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A nice asymptotic reproducing kernel

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ABSTRACT

We extend the assertion of Problem 12340 in Amer. Math. Monthly 129 (2022), 686, by deriving some additional asymptotic behaviour of that special kernel.

RESUMEN

Extendemos la formulación del Problema 12340 en Amer. Math. Monthly 129 (2022), 686, derivando un comportamiento asintótico adicional de dicho núcleo especial.

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1 An integral operator

The solution to Problem 12340 in [1, p. 686] tells us that for $g : [0,1] \to \mathbb{R}$ continuous,

$$\lim_{n\to\infty} \frac{n}{2^n} \int_0^1 \frac{g(x)}{x^n + (1-x)^n}\, dx = \frac{\pi}{4}\, g\left(\frac{1}{2}\right). \tag{1.1}$$

We shall prove the following additional properties:

Proposition 1.1. *Let $f \in L^1_{loc}(\mathbb{R})$ and for $t > 0$ and $x \in [0,1]$, let*

$$k_t(x) := \frac{1}{\pi}\,\frac{1}{2^t}\,\frac{4t}{x^t + (1-x)^t}.$$

Then

(i) $\lim\limits_{t\to\infty} k_t(x) = \begin{cases} 0 & \text{if } x \neq 1/2 \\ \infty & \text{if } x = 1/2. \end{cases}$

(ii) $\lim\limits_{t\to\infty} \int_0^1 k_t(x)\, dx = 1.$

(iii) $\lim\limits_{t\to\infty} \int_0^1 k_t(x) f\left(s - \frac{1}{2} + x\right)\, dx = f(s)$ *for each continuity point s of f.*

(iv) $\lim\limits_{t\to\infty} \int_{-\infty}^{\infty} K_t\left(y - s + \frac{1}{2}\right) f(y)\, dy = f(s)$ *for each continuity point s of f, where K_t coincides with k_t extended outside $[0,1]$ by 0.*

Thus we may call $k(t,x) := k_t(x)$ a *shifted asymptotic reproducing kernel* for $L^1[0,1]$ or $C[0,1]$ for example (see also at the end of this note).

Proof. (i) is evident

(ii) We show, more generally, that for any continuous function g on $[0,1]$ we have

$$\lim_{t\to\infty} \int_0^1 k_t(x) g(x)\, dx = g(1/2). \tag{1.2}$$

Note that (1.1) is just the discrete version of (1.2) by taking $t = n$ [1]. So, to prove (1.2), we split the integral into two parts and use two different changes of variables. Let $t \geq 1$. Then

[1]This was submitted by myself and Rudolf Rupp as solution to the Monthly problem above.

$$
\begin{aligned}
I_t &:= \frac{t}{2^t}\int_0^{1/2} \underbrace{\frac{g(x)}{x^t+(1-x)^t}\,dx}_{x=:\frac{1}{2}-\frac{s}{2t}} + \frac{t}{2^t}\int_{1/2}^1 \underbrace{\frac{g(x)}{x^t+(1-x)^t}\,dx}_{x=:\frac{1}{2}+\frac{s}{2t}} \\
&= \frac{t}{2^t}\int_0^t \frac{g(\frac{1}{2}-\frac{s}{2t})}{(\frac{1}{2}-\frac{s}{2t})^t+(\frac{1}{2}+\frac{s}{2t})^t}\,\frac{1}{2t}ds + \frac{t}{2^t}\int_0^t \frac{g(\frac{1}{2}+\frac{s}{2t})}{(\frac{1}{2}+\frac{s}{2t})^t+(\frac{1}{2}-\frac{s}{2t})^t}\,\frac{1}{2t}ds \\
&= \frac{1}{2}\int_0^t \frac{g(\frac{1}{2}-\frac{s}{2t})+g(\frac{1}{2}+\frac{s}{2t})}{(1-\frac{s}{t})^t+(1+\frac{s}{t})^t}\,ds.
\end{aligned}
$$

Note that $t\mapsto(1+\frac{s}{t})^t$ is increasing; so the integrand is dominated for $s\geq 1$ by

$$\frac{||g||_\infty}{(1+\frac{s}{2})^2}\leq ||g||_\infty 4s^{-2}.$$

Hence, as $t\to\infty$,

$$
\begin{aligned}
\lim_{t\to\infty} I_t &= \frac{1}{2}2g(1/2)\int_0^\infty \frac{1}{e^{-s}+e^s}\,ds \\
&= g(1/2)\int_0^\infty \frac{e^s}{1+(e^s)^2}\,ds \\
&= g(1/2)\big[\arctan e^s\big]_0^\infty \\
&= g(1/2)\left(\frac{\pi}{2}-\frac{\pi}{4}\right) \\
&= \frac{\pi}{4}g(1/2).
\end{aligned}
$$

If we take $g\equiv 1$, we finally obtain (ii):

$$\int_0^1 k_t(x)dx = \frac{4}{\pi}I_t\to 1.$$

(iii) Let $f\in L^1[0,1]$ and suppose that $1/2$ is a continuity point of f. Given $\epsilon>0$, choose $\delta>0$ so that $|f(x)-f(1/2)|<\epsilon$ for $|x-1/2|<\delta$. For $0\leq x\leq 1$ and $t\geq 1$, let $h(x):=x^t+(1-x)^t$. Then h is a convex function with minimum at $x=1/2$. Hence, whenever $0<\delta<1/2$, the condition $|x-1/2|\geq\delta$ with $0\leq x\leq 1$ implies that

$$x^t+(1-x)^t\geq(1/2+\delta)^t+(1/2-\delta)^t.$$

Thus, as $\delta\neq 0$,

$$\frac{t}{2^t}\,\frac{1}{x^t+(1-x)^t}\leq\frac{t}{(1+2\delta)^t+(1-2\delta)^t}=:m_t\to 0\quad\text{as}\quad t\to\infty.$$

Consequently $\left|\int_0^1 k_t(x)f(x)\,dx - \int_0^1 k_t(x)f(1/2)\,dx\right|$

$$\begin{aligned}
&\le \left|\left(\int_{\substack{|x-1/2|\ge\delta\\ 0\le x\le 1}} + \int_{|x-1/2|\le\delta}\right) k_t(x)|f(x)-f(1/2)|\,dx\right|\\
&\le \frac{4\,m_t}{\pi}\int_{\substack{|x-1/2|\ge\delta\\ 0\le x\le 1}} \big(|f(x)|+|f(1/2)|\big)dx + \epsilon\int_{|x-1/2|\le\delta} k_t(x)dx\\
&\le \frac{4\,m_t}{\pi}\big(||f||_1+|f(1/2)|\big) + \epsilon\int_0^1 k_t(x)dx\\
&\le 2\epsilon \quad \text{for } t\ge t_0.
\end{aligned}$$

As $\lim\limits_{t\to\infty}\int_0^1 k_t(x)f\left(\frac{1}{2}\right)\,dx = f\left(\frac{1}{2}\right)$ by (ii), we deduce that

$$\lim_{t\to\infty}\int_0^1 k_t(x)f(x)\,dx = f\left(\frac{1}{2}\right).$$

If $f\in L^1_{loc}(\mathbb{R})$ satisfies the assumptions above, we put $F(x) := f(s-\frac{1}{2}+x)$. Then $F\in L^1[0,1]$ and $1/2$ is a continuity point of F. Hence

$$\lim_{t\to\infty}\int_0^1 k_t(x)F(x)\,dx = F(1/2) = f(s).$$

(iv) is obtained from (iii) by a linear change of the variable. □

We may ask what happens if s is a jump point. Do we have a similar behaviour as in the Dirichlet-Jordan Theorem for Fourier series?

It is interesting to discuss the relations that exist between our shifted asymptotic reproducing kernel $k_t(x) = k(t,x)$ and the so-called "summability kernels" in [2, p. 9], respectively "good kernels" in [3, p. 48], the most prominent examples being the Fejér kernel and the Poisson kernel for $L^1(\mathbb{T})$ concerning 2π-periodic functions. In fact, using suitable transformations, in particular the new variable $y = 2\pi(x-\frac{1}{2})$, equivalently $x = \frac{1}{2}+\frac{y}{2\pi}$, we get the following relations (we restrict w.l.o.g. to the discrete case): let $I_n := \int_0^1 k_n(x)dx$ and

$$K_n^*(y) := I_n^{-1}\cdot k_n\left(\frac{1}{2}+\frac{y}{2\pi}\right), \quad -\pi\le y<\pi,$$

and extend this function 2π-periodically. Then K_n^* is continuous on $\mathbb{R}$ as

$$K_n^*(-\pi) = \lim_{y\to-\pi} K_n^*(y) = k_n(0) = k_n(1) = \lim_{y\to\pi} K_n^*(y) = K_n^*(\pi).$$

Observe that for $|y| < \pi$,

$$K_n^*(y) = I_n^{-1} \cdot \frac{4n\pi^{n-1}}{(\pi+y)^n + (\pi-y)^n}.$$

Moreover, $K_n^* \geq 0$,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} K_n^*(y)dy = 1,$$

and, by the proof of (iii) and (ii),

$$\int_{\delta \leq |y| \leq \pi} K_n^*(y)dy \to 0 \quad \text{as } n \to \infty$$

for every $\delta > 0$, δ small. Hence, according to [3, p. 48], (K_n^*) is a family of good kernels. Consequently, by [3, Theorem 4.1, p. 49],

$$(f * K_n^*)(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-y)K_n^*(y)dy \to f(x)$$

for every continuity point x of $f \in L_{loc}(\mathbb{R})$, f 2π-periodic.

Readers having a good command of the Chinese language (unfortunately I don't), may also consult the classroom survey [4] for studies on summability/good kernels.

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