ satisfies all assumptions of Theorem 2.12. The proof is divided into several steps:

Step 1. K is continuous.

Let $\{u_n\}_{n\in N}$ be a sequence such that $u_n \to u$ in B_{R_*}.

Then, for each $t \in \mathbb{R}_+$, we have

$$\begin{aligned}
&\left|((\psi(t)-\psi(0))^{1-\varsigma}(Ku_n)(t) - ((\psi(t)-\psi(0))^{1-\varsigma}(Ku)(t)\right| \\
\leq&\left|v(t,u_n(t))\left\{\frac{u_0}{v(0,u(0))} + \frac{(\psi(t)-\psi(0))^{1-\varsigma}}{\Gamma(\lambda)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\lambda-1}w(s,u_n(s))ds\right\}\right. \\
&\left.-v(t,u(t))\left\{\frac{u_0}{v(0,u(0))} + \frac{(\psi(t)-\psi(0))^{1-\varsigma}}{\Gamma(\lambda)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\lambda-1}w(s,u(s))ds\right\}\right| \\
\leq&\left|v(t,u_n(t))\left\{\frac{u_0}{v(0,u(0))} + \frac{(\psi(t)-\psi(0))^{1-\varsigma}}{\Gamma(\lambda)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\lambda-1}w(s,u_n(s))ds\right\}\right. \\
&-v(t,u(t))\left\{\frac{u_0}{v(0,u(0))} + \frac{(\psi(t)-\psi(0))^{1-\varsigma}}{\Gamma(\lambda)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\lambda-1}w(s,u_n(s))ds\right\} \\
&+v(t,u(t))\left\{\frac{u_0}{v(0,u(0))} + \frac{(\psi(t)-\psi(0))^{1-\varsigma}}{\Gamma(\lambda)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\lambda-1}w(s,u_n(s))ds\right\} \\
&\left.-v(t,u(t))\left\{\frac{u_0}{v(0,u(0))} + \frac{(\psi(t)-\psi(0))^{1-\varsigma}}{\Gamma(\lambda)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\lambda-1}w(s,u(s))ds\right\}\right|
\end{aligned}$$

$$\leq \left|v(t,u_n(t)) - v(t,u(t))\right|\left|\frac{u_0}{v(0,u(0))} + \frac{(\psi(t)-\psi(0))^{1-\varsigma}}{\Gamma(\lambda)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\lambda-1}\right.$$
$$\left.\times w(s,u_n(s))ds\right| + |v(t,u(t))|\frac{(\psi(t)-\psi(0))^{1-\varsigma}}{\Gamma(\lambda)}$$
$$\times \int_0^t \psi'(s)(\psi(t)-\psi(s))^{\lambda-1}|w(s,u_n(s)) - w(s,u(s))|ds.$$

Hence

$$\begin{aligned}
&\left|(\psi(t)-\psi(0))^{1-\varsigma}(Ku_n)(t) - (\psi(t)-\psi(0))^{1-\varsigma}(Ku)(t)\right| \\
&\leq \left|v(t,u_n(t)) - v(t,u(t))\right|\left\{\left|\frac{u_0}{v(0,u(0))}\right|\right. \\
&\left.+ \frac{(\psi(t)-\psi(0))^{1-\varsigma}}{\Gamma(\lambda)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\lambda-1}|w(s,u_n(s))|ds\right\} \\
&+ L\frac{(\psi(t)-\psi(0))^{1-\varsigma}}{\Gamma(\lambda)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\lambda-1}|w(s,u_n(s)) - w(s,u(s))|ds. \qquad (3.2)
\end{aligned}$$

Case 1. If $t \in [0,d]$, then, in view of the facts that $u_n \to u$ as $n \to \infty$, v and w are continuous, by the Lebesgue dominated convergence theorem, from the equation (3.2), we have

$$\|K(u_n) - K(u)\|_{BC_\varsigma} \to 0 \quad \text{as} \quad n \to \infty.$$

Case 2. If $t \in (d,\infty)$, then, from the hypotheses and (3.2), we have

$$\begin{aligned}
&\left|(\psi(t)-\psi(0))^{1-\varsigma}(Ku_n)(t) - (\psi(t)-\psi(0))^{1-\varsigma}(Ku)(t)\right| \\
&\leq \left|v(t,u_n(t)) - v(t,u(t))\right|\left\{\left|\frac{u_0}{v(0,u(0))}\right|\right. \\
&\left.+ \frac{(\psi(t)-\psi(0))^{1-\varsigma}}{\Gamma(\lambda)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\lambda-1}T(s)ds\right\} \\
&+ 2L\frac{(\psi(t)-\psi(0))^{1-\varsigma}}{\Gamma(\lambda)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\lambda-1}T(s)ds.
\end{aligned}$$

Then

$$\begin{aligned}
&\left|(\psi(t)-\psi(0))^{1-\varsigma}(Ku_n)(t) - (\psi(t)-\psi(0))^{1-\varsigma}(Ku)(t)\right| \\
&\leq \left|v(t,u_n(t)) - v(t,u(t))\right|\left\{\left|\frac{u_0}{v(0,u(0))}\right| + ((\psi(t)-\psi(0))^{1-\varsigma}\left(I_{0^+}^{\lambda;\psi}T\right)(t)\right\} \\
&+ 2L((\psi(t)-\psi(0))^{1-\varsigma}\left(I_{0^+}^{\lambda;\psi}T\right)(t). \qquad (3.3)
\end{aligned}$$

Since $u_n \to u$ as $n \to \infty$, v is continuous and $(\psi(t)-\psi(0))^{1-\varsigma}\left(I_{0^+}^{\lambda;\psi}T\right)(t) \to 0$ as $t \to \infty$, it follows from (3.3) that

$$\|K(u_n) - K(u)\|_{BC_\varsigma} \to 0 \quad \text{as} \quad n \to \infty.$$

Step 2. $L(B_{R_*})$ is uniformly bounded, and equicontinuous on every compact subset $[0,d]$ of $\mathbb{R}_+$, $d > 0$.

We have $L(B_{R_*}) \subset B_{R_*}$ and B_{R_*} is bounded, so $L(B_{R_*})$ is uniformly bounded.

Next, for each $t_1, t_2 \in [0, d], t_1 < t_2$, and $u \in B_{R_*}$, we have

$$\begin{aligned}
&\left|(\psi(t_2)-\psi(0))^{1-\varsigma}(Ku)(t_2)-(\psi(t_1)-\psi(0))^{1-\varsigma}(Ku)(t_1)\right| \\
&\leq \left|v(t_2,u(t_2))\left\{\frac{u_0}{v(0,u(0))}+\frac{(\psi(t_2)-\psi(0))^{1-\varsigma}}{\Gamma(\lambda)}\int_0^{t_2}\psi'(s)(\psi(t_2)-\psi(s))^{\lambda-1}w(s,u(s))ds\right\}\right. \\
&\left.-v(t_1,u(t_1))\left\{\frac{u_0}{v(0,u(0))}+\frac{(\psi(t_1)-\psi(0))^{1-\varsigma}}{\Gamma(\lambda)}\int_0^{t_1}\psi'(s)(\psi(t_1)-\psi(s))^{\lambda-1}w(s,u(s))ds\right\}\right| \\
&\leq \left|v(t_2,u(t_2))\left\{\frac{u_0}{v(0,u(0))}+\frac{(\psi(t_2)-\psi(0))^{1-\varsigma}}{\Gamma(\lambda)}\int_0^{t_2}\psi'(s)(\psi(t_2)-\psi(s))^{\lambda-1}w(s,u(s))ds\right\}\right. \\
&-v(t_1,u(t_1))\left\{\frac{u_0}{v(0,u(0))}+\frac{(\psi(t_2)-\psi(0))^{1-\varsigma}}{\Gamma(\lambda)}\int_0^{t_2}\psi'(s)(\psi(t_2)-\psi(s))^{\lambda-1}w(s,u(s))ds\right\} \\
&+v(t_1,u(t_1))\left\{\frac{u_0}{v(0,u(0))}+\frac{(\psi(t_2)-\psi(0))^{1-\varsigma}}{\Gamma(\lambda)}\int_0^{t_2}\psi'(s)(\psi(t_2)-\psi(s))^{\lambda-1}w(s,u(s))ds\right\} \\
&\left.-v(t_1,u(t_1))\left\{\frac{u_0}{v(0,u(0))}+\frac{(\psi(t_1)-\psi(0))^{1-\varsigma}}{\Gamma(\lambda)}\int_0^{t_1}\psi'(s)(\psi(t_1)-\psi(s))^{\lambda-1}w(s,u(s))ds\right\}\right|.
\end{aligned}$$

Thus

$$\begin{aligned}
&\left|(\psi(t_2)-\psi(0))^{1-\varsigma}(Ku)(t_2)-(\psi(t_1)-\psi(0))^{1-\varsigma}(Ku)(t_1)\right| \\
&\leq |v(t_2,u(t_2))-v(t_1,u(t_1))|\left|\frac{u_0}{v(0,u(0))}\right. \\
&\left.+\frac{(\psi(t_2)-\psi(0))^{1-\varsigma}}{\Gamma(\lambda)}\int_0^{t_2}\psi'(s)(\psi(t_2)-\psi(s))^{\lambda-1}w(s,u(s))ds\right| \\
&+|v(t_1,u(t_1))|\left|\frac{(\psi(t_2)-\psi(0))^{1-\varsigma}}{\Gamma(\lambda)}\int_0^{t_1}\psi'(s)(\psi(t_2)-\psi(s))^{\lambda-1}w(s,u(s))ds\right. \\
&+\frac{(\psi(t_2)-\psi(0))^{1-\varsigma}}{\Gamma(\lambda)}\int_{t_1}^{t_2}\psi'(s)(\psi(t_2)-\psi(s))^{\lambda-1}w(s,u(s))ds \\
&\left.-\frac{(\psi(t_1)-\psi(0))^{1-\varsigma}}{\Gamma(\lambda)}\int_0^{t_1}\psi'(s)(\psi(t_1)-\psi(s))^{\lambda-1}w(s,u(s))ds\right|.
\end{aligned}$$

Hence

$$\begin{aligned}
&\left|(\psi(t_2)-\psi(0))^{1-\varsigma}(Ku)(t_2)-(\psi(t_1)-\psi(0))^{1-\varsigma}(Ku)(t_1)\right| \\
&\leq |v(t_2,u(t_2))-v(t_1,u(t_1))|\left(\left|\frac{u_0}{v(0,u(0))}\right|\right. \\
&\left.+\frac{(\psi(t_2)-\psi(0))^{1-\varsigma}}{\Gamma(\lambda)}\int_0^{t_2}\psi'(s)(\psi(t_2)-\psi(s))^{\lambda-1}|w(s,u(s))|ds\right) \\
&+L\left(\int_0^{t_1}\left|\frac{(\psi(t_2)-\psi(0))^{1-\varsigma}}{\Gamma(\lambda)}\psi'(s)(\psi(t_2)-\psi(s))^{\lambda-1}\right.\right. \\
&\left.-\frac{(\psi(t_1)-\psi(0))^{1-\varsigma}}{\Gamma(\lambda)}\psi'(s)(\psi(t_1)-\psi(s))^{\lambda-1}\right| \\
&\left.|w(s,u(s))|ds+\frac{(\psi(t_2)-\psi(0))^{1-\varsigma}}{\Gamma(\lambda)}\int_{t_1}^{t_2}\psi'(s)(\psi(t_2)-\psi(s))^{\lambda-1}|w(s,u(s))|ds\right)
\end{aligned}$$

$$
\begin{aligned}
&\leq |v(t_2,u(t_2))-v(t_1,u(t_1))|\left(\left|\frac{u_0}{v(0,u(0))}\right|\right.\\
&+\left.\frac{(\psi(t_2)-\psi(0))^{1-\varsigma}}{\Gamma(\lambda)}\int_0^{t_2}\psi'(s)(\psi(t_2)-\psi(s))^{\lambda-1}T(s)ds\right)\\
&+L\Bigg(\int_0^{t_1}\left|\frac{(\psi(t_2)-\psi(0))^{1-\varsigma}}{\Gamma(\lambda)}\psi'(s)(\psi(t_2)-\psi(s))^{\lambda-1}\right.\\
&-\left.\frac{(\psi(t_1)-\psi(0))^{1-\varsigma}}{\Gamma(\lambda)}\psi'(s)(\psi(t_1)-\psi(s))^{\lambda-1}\right|\\
&T(s)ds+\frac{(\psi(t_2)-\psi(0))^{1-\varsigma}}{\Gamma(\lambda)}\int_{t_1}^{t_2}\psi'(s)(\psi(t_2)-\psi(s))^{\lambda-1}T(s)ds\Bigg).
\end{aligned}
$$

From the continuity of the functions T and v, by setting $T_*=\sup_{t\in[0,d]}T(t)$, we obtain

$$
\begin{aligned}
&\left|(\psi(t_2)-\psi(0))^{1-\varsigma}\,(Ku)\,(t_2)-(\psi(t_1)-\psi(0))^{1-\varsigma}(Ku)(t_1)\right|\\
&\leq |v(t_2,u(t_2))-v(t_1,u(t_1))|\left(\left|\frac{u_0}{v(0,u(0))}\right|+\frac{T_*(\psi(t_2)-\psi(0))^{1-\varsigma}}{\Gamma(\lambda)}\int_0^{t_2}\psi'(s)(\psi(t_2)-\psi(s))^{\lambda-1}ds\right)\\
&+LT_*\Bigg(\int_0^{t_1}\left|\frac{(\psi(t_2)-\psi(0))^{1-\varsigma}}{\Gamma(\lambda)}\psi'(s)(\psi(t_2)-\psi(s))^{\lambda-1}\right.\\
&-\left.\frac{(\psi(t_1)-\psi(0))^{1-\varsigma}}{\Gamma(\lambda)}\psi'(s)(\psi(t_1)-\psi(s))^{\lambda-1}\right|ds\\
&+\frac{(\psi(t_2)-\psi(0))^{1-\varsigma}}{\Gamma(\lambda)}\int_{t_1}^{t_2}\psi'(s)(\psi(t_2)-\psi(s))^{\lambda-1}ds\Bigg)\\
&\leq |v(t_2,u(t_2))-v(t_1,u(t_1))|\left(\left|\frac{u_0}{v(0,u(0))}\right|+\frac{T_*(\psi(t_2)-\psi(0))^{1-\varsigma+\lambda}}{\Gamma(\lambda+1)}\right)\\
&+LT_*\Bigg(\int_0^{t_1}\left|\frac{(\psi(t_2)-\psi(0))^{1-\varsigma}}{\Gamma(\lambda)}\psi'(s)(\psi(t_2)-\psi(s))^{\lambda-1}\right.\\
&-\left.\frac{(\psi(t_1)-\psi(0))^{1-\varsigma}}{\Gamma(\lambda)}\psi'(s)(\psi(t_1)-\psi(s))^{\lambda-1}\right|ds+\frac{(\psi(t_2)-\psi(0))^{1-\varsigma}}{\Gamma(\lambda+1)}(\psi(t_2)-\psi(t_1))^{\lambda}\Bigg).
\end{aligned}
$$

As $t_1\to t_2$, the right-hand side of the inequality tends to zero.

Step 3. $L\,(B_R)$ is equiconvergent.

Let $u\in B_{R*}$. Then, for each $t\in\mathbb{R}_+$ we have

$$
\begin{aligned}
\left|(\psi(t)-\psi(0))^{1-\varsigma}(Ku)(t)\right| &\leq |v(t,u(t))|\Bigg\{\left|\frac{u_0}{v(0,u(0))}\right|\\
&+\left|\frac{(\psi(t)-\psi(0))^{1-\varsigma}}{\Gamma(\lambda)}\int_0^t\psi'(s)(\psi(t)-\psi(s))^{\lambda-1}w(s,u(s))ds\right|\Bigg\}\\
&\leq |v(t,u(t))|\Bigg\{\left|\frac{u_0}{v(0,u(0))}\right|+\left|\frac{(\psi(t)-\psi(0))^{1-\varsigma}}{\Gamma(\lambda)}\int_0^t\psi'(s)(\psi(t)-\psi(s))^{\lambda-1}T(s)ds\right|\Bigg\}\\
&\leq L\Bigg\{\left|\frac{u_0}{v(0,u(0))}\right|+(\psi(t)-\psi(0))^{1-\varsigma}\left(I_{0^+}^{\lambda;\psi}T\right)(t)\Bigg\}.
\end{aligned}
$$

Since

$$(\psi(t)-\psi(0))^{1-\varsigma}\left(I_{0^+}^{\lambda;\psi}T\right)(t)\to 0\;as\;\;t\to+\infty,$$

we find

$$|(Ku)(t)| \leq L\Bigg\{\left|\frac{u_0}{(\psi(t)-\psi(0))^{1-\varsigma}v(0,u(0))}\right| + \frac{(\psi(t)-\psi(0))^{1-\varsigma}\left(I_{0^+}^{\lambda;\psi}T\right)(t)}{(\psi(t)-\psi(0))^{1-\varsigma}}\Bigg\}.$$

Hence,

$$|(Lu)(t)-(Lu)(+\infty)| \to 0 \quad \text{as} \quad t \to +\infty,$$

in view of Lemma 2.11 as a consequence of Steps $1-4$, we conclude that $K : B_{R_*} \to B_{R_*}$ is compact and continuous. Applying the Theorem 2.12, we have that K has a fixed point u, which is a solution of problem (1.1) on $\mathbb{R}_+$.

Step 4. The uniform local attractivity of solutions.
We assume that u_* is a solution of problem (1.1) under the conditions of this theorem.
Set $u \in B\left(u_*, 2L\Big\{\left|\frac{u_0}{v(0,u(0))}\right| + 2T^*\Big\}\right)$, we have

$$\begin{aligned}
&\left|(\psi(t)-\psi(0))^{1-\varsigma}(Ku)(t) - (\psi(t)-\psi(0))^{1-\varsigma}(u_*)(t)\right| \\
&\leq \left|(\psi(t)-\psi(0))^{1-\varsigma}(Ku)(t) - (\psi(t)-\psi(0))^{1-\varsigma}(Ku_*)(t)\right| \\
&\leq |v(t,u(t)) - v(t,u_*(t))|\Bigg\{\left|\frac{u_0}{v(0,u(0))}\right| \\
&+ \frac{(\psi(t)-\psi(0))^{1-\varsigma}}{\Gamma(\lambda)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\lambda-1}|w(s,u(s))|ds\Bigg\} \\
&+ L\frac{(\psi(t)-\psi(0))^{1-\varsigma}}{\Gamma(\lambda)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\lambda-1}|w(s,u(s)) - w(s,u_*(s))|ds \\
&\leq 2L\Bigg\{\left|\frac{u_0}{v(0,u(0))}\right| + \frac{(\psi(t)-\psi(0))^{1-\varsigma}}{\Gamma(\lambda)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\lambda-1}T(s)ds\Bigg\} \\
&+ 2L\frac{(\psi(t)-\psi(0))^{1-\varsigma}}{\Gamma(\lambda)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\lambda-1}T(s)ds \\
&\leq 2L\Bigg\{\left|\frac{u_0}{v(0,u(0))}\right| + 2T^*\Bigg\}.
\end{aligned}$$

Thus, we get

$$\|K(u)-u_*\|_{BC_\varsigma} \leq 2L\Bigg\{\left|\frac{u_0}{v(0,u(0))}\right| + 2T^*\Bigg\}.$$

So, we conclude that K is a continuous function such that

$$K\left(B\left(u_*, 2L\Bigg\{\left|\frac{u_0}{v(0,u(0))}\right| + 2T^*\Bigg\}\right)\right) \subset B\left(u_*, 2L\Bigg\{\left|\frac{u_0}{v(0,u(0))}\right| + 2T^*\Bigg\}\right).$$

Moreover, if u is a solution of problem (1.1), then

$$\begin{aligned}
|u(t)-u_*(t)| &= |(Ku)(t)-(Ku_*)(t)| \\
&\leq |v(t,u(t))-v(t,u_*(t))| \left\{ (\psi(t)-\psi(0))^{\varsigma-1} \left| \frac{u_0}{v(0,u(0))} \right| \right. \\
&\left. + \frac{1}{\Gamma(\lambda)} \int_0^t \psi'(s)(\psi(t)-\psi(s))^{\lambda-1} |w(s,u(s))| ds \right\} \\
&+ \frac{L}{\Gamma(\lambda)} \int_0^t \psi'(s)(\psi(t)-\psi(s))^{\lambda-1} |w(s,u(s))-w(s,u_*(s))| ds \\
&\leq 2L \left\{ (\psi(t)-\psi(0))^{\varsigma-1} \left| \frac{u_0}{v(0,u(0))} \right| \right. \\
&\left. + \frac{1}{\Gamma(\lambda)} \int_0^t \psi'(s)(\psi(t)-\psi(s))^{\lambda-1} |w(s,u(s))| ds \right\} \\
&+ \frac{L}{\Gamma(\lambda)} \int_0^t \psi'(s)(\psi(t)-\psi(s))^{\lambda-1} |w(s,u(s))-w(s,u_*(s))| ds \\
&\leq 2L \left\{ (\psi(t)-\psi(0))^{\varsigma-1} \left| \frac{u_0}{v(0,u(0))} \right| + 2(I_{0^+}^{\lambda;\psi} T)(t) \right\}.
\end{aligned}$$

Therefore,

$$|u(t)-u_*(t)| \leq 2L \left\{ (\psi(t)-\psi(0))^{\varsigma-1} \left| \frac{u_0}{v(0,u(0))} \right| + 2 \frac{(\psi(t)-\psi(0))^{1-\varsigma} (I_{0^+}^{\lambda;\psi} T)(t)}{(\psi(t)-\psi(0))^{1-\varsigma}} \right\}. \tag{3.4}$$

By using (3.4) and the fact that

$$\lim_{t\to\infty} (\psi(t)-\psi(0))^{1-\varsigma} (I_{0^+}^{\lambda;\psi} T)(t) = 0,$$

we conclude

$$\lim_{t\to\infty} |u(t)-u_*(t)| = 0.$$

Consequently, all solutions of problem (1.1) are uniformly locally attractive. □

4 An Example

As an application of our results, we consider the following problem for a ψ-Hilfer fractional differential equation

$$\begin{cases} D_{0^+}^{\frac{1}{2},\frac{1}{2};\psi} \dfrac{u(t)}{v(t,u(t))} = w(t,u(t)), a.e. \quad t \in \mathbb{R}_+, \\ (\psi(t)-\psi(0))^{\frac{1}{4}} u(t) \mid_{t=0} = 1, \end{cases} \tag{4.1}$$

where $\psi : [0,1] \to \mathbb{R}$ with $\psi(t) = \sqrt{t+3}$,

$$v(t,u) = \frac{1}{(1+t)(1+|u|)},$$

$$\begin{cases} w(t,u) = \dfrac{\beta(\psi(t)-\psi(0))^{\frac{-1}{4}} \sin t}{64(1+\sqrt{t})(1+|u|)}, \quad t \in (0,\infty), \quad u \in \mathbb{R}, \\ w(0,u) = 0, \quad u \in \mathbb{R}, \end{cases}$$

and

$$\beta = \frac{9\sqrt{\pi}}{16}.$$

Clearly, the function w is continuous. The hypothesis (H_2) is satisfied with

$$\begin{cases} T(t) = \dfrac{\beta(\psi(t)-\psi(0))^{\frac{-1}{4}}|\sin t|}{64(1+\sqrt{t})}, & t \in (0,\infty), \\ T(0) = 0. \end{cases}$$

In addition, we have

$$\begin{aligned} (\psi(t)-\psi(0))^{\frac{1}{4}} \left(I_{0^+}^{\frac{1}{2};\psi} T\right)(t) &= \frac{(\psi(t)-\psi(0))^{\frac{1}{4}}}{\Gamma\left(\frac{1}{2}\right)} \int_0^t \psi'(\tau)(\psi(t)-\psi(\tau))^{\frac{-1}{2}} T(\tau)d\tau \\ &\leq \frac{1}{4}(\psi(t)-\psi(0))^{\frac{-1}{4}} \to 0 \quad \text{as} \quad t \to \infty. \end{aligned}$$

Simple computations show that all conditions of Theorem 3.2 are satisfied. Consequently, our problem (4.1) has at least one solution defined on $\mathbb{R}_+$, and all solutions of this problem are uniformly locally attractive.

5 Conclusion

In this paper, we provided some sufficient conditions ensuring the existence and the uniform locally attractivity of solutions of some ψ-Hilfer fractional differential equations. The technique used is based on Schauder's fixed point theory theorem.

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