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Vlasov-Poisson equation in weighted Sobolev space $W^{m,p}(w)$

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ABSTRACT

In this paper, we are concerned about the well-posedness of Vlasov-Poisson equation near vaccum in weighted Sobolev space $W^{m,p}(w)$. The most difficult part comes from estimates of the electronic term $\nabla_x \phi$. To overcome this difficulty, we establish the L^p-L^q estimates of the electronic term $\nabla_x \phi$; some weight is introduced as well to obtain the off-diagonal estimate. The weight is also useful when it comes to control the higher-order derivative term.

RESUMEN

En este artículo, estamos interesados en que la ecuación de Vlasov-Poisson está bien puesta cercana al vacío en el espacio de Sobolev $W^{m,p}(w)$ con peso. La parte más difícil proviene de estimaciones del término electrónico $\nabla_x \phi$. Para superar esta dificultad, establecemos las estimaciones L^p-L^q del término electrónico $\nabla_x \phi$; donde algún peso es también introducido para obtener la estimación fuera de la diagonal. El peso es también útil cuando se trata de controlar el término de la derivada de alto orden.

Keywords and Phrases: Vlasov-Poisson, L^p-Sobolev, weighted estimates, L^p-L^q estimates.

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1 Introduction

Understanding the evolution of a distribution of particles over time is a major research area of statistical physics. The Vlasov-Poisson equation is one of the key equations governing this evolution. Specifically, it models particle behaviors with long range interactions in a non-relativistic zero-magnetic field setting. Two principal types of long range interactions are Coulomb's forces, the electrostatic repulsion of similarly charged particles in a plasma, and Newtonian's forces, the gravitational attraction of stars in a galaxy. The general Cauchy's problem for the Vlasov-Poisson equation (VP equation) in n dimensional space is as follows:

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + \nabla_x \phi \cdot \nabla_v f = 0, \\ -\Delta_x \phi = \displaystyle\int_{\mathbb{R}^n} f \ dv, \\ f(0,x,v) = f_0(x,v), \end{cases} \tag{1.1}$$

where $f(t,x,v)$ denotes the distribution function of particles, $x \in \mathbb{R}^n$ is the position, $v \in \mathbb{R}^n$ is the velocity, and $t > 0$ is the time and $n \geq 3$.

The Cauchy problem for the Vlasov-Poisson equation has been studied for several decades. The first paper on global existence is due to Arsen'ev [3]. He showed the global existence of weak solutions. Then in 1977 Batt [5] established the global existence for spherically symmetric data. In 1981 Horst [9] extended the global classical solvability to cylindrically symmetric data. Next, in 1985, Bardos and Degond [4] obtained the global existence for "small" data. Finally, in 1989 Pfaffelmoser [12] proved the global existence of a smooth solution with large data. Later, simpler proofs of the same results were published by Schaeffer [13], Horst [10], and Lions and Pertharne [11]. Nevertheless, most of them were concerned about solutions in L^∞ or continuous function spaces. Also, there are many papers studying Vlasov-Poisson-Boltzmman (Landau) equation in L^2 setting, see [2, 6, 7, 8] and the references therein. A natural question is whether we can obtain the solutions in L^p context, for example, $W^{m,p}$ spaces. This becomes our main theme in this paper.

In this paper, our aim is to construct the solution to (1.1) in $W^{m,p}$ space. The difficulty lies in the absence of L^p estimates of the electronic term $\nabla_x \phi$. To handle this issue, we establish the L^p-L^q off-diagonal estimates of $\nabla_x \phi$ which is highly important in estimating the higher order derivative term. Also, it is necessary to introduce a weight w in order to obtain this off-diagonal estimate. It is worthy to mention that this weight is crucial to deal with the higher-order derivative term.

2 Preliminaries and main theorem

2.1 Notations and definitions

We first would like to introduce some notations.

- Given a locally integrable function f, the maximal function Mf is defined by

$$(Mf)(x) = \sup_{\delta>0} \frac{1}{|B(x,\delta)|} \int_{B(x,\delta)} |f(y)|dy, \tag{2.1}$$

  where $|B(x,\delta)|$ is the volume of the ball of $B(x,\delta)$ with center x and radius δ.

- Weight $w(v) = \langle v\rangle^{\gamma}$, $\gamma \cdot \frac{p'}{p} > n$, $\frac{1}{p} + \frac{1}{p'} = 1$, n is the dimension.

- $\|f\|^p_{L^p_{x,v}(w)} =: \int_{\mathbb{R}^{2n}} |f|^p w\, dxdv$.

- Define the higher-order energy norm as

$$\mathcal{E}(f(t)) =: \|f\|_{W^{m,p}(w)} = \sum_{|\alpha|+|\beta|\leq m} \|\partial_x^{\alpha}\partial_v^{\beta} f(t)\|^p_{L^p_{x,v}(w)},$$

  and

$$\mathcal{E}(f_0) =: \mathcal{E}(f(0)) = \sum_{|\alpha|+|\beta|\leq m} \|\partial_x^{\alpha}\partial_v^{\beta} f_0\|^p_{L^p_{x,v}(w)},$$

  where $m \geq 5$ and $\frac{n}{3} < p < \frac{n}{2}$, $n \geq 3$. Here α and β denote multi-indices with length $|\alpha|$ and $|\beta|$, respectively. If each component of α_1 is not greater than that of α, we denote the condition by $\alpha_1 \leq \alpha$. We also define $\alpha_1 < \alpha$ if $\alpha_1 \leq \alpha$ and $|\alpha_1| < |\alpha|$. We also denote $\binom{\alpha_1}{\alpha}$ by $C_{\alpha}^{\alpha_1}$.

- $A \lesssim B$ means there exists a constant $c > 1$ independent of the main parameters such that $A \leq cB$. $A \sim B$ means $A \lesssim B$ and $B \lesssim A$.

Now we are ready to state our main theorem.

Theorem 2.1. *For any sufficiently small $M > 0$, there exists $T^*(M) > 0$ such that if*

$$\mathcal{E}(f_0) = \sum_{|\alpha|+|\beta|\leq m} \|\partial_x^{\alpha}\partial_v^{\beta} f_0\|^p_{L^p_{x,v}(w)} \leq \frac{M}{2},$$

then there is a unique solution $f(t,x,v)$ to Vlasov-Poisson system (1.1) in $[0, T^*(M)) \times \mathbb{R}^n \times \mathbb{R}^n$ such that $\sup_{0\leq t\leq T^*} \mathcal{E}(f(t)) \leq M$, where $m > \frac{n}{p} + 1$ with $n \geq 3$ and $\frac{n}{3} < p < \frac{n}{2}$.

Remark 2.2.

- *One should pay attention to the differential index m in $W^{m,p}(\mathbb{R}^n)$ which represents the weak derivative, is not the classical derivative in $C^2(\mathbb{R}^n)$. Indeed, for the space $W^{4,1.4}(\mathbb{R}^6)$ in which we could obtain solutions that could not be embedded into $C(\mathbb{R}^6)$ (the continuous function space) or $L^\infty(\mathbb{R}^6)$, not to mention $C^2(\mathbb{R}^6)$ (the twice continuously differentiable function space) due to the fact $4 \cdot 1.4 < 6$, i.e. $W^{4,1.4}(\mathbb{R}^6) \not\hookrightarrow C^2(\mathbb{R}^6)$ which implies that the classical results in [3, 4] and [9]-[13] could not cover our results.*

- *In [4], C. Bardos and P. Degond also imposed the pointwise condition like*

$$0 \leq u_{\alpha,0}(x,v) \leq \frac{\epsilon}{(1+|x|)^4 \cdot (1+|v|)^4}.$$

  However, the polynomial decay in the x variable is not needed at all in our proofs.

- *Our working space $W^{m,p}(\mathbb{R}^n)$ has more flexibility than $C^2(\mathbb{R}^n)$ because of the triplet (m,n,p) which implies that we can obtain the solutions in more spaces.*

Let us illustrate our strategies for proving Theorem 2.1. As is known, the routine to prove the existence of solution is to get a uniform-in-k estimate for the energy norm $\mathcal{E}(f^{k+1}(t))$. In this paper, we adopt the L^p version energy method, *i.e.* to do the dual with $|\partial_x^\alpha \partial_v^\beta f^{k+1}|^{p-2}(\partial_x^\alpha \partial_v^\beta f^{k+1})w$ (see (4.5)). We expect all the estimates J_i in Section 4 can be controlled by

$$\mathcal{E}(f(t)) =: \sum_{|\alpha|+|\beta|\leq m} \|\partial_x^\alpha \partial_v^\beta f(t)\|^p_{L^p_{x,v}(w)}.$$

To achieve our goal, some estimates related to the electronic term $\nabla_x \phi$ are needed. The L^p-L^q estimate is established to deal with the higher-order derivative. For instance, when $|\alpha| = m$, the L^p-L^q estimate comes in to handle the highest order derivative term $\partial_x^\alpha \nabla_x \phi^k$:

$$\begin{aligned} &\Big\langle \partial_x^\alpha \nabla_x \phi^k \cdot \nabla_v f^{k+1},\ |\partial_x^\alpha f^{k+1}|^{p-2} \cdot \partial_x^\alpha f^{k+1} \cdot w \Big\rangle \\ &\lesssim \|\partial_x^\alpha \nabla_x \phi^k\|_{L^q_x} \|\nabla_v f^{k+1}\|_{L^n_{x,v}(w)} \|\partial_x^\alpha f^{k+1}\|^{p-1}_{L^p_{x,v}(w)}. \end{aligned} \tag{2.2}$$

In turn, in order to get this L^p-L^q estimate involving $\nabla_x \phi$, we introduce weight w; surprisingly, this weight w also plays another crucial role to deal with the higher order derivative. More precisely, we do this trick when $|\alpha| + |\beta| = m$, w could "absorb" the extra derivative in ∇_v as follows:

$$\begin{aligned} &\Big\langle \nabla_x \phi^k \cdot \nabla_v \partial_x^\alpha \partial_v^\beta f^{k+1},\ |\partial_x^\alpha \partial_v^\beta f^{k+1}|^{p-2}(\partial_x^\alpha \partial_v^\beta f^{k+1})w \Big\rangle \\ &\sim \Big\langle \nabla_x \phi^k \cdot \nabla_v |\partial_x^\alpha \partial_v^\beta f^{k+1}|^p,\ w \Big\rangle \sim -\Big\langle \nabla_x \phi^k \cdot |\partial_x^\alpha \partial_v^\beta f^{k+1}|^p,\ \nabla_v w \Big\rangle. \end{aligned} \tag{2.3}$$

Before we give the proof of the main theorem, we would like to establish the following L^p-L^q estimates.

3 L^p-L^q estimates

In this section, we are going to prove the L^p-L^q estimate which plays an essentially important role in our proofs.

Lemma 3.1. *Suppose* $1 < p < \frac{n}{2}$ *and* $\frac{1}{q} = \frac{1}{p} - \frac{1}{n}$. *If* $-\Delta\phi = \int_{\mathbb{R}^n} f dv =: g$, *then it holds that*

$$\|\nabla_x \phi\|_{L^q(\mathbb{R}^n)} \lesssim \|g\|_{L^p(\mathbb{R}^n)}, \tag{3.1}$$

Proof. Note that $\nabla_x \phi = \nabla_x (I_2 * g)$, with $I_2(x) = \frac{1}{(n-2)\omega_{n-1}} \cdot \frac{1}{|x|^{n-2}}$, for more details, see the last section Appendix. Therefore there holds

$$\begin{aligned}
\|\nabla_x \phi\|_{L^q(\mathbb{R}^n)} &= \|\nabla_x (I_2 * g)\|_{L^q(\mathbb{R}^n)} \\
&\lesssim \|(Mg)^{\frac{1}{2}} \cdot (I_2 * |g|)^{\frac{1}{2}}\|_{L^q(\mathbb{R}^n)} \\
&\lesssim \|(Mg)^{\frac{1}{2}}\|_{L^{q_1}(\mathbb{R}^n)} \cdot \|(I_2 * |g|)^{\frac{1}{2}}\|_{L^{q_2}(\mathbb{R}^n)} \\
&\lesssim \|Mg\|^{\frac{1}{2}}_{L^{\frac{q_1}{2}}(\mathbb{R}^n)} \cdot \|I_2 * |g|\|^{\frac{1}{2}}_{L^{\frac{q_2}{2}}(\mathbb{R}^n)},
\end{aligned}$$

where we applied (5.3) in the second line, and Hölder's inequality with

$$\frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q}, \quad q_i > 1,$$

in the third line separately.

On the one hand, the boundedness of Hardy-Littlewood operator M as defined by identity (2.1) yields that

$$\|Mg\|_{L^{\frac{q_1}{2}}(\mathbb{R}^n)} \lesssim \|g\|_{L^{\frac{q_1}{2}}(\mathbb{R}^n)} = \|g\|_{L^p(\mathbb{R}^n)}, \tag{3.2}$$

since we require that $\frac{q_1}{2} = p$, *i.e.*

$$\frac{2}{q_1} = \frac{1}{p}. \tag{3.3}$$

On the other hand, by Lemma 5.3, we have

$$\|I_2 * |g|\|_{L^{\frac{q_2}{2}}(\mathbb{R}^n)} \lesssim \|g\|_{L^p(\mathbb{R}^n)}, \tag{3.4}$$

where

$$\frac{2}{q_2} = \frac{1}{p} - \frac{2}{n}. \tag{3.5}$$

Consequently, $\|\nabla_x \phi\|_{L^q(\mathbb{R}^n)} \lesssim \|g\|^{\frac{1}{2}}_{L^p(\mathbb{R}^n)} \cdot \|g\|^{\frac{1}{2}}_{L^p(\mathbb{R}^n)} = \|g\|_{L^p(\mathbb{R}^n)}$. □

A "derivative version" is immediate:

Corollary 3.2. *With the same assumptions as in Lemma 3.1, we have*

$$\|\partial_x^\alpha \nabla_x \phi\|_{L^q(\mathbb{R}^n)} \lesssim \|\partial_x^\alpha g\|_{L^p(\mathbb{R}^n)}.$$

Proof. One only needs to observe that

$$\nabla_x \partial_x^\alpha \phi = \partial_x^\alpha \nabla_x \phi = \partial_x^\alpha \nabla_x (I_2 * g) = \nabla_x (I_2 * \partial_x^\alpha g).$$

Applying Lemma 3.1 with ϕ and g replaced by $\partial^\alpha \phi$ and $\partial^\alpha g$ respectively, the desired result is immediate. □

Now we adapt Corollary 3.2 to the "kinetic version". To achieve this goal, we need to introduce a weight w.

Corollary 3.3. *Take* $g = \int_{\mathbb{R}^n} f dv$ *in Corollary 3.2, then we have*

$$\|\partial_x^\alpha \nabla_x \phi\|_{L_x^q(\mathbb{R}^n)} \lesssim \|\partial_x^\alpha f\|_{L_{x,v}^p(w)}.$$

Proof. Hölder's inequality leads to

$$\left| \int_{\mathbb{R}^n} \partial_x^\alpha f dv \right| \lesssim \left(\int_{\mathbb{R}^n} |\partial_x^\alpha f|^p w dv \right)^{\frac{1}{p}} \left(\int_{\mathbb{R}^n} w^{-\frac{p'}{p}} dv \right)^{\frac{1}{p'}}.$$

Note that $w = \langle v \rangle^\gamma$ and $\gamma \cdot \frac{p'}{p} > n$, which implies that

$$\left(\int_{\mathbb{R}^n} w^{-\frac{p'}{p}} dv \right)^{\frac{1}{p'}} \leq c.$$

Thus we end the proof of Corollary 3.3. □

An L^∞ estimate is also needed in the proof of the main Theorem 2.1.

Lemma 3.4. *Suppose* $-\Delta \phi = \int_{\mathbb{R}^n} f dv$. *If* $0 \leq |\alpha| \leq m-2, m \geq 3$, *then*

$$\|\partial_x^\alpha \nabla_x \phi\|_{L_x^\infty} \lesssim \sum_{|i| \leq 2} \|\partial_x^{i+\alpha} f\|_{L_{x,v}^p(w)}. \tag{3.6}$$

Proof. Choose a q such that $q > \frac{n}{2}$ and $p \leq q$, then $W^{2,q} \hookrightarrow L^\infty$. Thus we have

$$\|\partial_x^\alpha \nabla_x \phi\|_{L_x^\infty} \lesssim \|\partial_x^\alpha \nabla_x \phi\|_{W^{2,q}(\mathbb{R}_x^n)}.$$

Combining Corollary 3.2 and Corollary 3.3 leads to

$$\|\partial_x^\alpha \nabla_x \phi\|_{W^{2,q}} = \sum_{|i| \leq 2} \|\partial_x^i \partial_x^\alpha \nabla_x \phi\|_{L_x^q} \lesssim \sum_{|i| \leq 2} \|\partial_x^{i+\alpha} f\|_{L_{x,v}^p(w)},$$

i.e.

$$\|\partial_x^\alpha \nabla_x \phi\|_{L_x^\infty} \lesssim \sum_{|i| \leq 2} \|\partial_x^{i+\alpha} f\|_{L_{x,v}^p(w)}.$$ □

4 Proof of main theorem

Now we are in the position to prove Theorem 2.1. We split the proof into two parts which are existence and uniqueness.

Part I: Proof of existence. To prove the existence of the solution to (1.1), we adopt the L^p-version energy method and iteration method. In this process, we will apply the L^p-L^q estimate of electronic term $\nabla_x\phi$ proved in Lemma 3.1 to estimate J_3.

Proof. We consider the following iterating sequence for solving the Vlasov-Poisson system (1.1),

$$\begin{cases} \partial_t f^{k+1} + v\cdot\nabla_x f^{k+1} + \nabla_x\phi^k\cdot\nabla_v f^{k+1} = 0, & (4.1)\\ -\Delta\phi^k = \int_{\mathbb{R}^n} f^k dv, & (4.2)\\ f^{k+1}(0,x,v) = f_0(x,v). & (4.3)\end{cases}$$

Step 1. Applying $\partial_x^\alpha\partial_v^\beta$ to (4.1) with $\beta\neq 0, |\alpha|+|\beta|\le m$, starting with $f^0(t,x,v)=f_0(x,v)$, we have

$$\begin{aligned}&(\partial_t + v\cdot\nabla_x + \nabla_x\phi^k\cdot\nabla_v)\partial_x^\alpha\partial_v^\beta f^{k+1} + \sum_{\beta_1<\beta} C_\beta^{\beta_1}\partial_v^{\beta-\beta_1}v\cdot\partial_v^{\beta_1}\nabla_x\partial_x^\alpha f^{k+1}\\ &= -\sum_{0\neq\alpha_1\le\alpha} C_\alpha^{\alpha_1}\partial_x^{\alpha_1}\nabla_x\phi^k\cdot\partial_x^{\alpha-\alpha_1}\partial_v^\beta\nabla_v f^{k+1}.\end{aligned} \tag{4.4}$$

Multiplying $|\partial_x^\alpha\partial_v^\beta f^{k+1}|^{p-2}(\partial_x^\alpha\partial_v^\beta f^{k+1})w$ on both sides of (4.4), and then integrating over $\mathbb{R}_x^n\times\mathbb{R}_v^n$ yields that

$$\begin{aligned}&\frac{1}{p}\frac{d}{dt}\|\partial_x^\alpha\partial_v^\beta f^{k+1}\|_{L^p_{x,v}(w)}^p\\ &+\underbrace{\sum_{\beta_1<\beta} C_\beta^{\beta_1}\left\langle\partial_v^{\beta-\beta_1}v\cdot\partial_v^{\beta_1}\nabla_x\partial_x^\alpha f^{k+1}, |\partial_x^\alpha\partial_v^\beta f^{k+1}|^{p-2}(\partial_x^\alpha\partial_v^\beta f^{k+1})w\right\rangle}_{J_1}\\ &=\underbrace{\left\langle\nabla_x\phi^k\cdot|\partial_x^\alpha\partial_v^\beta f^{k+1}|^p, \nabla_v w\right\rangle}_{J_2}\\ &-\underbrace{\sum_{0\neq\alpha_1\le\alpha} C_\alpha^{\alpha_1}\left\langle\partial_x^{\alpha_1}\nabla_x\phi^k\cdot\partial_x^{\alpha-\alpha_1}\partial_v^\beta\nabla_v f^{k+1}, |\partial_x^\alpha\partial_v^\beta f^{k+1}|^{p-2}(\partial_x^\alpha\partial_v^\beta f^{k+1})w\right\rangle}_{J_3}.\end{aligned} \tag{4.5}$$

We now estimate (4.5) term by term.

For J_1, note that $|\partial_v^{\beta-\beta_1} v| \leq c$, $\beta_1 < \beta$. Thus,

$$\begin{aligned}
J_1 &\lesssim \sum_{\beta_1<\beta} \int_{\mathbb{R}^{2n}} |\partial_v^{\beta_1} \nabla_x \partial_x^{\alpha} f^{k+1}| w^{\frac{1}{p}} \cdot |\partial_x^{\alpha} \partial_v^{\beta} f^{k+1}|^{p-1} w^{\frac{1}{p'}} dxdv \\
&\lesssim \sum_{\beta_1<\beta} \left(\int_{\mathbb{R}^{2n}} |\partial_v^{\beta_1} \nabla_x \partial_x^{\alpha} f^{k+1}|^p w \ dxdv \right)^{\frac{1}{p}} \left(\int_{\mathbb{R}^{2n}} |\partial_x^{\alpha} \partial_v^{\beta} f^{k+1}|^{(p-1)p'} w \ dxdv \right)^{\frac{1}{p'}} \\
&\lesssim \sum_{\beta_1<\beta} \|\partial_v^{\beta_1} \nabla_x \partial_x^{\alpha} f^{k+1}\|_{L^p_{x,v}(w)} \cdot \|\partial_x^{\alpha} \partial_v^{\beta} f^{k+1}\|^{p-1}_{L^p_{x,v}(w)},
\end{aligned}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$, *i.e.* $(p-1)p' = p$, $\frac{p}{p'} = p - 1$.

For J_2, note $|\nabla_v w| \leq w$, by Lemma 3.4, we have

$$\begin{aligned}
J_2 &\lesssim \|\nabla_x \phi^k\|_{L^\infty_x} \|\partial_x^{\alpha} \partial_v^{\beta} f^{k+1}\|^p_{L^p_{x,v}(w)} \\
&\lesssim \sum_{|i|\leq 2} \|\partial_x^i f^k\|_{L^p_{x,v}(w)} \|\partial_x^{\alpha} \partial_v^{\beta} f^{k+1}\|^p_{L^p_{x,v}(w)}.
\end{aligned}$$

For J_3, we consider two cases individually.

Case 1: Recall $|\alpha| \leq m-1$, if $0 < |\alpha_1| \leq m-2, m \geq 3$, Lemma 3.4 leads to

$$\|\partial_x^{\alpha_1} \nabla_x \phi^k\|_{L^\infty_x} \lesssim \sum_{|i|\leq 2} \|\partial_x^{i+\alpha_1} f^k\|_{L^p_{x,v}(w)}.$$

Note $|i| + |\alpha_1| \leq m - 2 + 2 = m$, the order of the derivatives does not exceed m, then we obtain,

$$\begin{aligned}
J_3 &\lesssim \sum_{0<|\alpha_1|\leq m-2} \int_{\mathbb{R}^n} \|\partial_x^{\alpha_1} \nabla_x \phi^k\|_{L^\infty_x} \|\partial_x^{\alpha-\alpha_1} \partial_v^{\beta} \nabla_v f^{k+1}\|_{L^p_x} \left\| |\partial_x^{\alpha} \partial_v^{\beta} f^{k+1}|^{p-1} \right\|_{L^{p'}_x} w \ dv \\
&\lesssim \sum_{0<|\alpha_1|\leq m-2} \sum_{|i|\leq 2} \|\partial_x^{i+\alpha_1} f^k\|_{L^p_{x,v}(w)} \|\partial_x^{\alpha-\alpha_1} \partial_v^{\beta} \nabla_v f^{k+1}\|_{L^p_{x,v}(w)} \|\partial_x^{\alpha} \partial_v^{\beta} f^{k+1}\|^{p-1}_{L^p_{x,v}(w)},
\end{aligned}$$

where $|\alpha - \alpha_1| + |\beta| + 1 \leq |\alpha| + |\beta| \leq m$.

Case 2: $|\alpha_1| = m - 1$, we have

$$\begin{aligned}
J_3 &\lesssim \sum_{|\alpha_1|=m-1} \int_{\mathbb{R}^n} w(v) \|\partial_x^{\alpha_1} \nabla_x \phi^k\|_{L^q_x} \|\partial_x^{\alpha-\alpha_1} \partial_v^{\beta} \nabla_v f^{k+1}\|_{L^n_x} \left\| |\partial_x^{\alpha} \partial_v^{\beta} f^{k+1}|^{p-1} \right\|_{L^{p'}_x} dv \\
&\lesssim \sum_{|\alpha_1|=m-1} \|\partial_x^{\alpha_1} \nabla_x \phi^k\|_{L^q_x} \int_{\mathbb{R}^n} w(v) \sum_{|i|\leq m-2} \|\partial_x^i \partial_x^{\alpha-\alpha_1} \partial_v^{\beta} \nabla_v f^{k+1}\|_{L^p_x} \left\| |\partial_x^{\alpha} \partial_v^{\beta} f^{k+1}|^{p-1} \right\|_{L^{p'}_x} dv \\
&\lesssim \sum_{|\alpha_1|=m-1} \|\partial_x^{\alpha_1} f^k\|_{L^p_{x,v}(w)} \sum_{|i|\leq m-2} \|\partial_x^i \partial_x^{\alpha-\alpha_1} \partial_v^{\beta} \nabla_v f^{k+1}\|_{L^p_{x,v}(w)} \|\partial_x^{\alpha} \partial_v^{\beta} f^{k+1}\|^{p-1}_{L^p_{x,v}(w)},
\end{aligned}$$

where in the first inequality, we applied Hölder's inequality with respect to x with

$$\frac{1}{q} + \frac{1}{n} + \frac{1}{p'} = 1, \quad \frac{1}{p} + \frac{1}{p'} = 1.$$

And in the second inequality, we used the embedding $W^{m-2,p} \hookrightarrow L^n$, with

$$m > \frac{n}{p} + 1, \quad p \leq n. \tag{4.6}$$

In the third inequality, we applied Corollary 3.3 and Hölder's inequality in v.

Finally, plugging all the estimates of J_1, J_2, and J_3 into (4.5) yields that

$$\begin{aligned}
&\frac{d}{dt}\|\partial_x^\alpha\partial_v^\beta f^{k+1}\|^p_{L^p_{x,v}(w)} \lesssim \sum_{\beta_1<\beta}\|\partial_v^{\beta_1}\nabla_x\partial_x^\alpha f^{k+1}\|_{L^p_{x,v}(w)}\cdot\|\partial_x^\alpha\partial_v^\beta f^{k+1}\|^{p-1}_{L^p_{x,v}(w)}\\
&+\sum_{|i|\le 2}\|\partial_x^i f^k\|_{L^p_{x,v}(w)}\|\partial_x^\alpha\partial_v^\beta f^{k+1}\|^p_{L^p_{x,v}(w)}\\
&+\sum_{0<|\alpha_1|\le m-2}\sum_{|i|\le 2}\|\partial_x^{i+\alpha_1}f^k\|_{L^p_{x,v}(w)}\|\partial_x^{\alpha-\alpha_1}\partial_v^\beta\nabla_v f^{k+1}\|_{L^p_{x,v}(w)}\|\partial_x^\alpha\partial_v^\beta f^{k+1}\|^{p-1}_{L^p_{x,v}(w)}\\
&+\sum_{|\alpha_1|=m-1}\sum_{|i|\le m-2}\|\partial_x^{\alpha_1}f^k\|_{L^p_{x,v}(w)}\|\partial_x^i\partial_x^{\alpha-\alpha_1}\partial_v^\beta\nabla_v f^{k+1}\|_{L^p_{x,v}(w)}\|\partial_x^\alpha\partial_v^\beta f^{k+1}\|^{p-1}_{L^p_{x,v}(w)}.
\end{aligned}\tag{4.7}$$

Step 2. $\beta=0, |\alpha|\le m$, applying ∂_x^α to (4.1) on both sides, we have

$$(\partial_t+v\cdot\nabla_x+\nabla_x\phi^k\cdot\nabla_v)\partial_x^\alpha f^{k+1}=-\sum_{0\neq\alpha_1\le\alpha}C_\alpha^{\alpha_1}\partial_x^{\alpha_1}\nabla_x\phi^k\cdot\partial_x^{\alpha-\alpha_1}\nabla_v f^{k+1}. \tag{4.8}$$

We could completely repeat the process of step 1, the only difference is that we do not need to estimate J_1, thus we give the estimates as below but omit the process of proof in details.

$$\begin{aligned}
&\frac{d}{dt}\|\partial_x^\alpha f^{k+1}\|^p_{L^p_{x,v}(w)} \lesssim \sum_{|i|\le 2}\|\partial_x^i f^k\|_{L^p_{x,v}(w)}\|\partial_x^\alpha f^{k+1}\|^p_{L^p_{x,v}(w)}\\
&+\sum_{0<|\alpha_1|\le m-2}\sum_{|i|\le 2}\|\partial_x^{i+\alpha_1}f^k\|_{L^p_{x,v}(w)}\|\partial_x^{\alpha-\alpha_1}\nabla_v f^{k+1}\|_{L^p_{x,v}(w)}\|\partial_x^\alpha f^{k+1}\|^{p-1}_{L^p_{x,v}(w)}\\
&+\sum_{m-1\le|\alpha_1|\le m}\sum_{|i|\le m-2}\|\partial_x^{\alpha_1}f^k\|_{L^p_{x,v}(w)}\|\partial_x^i\partial_x^{\alpha-\alpha_1}\nabla_v f^{k+1}\|_{L^p_{x,v}(w)}\|\partial_x^\alpha f^{k+1}\|^{p-1}_{L^p_{x,v}(w)}.
\end{aligned}\tag{4.9}$$

Collecting the estimates of J_1, J_2 and J_3 and integrating over $[0,t]$ of (4.5), summing over $|\alpha|+|\beta|\le m$, we deduce from the definition of $\mathcal{E}(f(t))$ that

$$\mathcal{E}(f^{k+1}(t))\le\mathcal{E}(f_0)+Ct\sup_{0\le s\le t}\mathcal{E}(f^{k+1}(s))+Ct\sup_{0\le s\le t}(\mathcal{E}(f^k(s)))^{\frac{1}{p}}\cdot\sup_{0\le s\le t}\mathcal{E}(f^{k+1}(s)).$$

Inductively, assume $\sup\limits_{0\le s\le T^*(M)}\mathcal{E}(f^k(s))\le M$, $T^*(M)$ and M are sufficiently small; note that $f^0(t,x,v)\equiv f_0(x,v)$, $\mathcal{E}(f_0)\le\frac{M}{2}$, we have

$$\mathcal{E}(f^{k+1}(t))\le\frac{M}{2}+Ct\sup_{0\le s\le t}\mathcal{E}(f^{k+1}(s))+CM^{\frac{1}{p}}\cdot t\sup_{0\le s\le t}\mathcal{E}(f^{k+1}(s)),$$

i.e.

$$(1-CT^*-CM^{\frac{1}{p}}T^*(M))\sup_{0\le s\le T^*(M)}\mathcal{E}(f^{k+1}(s))\le\frac{M}{2}.$$

Thus $\sup\limits_k\sup\limits_{0\le s\le T^*(M)}\mathcal{E}(f^k(s))\le M$, *i.e.* we get a uniform-in-k estimate.

As a routine, let $k\to\infty$, we obtain the solution and complete the proof of existence.

Remark 4.1. *We summarize the indices as follows:*

$$\begin{cases} \dfrac{2}{q_1} = \dfrac{1}{p}, \quad 1 < p < \dfrac{n}{2}, \quad n \geq 3, & (4.10) \\ \dfrac{2}{q_2} = \dfrac{1}{p} - \dfrac{2}{n}, & (4.11) \\ \dfrac{1}{q_1} + \dfrac{1}{q_2} = \dfrac{1}{q}, \quad q > 1, \quad q_i > 1,\ i = 1, 2, & (4.12) \\ q > \dfrac{n}{2}, & (4.13) \\ m > \dfrac{n}{p} + 1, \quad m, n \in \mathbb{N}, & (4.14) \\ \dfrac{1}{q} + \dfrac{1}{p_1} = \dfrac{1}{p}, & (4.15) \\ \gamma > n(p-1). & (4.16) \end{cases}$$

In fact, for any given (m, n, p) satisfying

$$\begin{cases} m > \frac{n}{p} + 1, m \in \mathbb{N}, \\ n \geq 3, \\ \frac{n}{3} < p < \frac{n}{2}, \end{cases} \tag{4.17}$$

we could designate

$$\begin{cases} q_1 = 2p, \\ q_2 = \frac{2np}{n-2p}, \\ q = \frac{np}{n-1}. \end{cases} \tag{4.18}$$

□

Let us move on to proving the uniqueness.

Part II: Proof of uniqueness. The proof of the uniqueness is analogous to the existence part. However, we use a different energy norm $\mathcal{E}_1(f(t)) =: \sum\limits_{|\alpha|+|\beta|\leq m-1} \|\partial_x^\alpha \partial_v^\beta f(t)\|^p_{L^p_{x,v}(w)}$ because of a difficult term $\tilde{J}_4$. In $\tilde{J}_4$, there is a term

$$\Big\langle \nabla_x(\phi_f - \phi_g) \cdot \partial_x^\alpha \partial_v^\beta \nabla_v g,\ |\partial_x^\alpha \partial_v^\beta (f-g)|^{p-2} \cdot \partial_x^\alpha \partial_v^\beta (f-g) w \Big\rangle.$$

If we still work with $\mathcal{E}(f(t)) = \sum\limits_{|\alpha|+|\beta|\leq m} \|\partial_x^\alpha \partial_v^\beta f(t)\|^p_{L^p_{x,v}(w)}$ as in the existence part, the order of derivative of $\partial_x^\alpha \partial_v^\beta \nabla_v g$ will be $m+1$ which exceeds m when $|\alpha| + |\beta| = m$. This is the main reason we choose $\mathcal{E}_1(f(t))$ instead of $\mathcal{E}(f(t))$.

Proof. Assume another solution g exists such that $\sup_{0\le s\le T^*}\mathcal{E}(g(s))\le M$, taking the difference, we have

$$\begin{cases} (\partial_t + v\cdot\nabla_x + \nabla_x\phi_f\cdot\nabla_v)(f-g) + (\nabla_x\phi_f - \nabla_x\phi_g)\cdot\nabla_v g = 0, \\ -\Delta_x(\phi_f-\phi_g) = \displaystyle\int_{\mathbb{R}^n}(f-g)\, dv, \\ f(0,x,v) = g(0,x,v). \end{cases} \tag{4.19}$$

Step 1. Applying $\partial_x^\alpha\partial_v^\beta$ on both sides of $(4.19)_1$ with $\beta\neq 0$, $|\alpha|+|\beta|\le m-1$, we have

$$\begin{aligned} &(\partial_t + v\cdot\nabla_x + \nabla_x\phi_f\cdot\nabla_v)\cdot\partial_x^\alpha\partial_v^\beta(f-g) + \sum_{\beta_1<\beta} C_\beta^{\beta_1}\partial_v^{\beta-\beta_1}v\cdot\partial_v^{\beta_1}\nabla_x\partial_x^\alpha(f-g) \\ &= -\sum_{0\neq\alpha_1\le\alpha} C_\alpha^{\alpha_1}\partial_x^{\alpha_1}\nabla_x\phi_f\cdot\partial_x^{\alpha-\alpha_1}\partial_v^\beta\nabla_v(f-g) \\ &\quad -\sum_{0\le\alpha_1\le\alpha} C_\alpha^{\alpha_1}\partial_x^{\alpha_1}(\nabla_x\phi_f-\nabla_x\phi_g)\cdot\partial_x^{\alpha-\alpha_1}\partial_v^\beta\nabla_v g. \end{aligned} \tag{4.20}$$

Multiplying $|\partial_x^\alpha\partial_v^\beta(f-g)|^{p-2}\cdot\partial_x^\alpha\partial_v^\beta(f-g)w$ on both sides of (4.20), and then integrating over $\mathbb{R}_x^n\times\mathbb{R}_v^n$ yields that

$$\begin{aligned} &\frac{1}{p}\frac{d}{dt}\|\partial_x^\alpha\partial_v^\beta(f-g)\|_{L^p_{x,v}(w)}^p \\ &\quad + \underbrace{\sum_{\beta_1<\beta} C_\beta^{\beta_1}\Big\langle \partial_v^{\beta-\beta_1}v\cdot\partial_v^{\beta_1}\nabla_x\partial_x^\alpha(f-g), |\partial_x^\alpha\partial_v^\beta(f-g)|^{p-2}\cdot\partial_x^\alpha\partial_v^\beta(f-g)w\Big\rangle}_{\tilde{J}_1} \\ &= \underbrace{\Big\langle \nabla_x\phi_f\cdot|\partial_x^\alpha\partial_v^\beta(f-g)|^p, \nabla_v w\Big\rangle}_{\tilde{J}_2} \\ &\quad - \underbrace{\sum_{0\neq\alpha_1\le\alpha} C_\alpha^{\alpha_1}\Big\langle \partial_x^{\alpha_1}\nabla_x\phi_f\cdot\partial_x^{\alpha-\alpha_1}\partial_v^\beta\nabla_v(f-g), |\partial_x^\alpha\partial_v^\beta(f-g)|^{p-2}\cdot\partial_x^\alpha\partial_v^\beta(f-g)w\Big\rangle}_{\tilde{J}_3} \\ &\quad - \underbrace{\sum_{0\le\alpha_1\le\alpha} C_\alpha^{\alpha_1}\Big\langle \partial_x^{\alpha_1}(\nabla_x\phi_f-\nabla_x\phi_g)\cdot\partial_x^{\alpha-\alpha_1}\partial_v^\beta\nabla_v g, |\partial_x^\alpha\partial_v^\beta(f-g)|^{p-2}\cdot\partial_x^\alpha\partial_v^\beta(f-g)w\Big\rangle}_{\tilde{J}_4}. \end{aligned} \tag{4.21}$$

We could repeat the estimates in the proof of the existence except for some special term. Thus we would like to write down the estimates directly without the details.

For $\tilde{J}_1$, we have

$$\tilde{J}_1 \lesssim \sum_{\beta_1<\beta}\|\partial_v^{\beta_1}\nabla_x\partial_x^\alpha(f-g)\|_{L^p_{x,v}(w)}\cdot\|\partial_x^\alpha\partial_v^\beta(f-g)\|_{L^p_{x,v}(w)}^{p-1}.$$

For $\tilde{J}_2$, we get

$$\tilde{J}_2 \lesssim \sum_{|i|\le 2}\|\partial_x^i f\|_{L^p_{x,v}(w)}\cdot\|\partial_x^\alpha\partial_v^\beta(f-g)\|_{L^p_{x,v}(w)}^p.$$

For $\tilde{J}_3$, since $0 < |\alpha_1| \le m-2$, we have

$$\tilde{J}_3 \lesssim \sum_{0<|\alpha_1|\le m-2} \sum_{|i|\le 2} \|\partial_x^{i+\alpha_1} f\|_{L^p_{x,v}(w)} \cdot \|\partial_x^{\alpha-\alpha_1}\partial_v^\beta \nabla_v (f-g)\|_{L^p_{x,v}(w)} \cdot \|\partial_x^\alpha \partial_v^\beta (f-g)\|^{p-1}_{L^p_{x,v}(w)},$$

where $|\alpha-\alpha_1| + |\beta| + 1 \le |\alpha| + |\beta| \le m-1$.

For $\tilde{J}_4$, note that $-\Delta_x(\phi_f - \phi_g) = \int_{\mathbb{R}^n} (f-g)\, dv$. We consider two cases separately.

Case 1: $0 \le |\alpha_1| \le m-3$

$$\tilde{J}_4 \lesssim \sum_{0\le|\alpha_1|\le m-3} \sum_{|i|\le 2} \|\partial_x^{i+\alpha_1} (f-g)\|_{L^p_{x,v}(w)} \cdot \|\partial_x^{\alpha-\alpha_1}\partial_v^\beta \nabla_v g\|_{L^p_{x,v}(w)} \cdot \|\partial_x^\alpha \partial_v^\beta (f-g)\|^{p-1}_{L^p_{x,v}(w)},$$

where $|i| + |\alpha_1| \le 2 + m - 3 = m-1$ and

$$|\alpha-\alpha_1| + |\beta| + 1 \le |\alpha| + |\beta| - |\alpha_1| + 1 \le m - 1 - |\alpha_1| + 1 \le m.$$

Case 2: $|\alpha_1| = m-2$

$$\tilde{J}_4 \lesssim \sum_{|\alpha_1|=m-2} \sum_{|i|\le m-2} \|\partial_x^{\alpha_1} (f-g)\|_{L^p_{x,v}(w)} \cdot \ \|\partial_x^i \partial_x^{\alpha-\alpha_1}\partial_v^\beta \nabla_v g\|_{L^p_{x,v}(w)} \cdot \ \|\partial_x^\alpha \partial_v^\beta (f-g)\|^{p-1}_{L^p_{x,v}(w)},$$

where $|i| + |\alpha-\alpha_1| + |\beta| + 1 \le m - 2 + |\alpha| - |\alpha_1| + |\beta| + 1 \le m$.

Collecting all the estimates of $\tilde{J}_j$, $j = 1,2,3,4$, we have

$$\begin{aligned}
&\frac{d}{dt}\|\partial_x^\alpha \partial_v^\beta (f-g)\|^p_{L^p_{x,v}(w)} \lesssim \sum_{\beta_1<\beta} \|\partial_v^{\beta_1}\nabla_x \partial_x^\alpha (f-g)\|_{L^p_{x,v}(w)} \cdot \|\partial_x^\alpha \partial_v^\beta (f-g)\|^{p-1}_{L^p_{x,v}(w)} \\
&+ \sum_{|i|\le 2} \|\partial_x^i f\|_{L^p_{x,v}(w)} \cdot \|\partial_x^\alpha \partial_v^\beta (f-g)\|^p_{L^p_{x,v}(w)} \\
&+ \sum_{0<|\alpha_1|\le m-2} \sum_{|i|\le 2} \|\partial_x^{i+\alpha_1} f\|_{L^p_{x,v}(w)} \cdot \ \|\partial_x^{\alpha-\alpha_1}\partial_v^\beta \nabla_v (f-g)\|_{L^p_{x,v}(w)} \cdot \ \|\partial_x^\alpha \partial_v^\beta (f-g)\|^{p-1}_{L^p_{x,v}(w)} \\
&+ \sum_{0\le|\alpha_1|\le m-3} \sum_{|i|\le 2} \|\partial_x^{i+\alpha_1} (f-g)\|_{L^p_{x,v}(w)} \cdot \ \|\partial_x^{\alpha-\alpha_1}\partial_v^\beta \nabla_v g\|_{L^p_{x,v}(w)} \cdot \ \|\partial_x^\alpha \partial_v^\beta (f-g)\|^{p-1}_{L^p_{x,v}(w)} \\
&+ \sum_{|\alpha_1|=m-2} \sum_{|i|\le m-2} \|\partial_x^{\alpha_1} (f-g)\|_{L^p_{x,v}(w)} \cdot \ \|\partial_x^i \partial_x^{\alpha-\alpha_1}\partial_v^\beta \nabla_v g\|_{L^p_{x,v}(w)} \cdot \ \|\partial_x^\alpha \partial_v^\beta (f-g)\|^{p-1}_{L^p_{x,v}(w)}.
\end{aligned} \tag{4.22}$$

Step 2. $\beta = 0$, $|\alpha| \le m-1$, applying ∂_x^α on both sides of $(4.19)_1$ yields

$$\begin{aligned}
(\partial_t + v\cdot\nabla_x + \nabla_x\phi_f \cdot \nabla_v)\partial_x^\alpha (f-g) = &- \sum_{0\ne\alpha_1\le\alpha} C_\alpha^{\alpha_1} \partial_x^{\alpha_1}\nabla_x \phi_f \cdot \partial_x^{\alpha-\alpha_1}\nabla_v (f-g) \\
&- \sum_{0\le\alpha_1\le\alpha} C_\alpha^{\alpha_1} \partial_x^{\alpha_1}(\nabla_x\phi_f - \nabla_x\phi_g) \cdot \partial_x^{\alpha-\alpha_1}\nabla_v g.
\end{aligned} \tag{4.23}$$

Repeating the process of step 1, we get

$$\begin{aligned}
&\frac{d}{dt}\|\partial_x^\alpha(f-g)\|_{L^p_{x,v}(w)}^p \lesssim \sum_{|i|\le 2}\|\partial_x^i f\|_{L^p_{x,v}(w)}\cdot\ \|\partial_x^\alpha(f-g)\|_{L^p_{x,v}(w)}^p \\
&+\sum_{0<|\alpha_1|\le m-2}\ \sum_{|i|\le 2}\|\partial_x^{i+\alpha_1} f\|_{L^p_{x,v}(w)}\cdot\ \|\partial_x^{\alpha-\alpha_1}\nabla_v(f-g)\|_{L^p_{x,v}(w)}\cdot\ \|\partial_x^\alpha(f-g)\|_{L^p_{x,v}(w)}^{p-1} \\
&+\sum_{|\alpha_1|=m-1}\ \sum_{|i|\le m-2}\|\partial_x^{\alpha_1} f\|_{L^p_{x,v}(w)}\cdot\ \|\partial_x^i\partial_x^{\alpha-\alpha_1}\nabla_v(f-g)\|_{L^p_{x,v}(w)}\cdot\ \|\partial_x^\alpha(f-g)\|_{L^p_{x,v}(w)}^{p-1} \\
&+\sum_{0<|\alpha_1|\le m-3}\ \sum_{|i|\le 2}\|\partial_x^{i+\alpha_1}(f-g)\|_{L^p_{x,v}(w)}\cdot\ \|\partial_x^{\alpha-\alpha_1}\nabla_v g\|_{L^p_{x,v}(w)}\cdot\ \|\partial_x^\alpha\partial_v^\beta(f-g)\|_{L^p_{x,v}(w)}^{p-1} \\
&+\sum_{m-2\le|\alpha_1|\le m-1}\ \sum_{|i|\le m-2}\|\partial_x^{\alpha_1}(f-g)\|_{L^p_{x,v}(w)}\cdot\ \|\partial_x^i\partial_x^{\alpha-\alpha_1}\nabla_v g\|_{L^p_{x,v}(w)}\cdot\ \|\partial_x^\alpha(f-g)\|_{L^p_{x,v}(w)}^{p-1}.
\end{aligned} \tag{4.24}$$

Note $f(0,x,v)=g(0,x,v)$,

$$\sup_{0\le s\le t}\|\partial_x^{i+\alpha_1} f(s)\|_{L^p_{x,v}(w)}\le M, \quad \sup_{0\le s\le t}\|\partial_x^{\alpha-\alpha_1}\partial_v^\beta\nabla_v g(s)\|_{L^p_{x,v}(w)}\le M,$$

and

$$\sup_{0\le s\le t}\|\partial_x^i\partial_x^{\alpha-\alpha_1}\partial_v^\beta\nabla_v g(s)\|_{L^p_{x,v}(w)}\le M, \quad \sup_{0\le s\le t}\|\partial_x^i f(s)\|_{L^p_{x,v}(w)}\le M.$$

Integrating (4.22) and (4.24) over $[0,t]$, then summing over $|\alpha|+|\beta|\le m-1$, we deduce

$$\mathcal{E}_1((f-g)(t))\lesssim (1+M)\int_0^t \mathcal{E}_1((f-g)(s))\ ds,$$

where

$$\mathcal{E}_1(f(t)) =: \sum_{|\alpha|+|\beta|\le m-1}\|\partial_x^\alpha\partial_v^\beta f(t)\|_{L^p_{x,v}(w)}^p.$$

By Gronwall's inequality, we have $\mathcal{E}_1((f-g)(t))\equiv 0$ implying $f\equiv g$, which completes the proof of uniqueness. Thus we end the proof of Theorem 2.1. □

Remark 4.2. *All in all, we improved the results in [4] to the more general function space $W^{m,p}(\mathbb{R}^n)$ which does not have to be $C^2(\mathbb{R}^n)$ (too strong). Our results also shed light on exploring solutions in Sobolev spaces. We are very confident that our method could be applied in fractional Sobolev spaces, even the supercritical spaces which are far from being understood yet.*

5 Appendix

For the sake of completeness, we cite some known results about the estimate for the Riesz potential.

First of all, we give the pointwise estimate of the Riesz potential, for more details, see chapter 3, section 1, page 57 in [1].

Proposition 5.1 ([1])**.** *For any multi-index ξ with $|\xi| < \alpha < n$, there is a constant A such that for any $f \in L^p(\mathbb{R}^n)$, $1 \leq p < \infty$, and almost every x, we have*

$$|D^\xi(I_\alpha * f(x))| \leq AMf(x)^{\frac{|\xi|}{\alpha}} \cdot (I_\alpha * |f|(x))^{1-\frac{|\xi|}{\alpha}}, \tag{5.1}$$

where $I_\alpha = \dfrac{\gamma_\alpha}{|x|^{n-\alpha}}$, $\gamma_\alpha = \dfrac{\Gamma(n-\frac{\alpha}{2})}{\pi^{\frac{n}{2}}2^\alpha\Gamma(\frac{\alpha}{2})}$.

Remark 5.2. *In our paper, we consider $-\Delta\phi = \int_{\mathbb{R}^n} f dv =: g$, $n \geq 3$. Thus, in our context, I_α can be taken*

$$I_2(x) = \frac{1}{(n-2)\omega_{n-1}} \cdot \frac{1}{|x|^{n-2}}, \quad i.e. \quad \alpha = 2, \tag{5.2}$$

where $\omega_{n-1} = \dfrac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}$ is the $(n-1)-$dimensional area of the unit sphere in $\mathbb{R}^n$, then we have

$$|D^\xi(I_2 * g(x))| \leq cMg(x)^{\frac{|\xi|}{2}} \cdot (I_2 * |g|(x))^{1-\frac{|\xi|}{2}}. \tag{5.3}$$

Next, we give the off-diagonal estimate of the Riesz potential I_2. For the details, see chapter V, section 1 and page 119 in [14].

Lemma 5.3 ([14])**.** *If $-\Delta\phi = g \in L^p(\mathbb{R}^n)$, then $\phi = I_2 * g$ and*

$$\|I_2 * g\|_{L^{\tilde{q}}(\mathbb{R}^n)} \leq c\|g\|_{L^p(\mathbb{R}^n)}, \tag{5.4}$$

where $1 < p < \frac{n}{2}, c = c(p, \tilde{q})$ and

$$\frac{1}{\tilde{q}} = \frac{1}{p} - \frac{2}{n}. \tag{5.5}$$

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