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Estimation of sharp geometric inequality in D_α-homothetically deformed Kenmotsu manifolds

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ABSTRACT

In this article, we investigate the Kenmotsu manifold when applied to a D_α-homothetic deformation. Then, given a submanifold in a D_α-homothetically deformed Kenmotsu manifold, we derive the generalized Wintgen inequality. Additionally, we find this inequality for submanifolds such as slant, invariant, and anti-invariant in the same ambient space.

RESUMEN

En este artículo estudiamos la variedad de Kenmotsu cuando se aplica a una deformación D_α-homotética. Luego, dada una subvariedad en una variedad de Kenmotsu D_α-homotéticamente deformada, derivamos la desigualdad de Wintgen generalizada. Adicionalmente, encontramos esta desigualdad para subvariedades tales como oblicuas, invariantes y anti-invariantes en el mismo espacio ambiente.

Keywords and Phrases: Normalized scalar curvature, scalar curvature, mean curvature, D_α-homothetic deformation.

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1 Introduction

The Wintgen inequality is a sharp geometric inequality for surfaces in a 4-dimensional Euclidean space $\mathbb{E}^4$ involving Gauss curvature K (intrinsic invariants), normal curvature and square mean curvature (extrinsic invariants). The intrinsic and extrinsic curvature of a surface can be combined in the second fundamental form. This is a *quadratic form* in the tangent plane to the surface at a point.

Quadratic forms occupy a central place in various branches of mathematics, including number theory, linear algebra, group theory (orthogonal groups), differential geometry (the Riemannian metric, the second fundamental form), differential topology (intersection forms of four-manifolds), Lie theory (the Killing form), and statistics (where the exponent of a zero-mean multivariate normal distribution has the quadratic form $\mathbf{x}^{\mathsf{T}}\mathbf{\Sigma}^{-1}\mathbf{x}$).

P. Wintgen [25], proved that the Gauss curvature K, the normal curvature $K^{\perp}$ and the squared mean curvature $||H||^2$ for any surface $\mathcal{M}^2$ in E^4 satisfy the following inequality [1]:

$$||\mathcal{H}||^2 \geq K + |K^{\perp}|$$

and the equality holds if and only if the ellipse of curvature of $\tilde{\mathcal{M}}^2$ in $\mathbb{E}^4$ is a circle. Later, it was extended by I. V. Guadalupe *et al.* [12] for arbitrary codimension m in real space forms $\tilde{\mathcal{M}}^{(m+2)}(c)$ as

$$||\mathcal{H}||^2 + c \geq K + |K^{\perp}|.$$

In 1999, De Smet *et al.* conjectured the generalized Wintgen inequality for submanifolds in real space form. The conjecture is known as DDVV conjecture. It has been proved by Zhiqin Lu in [16] and Jianquan Ge-Zizhou Tang in [11], independently and differently. Ion Mihai [17, 18] established such inequality for Lagrangian submanifold in complex space form and for Legendrian submanifolds in Sasakian space forms. Since then numerous authors studied such inequality for several kinds of submanifolds in different ambient space forms (for example, see [3, 12, 19–22]).

In 1971, Kenmotsu investigated a class of contact Riemannian manifolds, named Kenmostu manifolds, which satisfy some special conditions [15]. After that Kenmotsu manifolds have been discussed by Jun *et al.* [14] and many authors.

In 1968 Tanno [24] introduced the notion of D-homothetic deformation (for more details see [23]). In [8] Carriazo and Martin-Molina studied D-homothetic deformation of generalized (k, μ)-space forms. De and Ghosh studied D-homothetic deformation of almost contact metric manifolds [10].

In the present article, we obtain the generalized Wintgen inequalities for submanifolds of a D_α-

homothetically deformed Kenmotsu manifold. We also discuss such inequality for various slant submanifolds as an application of the inequality obtained.

2 Preliminaries

An odd dimensional $(2n+1)$ smooth manifold $(\tilde{\mathcal{M}}, g)$ is said to be an almost contact metric manifold [5], if it admits a $(1,1)$-tensor field φ, a structure vector field ζ, a 1-form η and a Riemannian metric g such that [26]

$$\varphi^2 E = -E + \eta(E)\zeta, \tag{2.1}$$

$$\eta(\zeta) = 1, \quad \varphi(\zeta) = 0, \quad \eta \circ \varphi = 0, \tag{2.2}$$

$$\eta(E) = g(E, \zeta), \tag{2.3}$$

$$g(\varphi E, \varphi F) = g(E, F) - \eta(E)\eta(F), \tag{2.4}$$

for any vector fields E, F on $\tilde{\mathcal{M}}$.

If a contact metric manifold satisfies

$$(\tilde{\nabla}_E \varphi)F = -g(E, \varphi F)\zeta - \eta(F)\varphi E, \tag{2.5}$$

where $\tilde{\nabla}$ denotes the Levi-Civita connection with respect to g, then $\tilde{\mathcal{M}}$ is called a *Kenmotsu manifold* [15].

An almost contact metric manifold is Kenmotsu manifold if and only if

$$\tilde{\nabla}_E \zeta = E - \eta(E)\zeta. \tag{2.6}$$

Moreover, we suppose that the Riemannian curvature tensor $\tilde{R}$, the Ricci tensor $\tilde{S}$ of type $(0,2)$ in Kenmotsu manifold $\tilde{\mathcal{M}}$ with respect to $\tilde{\nabla}$ satisfy [15]

$$(\tilde{\nabla}_E \eta)F = g(\varphi E, \varphi F) = g(E, F) - \eta(E)\eta(F), \tag{2.7}$$

$$(\tilde{\nabla}_\zeta \eta)F = 0, \tag{2.8}$$

$$\tilde{R}(E, F)\zeta = \eta(E)F - \eta(F)E, \tag{2.9}$$

$$\tilde{R}(\zeta, E)F = \eta(F)E - g(E, F)\zeta, \tag{2.10}$$

$$\tilde{R}(\zeta, E)\zeta = -\tilde{R}(E, \zeta)\zeta = E - \eta(E)\zeta, \tag{2.11}$$

$$\eta(\tilde{R}(E,F)G) = g(E,G)\eta(F) - g(F,G)\eta(E), \tag{2.12}$$

$$\tilde{S}(E,\zeta) = -2n\eta(E), \tag{2.13}$$

$$\tilde{S}(\varphi E,\varphi F) = \tilde{S}(E,F) + 2n\eta(E)\eta(F). \tag{2.14}$$

An odd dimensional Kenmotsu manifold $\tilde{\mathcal{M}}(\varphi,\zeta,\eta,g)$ is said to be η-Einstein manifold if $\tilde{S}$ is of the form

$$\tilde{S} = ag + b\eta\otimes\eta,$$

where a and b are smooth functions on $\tilde{\mathcal{M}}$.

Definition 2.1 ([24])**.** *If an $(2n+1)$-dimensional contact metric manifold $\tilde{\mathcal{M}}$ with almost contact metric structure (φ,ζ,η,g) is transformed into $(\varphi^\sharp,\zeta^\sharp,\eta^\sharp,g^\sharp)$, where*

$$\varphi^\sharp = \varphi, \quad \zeta^\sharp = \frac{1}{\alpha}\zeta, \quad \eta^\sharp = \alpha\eta, \quad g^\sharp = \alpha g + \alpha(\alpha-1)\eta\otimes\eta \tag{2.15}$$

and α is a positive constant, then the transformation is called a D_α-homothetic deformation.

The relation between the Levi-Civita connection $\tilde{\nabla}$ of g and the Levi-Civita connection $\nabla^\sharp$ of $g^\sharp$ is given by [2]

$$\nabla^\sharp_E F = \tilde{\nabla}_E F + \frac{\alpha-1}{\alpha} g(\varphi E,\varphi F)\zeta \tag{2.16}$$

for all vector fields E,F on $\tilde{\mathcal{M}}$.

3 Curvature tensor on Kenmotsu manifold under a D_α-homothetic deformation

Let $\tilde{\mathcal{M}}$ be Kenmotsu manifold of dimension $(2n+1)$. The curvature tensor $\mathcal{R}^\sharp$ of $\tilde{\mathcal{M}}$ under a D_α-homothetic deformation $\nabla^\sharp$ is defined by [13]

$$\mathcal{R}^\sharp(E,F)G = \nabla^\sharp{}_E\nabla^\sharp{}_F G - \nabla^\sharp{}_F\nabla^\sharp{}_E G - \nabla^\sharp{}_{[E,F]}G. \tag{3.1}$$

In the work by Blaga [4], the curvature tensors of forms $(1,3)$ and $(0,4)$, along with the Ricci curvature tensor and scalar curvature are presented.

$$\mathcal{R}^\sharp(E,F,G,H) = \tilde{R}(E,F,G,H) + \frac{\alpha-1}{\alpha}[g(\varphi F,\varphi G)g(E,H) - g(\varphi E,\varphi G)g(F,H)], \tag{3.2}$$

$$\mathcal{R}^\sharp(E,F,G,H) = \alpha\tilde{R}(E,F,G,H) + (\alpha-1)\Big\{\eta(G)[\eta(E)g(F,H) - \eta(F)g(E,H)] - g(E,G)[g(F,H) - \eta(F)\eta(H)] + g(F,G)[g(E,H) - \eta(E)\eta(H)]\Big\}, \tag{3.3}$$

$$S^\sharp(E,F) = \tilde{S}(E,F) + 2n\left(\frac{\alpha-1}{\alpha}\right)g(\varphi E,\varphi F), \tag{3.4}$$

where $S^\sharp$ and $\tilde{S}$ indicate Ricci curvature tensors with respect to $\nabla^\sharp$ and $\tilde{\nabla}$. Also, the scalar curvatures $\tau^\sharp$ and $\tilde{\tau}$ with respect to $\nabla^\sharp$ and $\tilde{\nabla}$ are related by

$$\tau^\sharp = \frac{1}{\alpha}\tilde{\tau} + \frac{2n(2n+1)(\alpha-1)}{\alpha^2}. \tag{3.5}$$

Thus, we have the following result:

Proposition 3.1. *In an η-Einstein Kenmotsu manifold of dimension $(2n+1)$, the Ricci tensor is given by*

$$\tilde{S}(E,F) = \left[\frac{\tilde{\tau}+2n}{2n}\right]g(E,F) + \left[-(2n+1) - \frac{\tilde{\tau}}{2n}\right]\eta(E)\eta(F),$$

for any vector fields E, F on $\tilde{\mathcal{M}}$. Here $\tilde{Q}$ is the Ricci operator defined by $\tilde{S}(E,F) = g(\tilde{Q}E,F)$.

By equation (3.4) and Proposition 3.1, we have

Theorem 3.2. *Let $\tilde{\mathcal{M}}(\varphi,\zeta,\eta,g)$ be a $(2n+1)$-dimensional η-Einstein Kenmotsu manifold. Then the manifold $\tilde{\mathcal{M}}(\varphi^\sharp,\zeta^\sharp,\eta^\sharp,g^\sharp)$ is again an η-Einstein manifold under a D_α-homothetic deformation with*

$$S^\sharp(E,F) = \left[\frac{\tilde{\tau}+2n}{2n} + 2n\frac{\alpha-1}{\alpha}\right]g(E,F) + \left[2n\frac{1-\alpha}{\alpha} - (2n+1) - \frac{\tilde{\tau}}{2n}\right]\eta(E)\eta(F),$$

for all $E, F \in \Gamma(\tilde{\mathcal{M}})$.

4 Wintgen inequality for submanifolds in Kenmotsu manifold under D_α-homothetic deformation

The present section deals with the derivation of generalized Wintgen inequalities for submanifolds in D_α-homothetically deformed Kenmotsu manifold.

Let $\mathcal{M}$ be m-dimensional submanifold of $(2n+1)$-dimensional D_α-homothetically deformed Kenmotsu manifolds $\tilde{\mathcal{M}}$. Let ∇ and $\nabla^\perp$ represent the induced connections on the tangent bundle $T\mathcal{M}$ and $T^\perp\mathcal{M}$ of $\mathcal{M}$, respectively and denote by h the second fundamental form of $\mathcal{M}$ for all

$E, F \in \Gamma(T\mathcal{M})$ and $N \in \Gamma(T^{\perp}\mathcal{M})$, recall the Gauss and Weingarten formulas by

$$\tilde{\nabla}_E F = \nabla_E F + h(E, F),$$

and

$$\tilde{\nabla}_E N = -A_N E + \nabla_E^{\perp} N,$$

where A_N is used for notation of the shape operator of $\mathcal{M}$ with respect to N. The following equation is well known

$$g(A_N E, F) = g(h(E, F), N), \quad \text{for all} \;\; E, F \in \Gamma(T\mathcal{M}), \quad N \in \Gamma(T^{\perp}\mathcal{M}).$$

Let $\mathcal{R}$ is the Riemannian curvature tensor of $\mathcal{M}$. Then we recall the equation of Gauss given by

$$\tilde{R}(E, F, G, H) = \mathcal{R}(E, F, G, H) - g(h(E, H), h(F, G)) + g(h(E, G), h(F, H)), \tag{4.1}$$

for all $E, F, G, H \in \Gamma(T\mathcal{M})$.

On combining (3.2) and (4.1), we arrive at

$$\begin{aligned} \mathcal{R}^{\sharp}(E, F, G, H) &= \mathcal{R}(E, F, G, H) - g(h(E, H), h(F, G)) + g(h(E, G), h(F, H)) \\ &\quad + \frac{\alpha - 1}{\alpha}[g(\varphi F, \varphi G)g(E, H) - g(\varphi E, \varphi G)g(F, H)], \end{aligned}$$

which gives

$$\begin{aligned} \mathcal{R}(E, F, G, H) &= \mathcal{R}^{\sharp}(E, F, G, H) + g(h(E, H), h(F, G)) - g(h(E, G), h(F, H)) \\ &\quad - \left(\frac{\alpha - 1}{\alpha}\right)[g(\varphi F, \varphi G)g(E, H) - g(\varphi E, \varphi G)g(F, H)]. \end{aligned} \tag{4.2}$$

Assume that $\{e_1, \ldots, e_m\}$ and $\{e_{m+1}, \ldots, e_{2n+1}\}$ represent local orthonormal tangent frame of the tangent bundle $T\mathcal{M}$ of $\mathcal{M}$ and a local orthonormal normal frame of the normal bundle $T^{\perp}\mathcal{M}$ of $\mathcal{M}$ in $\tilde{\mathcal{M}}$. Define the mean curvature vector $\mathcal{H}$ of $\mathcal{M}$ by

$$\mathcal{H} = \sum_{i=1}^{m} \frac{1}{m} h(e_i, e_i) \tag{4.3}$$

and squared norm of second fundamental form by

$$||h||^2 = \sum_{i,j=1}^{m} g\big(h(e_i, e_j), h(e_i, e_j)\big)^2. \tag{4.4}$$

Here we note that a submanifold $\mathcal{M}$ in $\tilde{\mathcal{M}}$ is called *minimal* if $\mathcal{H} = 0$.

We write the scalar curvature τ of $\mathcal{M}$ at $p \in \mathcal{M}$ as

$$\tau = \sum_{1 \leq i < j \leq m} \mathcal{R}(e_i, e_j, e_j, e_i) \tag{4.5}$$

and define the normalized scalar curvature ρ of $\mathcal{M}$ by

$$\rho = \frac{2\tau}{m(m-1)} = \frac{2}{m(m-1)} \sum_{1 \leq i < j \leq m} \mathcal{K}(e_i \wedge e_j), \tag{4.6}$$

where $\mathcal{K}$ is the sectional curvature function on $\mathcal{M}$.

The scalar normal curvature $\mathcal{K}_{nor}$ in terms of the components of the second fundamental form by the following expression [17]:

$$\mathcal{K}_{nor} = \sum_{1 \leq r < s \leq 2n-m+1} \sum_{1 \leq i < j \leq m} \left(\sum_{k=1}^{m} h^r_{jk} h^s_{ik} - h^r_{ik} h^s_{jk} \right)^2 . \tag{4.7}$$

We also have the following relation for the normalized scalar normal curvature [17]

$$\rho_{nor} = \frac{2}{m(m-1)} \sqrt{\mathcal{K}_{nor}}. \tag{4.8}$$

Now, we prove the generalized Wintgen inequality for submanifolds of D_α-homothetically deformed Kenmotsu manifold $\tilde{\mathcal{M}}$.

Theorem 4.1. *Let $\mathcal{M}$ be an m-dimensional submanifold of a D_α-homothetically deformed Kenmotsu manifold $\tilde{\mathcal{M}}$ of dimension $(2n+1)$. Then*

$$\rho - \rho^\sharp + \rho_{nor} \leq ||\mathcal{H}||^2 - \left(\frac{m-1}{m}\right)\left(\frac{\alpha - 1}{\alpha}\right), \tag{4.9}$$

where $\rho^\sharp$ denotes the normalized scalar curvature with respect to $\nabla^\sharp$.

Moreover, the equality case holds uniformly in (4.9) if and only if the shape operators A_r, $r = \{1, \ldots, 2n-m+1\}$ take the following forms with the suitable orthonormal frames:

$$A = \begin{pmatrix} \mu_1 & \mu & 0 & \ldots & 0 \\ \mu & \mu_1 & 0 & \ldots & 0 \\ 0 & 0 & \mu_1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & \mu_1 \end{pmatrix}, \tag{4.10}$$

$$A_2 = \begin{pmatrix} \mu_2+\mu & 0 & 0 & \dots & 0 \\ 0 & \mu_2-\mu & 0 & \dots & 0 \\ 0 & 0 & \mu_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \mu_2 \end{pmatrix}, \tag{4.11}$$

$$A_3 = \begin{pmatrix} \mu_3 & 0 & 0 & \dots & 0 \\ 0 & \mu_3 & 0 & \dots & 0 \\ 0 & 0 & \mu_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \mu_3 \end{pmatrix}, \quad A_4 = \cdots = A_{2n-m+1} = 0, \tag{4.12}$$

for some suitable orthonormal basis $\{e_1, \dots, e_m\}$ *of* $T_p\mathcal{M}$ *and* $\{E_1, \dots, E_{2n-m+1}\}$ *of* $T_p^\perp\mathcal{M}$. *Here* μ_1, μ_2, μ_3, *and* μ *are real numbers.*

Proof. Assume that $\{e_1, \dots, e_m\}$ and $\{e_{m+1}, \dots, e_{2n+1} = \zeta\}$ denote the local orthonormal tangent frame and local orthonormal normal frame on $\mathcal{M}$ respectively. Then, in view of (4.2), we have

$$\begin{aligned}
\tau &= \sum_{1\le i<j\le m} \mathcal{R}(e_i, e_j, e_j, e_i) \\
&= \sum_{1\le i<j\le m} \Bigg\{ R^\sharp(e_i, e_j, e_j, e_i) + g(h(e_i, e_i), h(e_j, e_j)) - g(h(e_i, e_j), h(e_j, e_i)) \\
&- \left(\frac{\alpha-1}{\alpha}\right)\Big[g(\varphi e_j, \varphi e_j)g(e_i, e_i) - g(\varphi e_i, \varphi e_j)g(e_j, e_i)\Big]\Bigg\} \\
&= \tau^\sharp - (m-1)^2\left(\frac{\alpha-1}{2\alpha}\right) + \sum_{r=1}^{2n-m+1}\sum_{1\le i<j\le m}\Big[h_{ii}^r h_{jj}^r - (h_{ij}^r)^2\Big].
\end{aligned} \tag{4.13}$$

On the other hand, we note that

$$\begin{aligned}
m^2||\mathcal{H}||^2 &= \sum_{r=1}^{2n-m+1}\Big(\sum_{i=1}^{m} h_{ii}^r\Big)^2 \\
&= \frac{1}{m-1}\sum_{r=1}^{2n-m+1}\sum_{1\le i<j\le m}\Big(h_{ii}^r - h_{jj}^r\Big)^2 + \frac{2m}{m-1}\sum_{r=1}^{2n-m+1}\sum_{1\le i<j\le m} h_{ii}^r h_{jj}^r.
\end{aligned} \tag{4.14}$$

But from [16], it is known

$$\begin{aligned}
&\sum_{r=1}^{2n-m+1}\sum_{1\le i<j\le m}(h^r_{ii}-h^r_{jj})^2+2m\sum_{r=1}^{2n-m+1}\sum_{1\le i<j\le m}(h^r_{ij})^2\\
\ge\ & 2m\left[\sum_{1\le r<s\le 2n-m}\sum_{1\le i<j\le m}\left(\sum_{k=1}^{m}\left(h^r_{jk}h^s_{ik}-h^r_{ik}h^s_{jk}\right)\right)^2\right]^{\frac{1}{2}}. \qquad (4.15)
\end{aligned}$$

On combining (4.14), (4.15) and (4.7), we have

$$\begin{aligned}
m^2||\mathcal{H}||^2-m^2\rho_{nor} &\ge \frac{2m}{m-1}\sum_{r=1}^{2n-m+1}\sum_{1\le i<j\le m}\left(h^r_{ii}h^r_{jj}-(h^r_{ij})^2\right)\\
&= \frac{2m}{m-1}\left[\tau-\tau^\sharp+(m-1)^2\left(\frac{\alpha-1}{2\alpha}\right)\right]. \qquad (4.16)
\end{aligned}$$

Hence, by substituting (4.8), (4.13) into (4.16), we arrive

$$||\mathcal{H}||^2-\rho_{nor}\ge\rho-\rho^\sharp+\left(\frac{m-1}{m}\right)\left(\frac{\alpha-1}{\alpha}\right),$$

whereby proving the inequality (4.9). □

An immediate consequence of the Theorem 4.1 yields the following:

Corollary 4.2. *Let $\mathcal{M}$ be a minimal m-dimensional submanifold in a D_α-homothetically deformed Kenmotsu manifold $\tilde{\mathcal{M}}$ of dimension $(2n+1)$. Then*

$$\rho-\rho^\sharp+\rho_{nor}+\left(\frac{m-1}{m}\right)\left(\frac{\alpha-1}{\alpha}\right)\le 0.$$

5 Wintgen inequality for θ-slant submanifolds in Kenmotsu manifold under D_α-homothetic deformation

Let $\mathcal{M}$ be a submanifold of a D_α-homothetically deformed Kenmotsu manifold $\tilde{\mathcal{M}}$. For each nonzero vector U tangent to $\tilde{\mathcal{M}}$ at any point p if the slant angle between $T\mathcal{M}$ and φU is independent of the choice of $p\in\mathcal{M}$, then $\mathcal{M}$ is said to be *slant submanifold.* Observe that submanifold $\mathcal{M}$ becomes *φ-invariant* and *φ-anti-invariant* if the slant angle $\theta=0$ and $\theta=\frac{\pi}{2}$, respectively. A slant submanifold which is neither invariant nor anti-invariant is called proper slant (or θ-slant proper) submanifold.

Recall the results of [6, 7, 9] the following properties of slant submanifolds in an almost contact metric manifolds holds.

Theorem 5.1 ([7]). *Let $\mathcal{M}$ be a submanifold of an almost contact metric manifold $(M, \varphi, \eta, \zeta, g)$ such that $\zeta \in \Gamma(T\mathcal{M})$. Then*

(1) *$\mathcal{M}$ is slant if and only if there exists a constant $\delta \in [0,1]$ such that $P^2 = -\delta(I - \eta \otimes \zeta)$. Furthermore, if the θ is the slant angle of M, then $\delta = \cos^2\theta$*

(2) *$g(PU, PV) = \cos^2\theta[g(U,V) - \eta(U)\eta(V)]$, for any $U, V \in \Gamma(T\mathcal{M})$.*

Now, we prove the generalized Wintgen inequality for θ-slant submnaifolds of D_α-homothetically deformed Kenmotsu manifold $\tilde{\mathcal{M}}$.

Theorem 5.2. *Let $\mathcal{M}$ be an m-dimensional θ-slant submanifold of a D_α-homothetically deformed Kenmotsu manifold $\tilde{\mathcal{M}}$ of dimension $(2n+1)$. Then*

$$\rho_{nor} + \rho - \rho^\sharp \leq ||\mathcal{H}||^2 - \cos^2\theta\left(\frac{m-1}{m}\right)\left(\frac{\alpha-1}{\alpha}\right). \tag{5.1}$$

Proof. Suppose that the local orthonormal tangent frame field on $\mathcal{M}$ is as follows: $\{e_1, e_2 = \sec\theta P e_1, \ldots, e_{m-2}, e_{m-1} = \sec\theta P e_{m-2}, e_m = \zeta\}$ and the local orthonormal normal frame field on $\mathcal{M}$ is given by $\{e_{m+1}, \ldots, e_{2n+1}\}$. Then we have

$$\begin{aligned}
\tau &= \sum_{1\leq i<j\leq m} \mathcal{R}(e_i, e_j, e_j, e_i) \\
&= \sum_{1\leq i<j\leq m} \Big\{ R^\sharp(e_i, e_j, e_j, e_i) + g(h(e_i,e_i), h(e_j,e_j)) - g(h(e_i,e_j), h(e_j,e_i)) \\
&\quad - \left(\frac{\alpha-1}{\alpha}\right)\big[g(Pe_j, Pe_j)g(e_i,e_i) - g(Pe_i, Pe_j)g(e_j,e_i)\big]\Big\} \\
&= \tau^\sharp - (m-1)^2\cos^2\theta\left(\frac{\alpha-1}{2\alpha}\right) + \sum_{r=1}^{2n-m+1}\sum_{1\leq i<j\leq m}\Big[h^r_{ii}h^r_{jj} - (h^r_{ij})^2\Big].
\end{aligned} \tag{5.2}$$

By using similar arguments as in the proof of Theorem 4.1 and (5.2), we get the desired inequality (5.1). □

Remark 5.3. *In case of an m-dimensional θ-slant submanifold $\mathcal{M}$ of a D_α-homothetically deformed Kenmotsu manifold $\tilde{\mathcal{M}}$ of dimension $(2n+1)$. The equality case holds uniformly if and only if the shape operators take the following forms with the suitable orthonormal frames as in Theorem 4.1.*

Now, we classify the geometrical bearing of invariant and anti-invariant submanifolds a D_α-homothetically deformed Kenmotsu manifold $\tilde{\mathcal{M}}$ of dimension $(2n+1)$ in terms of slant angle θ and in light of Theorem 5.2.

If $\mathcal{M}$ is an invariant submanifold, then $\theta = 0$. Then we turn up

Corollary 5.4. *Let $\mathcal{M}$ be an m-dimensional invariant submanifold of a D_α-homothetically deformed Kenmotsu manifold $\tilde{\mathcal{M}}$ of dimension $(2n+1)$. Then*

$$\rho_{nor} + \rho - \rho^\sharp \leq ||\mathcal{H}||^2 - \left(\frac{m-1}{m}\right)\left(\frac{\alpha-1}{\alpha}\right). \tag{5.3}$$

If $\mathcal{M}$ is an ant-invariant submanifold, then $\theta = \frac{\pi}{2}$. Then we have

Corollary 5.5. *Let $\mathcal{M}$ be an m-dimensional anti-invariant submanifold of a D_α-homothetically deformed Kenmotsu manifold $\tilde{\mathcal{M}}$ of dimension $(2n+1)$. Then*

$$\rho_{nor} + \rho - \rho^\sharp \leq ||\mathcal{H}||^2. \tag{5.4}$$

Remark 5.6. *The equality case holds uniformly if and only if the shape operators take the following forms with the suitable orthonormal frames as in Theorem 4.1.*

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