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Note on the F_0-spaces

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ABSTRACT

A rationally elliptic space X is called an F_0-space if its rational cohomology is concentrated in even degrees. The aim of this paper is to characterize such a space in terms of the homotopy groups of its skeletons as well as the rational cohomology of its Postnikov sections.

RESUMEN

Un espacio racionalmente elíptico X se llama un espacio F_0 si su cohomología racional está concentrada en grados pares. El propósito de este artículo es caracterizar dichos espacios en términos de los grupos de homotopía tanto de sus esqueletos como de la cohomología racional de sus secciones de Postnikov.

Keywords and Phrases: Rationaly elliptic space, Sullivan model, Quillen model, Whitehead exact sequence, F_0-space.

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1 Introduction

Along this paper space means a simply connected CW-complex X of finite type, *i.e.*, dim $H^n(X;\mathbb{Q}) < \infty$ for all n. A space X is called rationally elliptic if both the graded vector spaces $H^*(X;\mathbb{Q})$ and $\pi_*(X)\otimes\mathbb{Q}$ are finite dimensional. Furthermore, if $H^{\mathrm{odd}}(X;\mathbb{Q})=0$, then X is called an F_0-space. For instance, products of even spheres, complex Grassmannian manifolds and homogeneous spaces G/H such that rank $G=$ rank H are F_0-spaces.

Given a rationally elliptic space X. For any positive integer n, let $X^{[n]}$ denote the n-Postnikov section of X and X^n its n-skeleton. The aim of this paper is to characterize an F_0-space in terms of the homotopy groups of its skeletons and the rational cohomology of its Postnikov sections. More precisely, let:

$$\Gamma_n(X)=\ker(\pi_n(X^n)\otimes\mathbb{Q}\longrightarrow\pi_n(X^n;X^{n-1})\otimes\mathbb{Q}),\quad n\geq 2.$$

By exploiting the properties of the Whitehead exact sequences associated respectively with the Sullivan model and the Quillen model of X, we prove the following result

Theorem 1.1. *Let X be a rationally elliptic space. If $\pi_{\mathrm{even}}(X)\otimes\mathbb{Q}\neq 0$, then the following statements are equivalent.*

(1) *X is an F_0-space.*

(2) *$\Gamma_{2n}(X)=0$ for all $n\geq 1$.*

(3) *$H^{2n+1}(X^{[2n-1]};\mathbb{Q})=0$ for all $n\geq 1$.*

Note that if X is a (non-trivial) rationally elliptic space such that $\pi_{\mathrm{even}}\,(X)\otimes\mathbb{Q}=0$, then X cannot be an F_0-space as it is mentioned in Remark 3.3.

We show our results using standard tools of rational homotopy theory by working algebraically on the models of Quillen and Sullivan of X. We refer to [8] for a general introduction to these techniques. We recall some of the notation here. By a Sullivan algebra we mean a free graded commutative algebra ΛV, for some finite-type graded vector space $V=(V^{\geq 2})$, *i.e.*, dim $V^n<\infty$ for all $n\geq 2$, together with a differential ∂ of degree $+1$ that is decomposable, *i.e.*, satisfies $\partial: V\to\Lambda^{\geq 2}V$. Here $\Lambda^{\geq 2}V$ denotes the graded vector space spanned by all the monomials $v_1\cdots v_r$ such that $v_1,\dots,v_r\in V$ and $r\geq 2$.

Every space X has a corresponding Sullivan algebra called the Sullivan model of X, unique up to isomorphism, that encodes the rational homotopy of X. In particular, we have the following identifications valid for every $n\geq 2$,

$$H^n(X;\mathbb{Q})\cong H^n(\Lambda V),\quad V^n\cong\mathrm{Hom}(\pi_n(X)\otimes\mathbb{Q},\mathbb{Q}).\tag{1.1}$$

Dually, by a free differential graded Lie algebra $(\mathbb{L}(W),\delta)$ (DGL for short), we mean a free graded Lie algebra $\mathbb{L}(W)$, for some finite-type vector space $W=(W_{\geq 1})$, together with a decomposable differential δ of degree -1, *i.e.*, $\delta(W)\to\mathbb{L}^{\geq 2}(W)$. Here $\mathbb{L}^{\geq 2}(W)$ denotes the graded vector space spanned by all the brackets of lengths ≥ 2.

Every space X has a corresponding DGL, called the Quillen model of X, unique up to isomorphism, and which determines completely the rational homotopy type of X. In particular, we have the following identifications valid for every $n\geq 2$,

$$\pi_n(X)\otimes\mathbb{Q}\cong H_{n-1}(\mathbb{L}(W)),\quad H_n(X;\mathbb{Q})\cong W_{n-1}. \tag{1.2}$$

2 Whitehead exact sequences in rational homotopy theory

2.1 Whitehead exact sequence of a DGL

Let $(\mathbb{L}(W),\delta)$ be a DGL. For any positive integer n, we define the linear maps

$$j_n: H_n(\mathbb{L}(W_{\leq n}))\to W_n,\quad b_n: W_n\to H_{n-1}(\mathbb{L}(W_{\leq n-1})),$$

by setting

$$j_n([w+y])=w,\quad b_n(w)=[\delta(w)], \tag{2.1}$$

were $[\delta(w)]$ denotes the homology class of $\delta(w)$ in the sub-Lie algebra $\mathbb{L}_{n-1}(W_{\leq n-1})$. Recall that if $x\in H_n(\mathbb{L}(W_{\leq n}))$, then $x=[w+y]$, where $w\in W_n$, $y\in\mathbb{L}_n(W_{\leq n-1})$ and $\delta(w+y)=0$.

To every DGL $(\mathbb{L}(W),\delta)$, we can assign (see [2, 6, 7] for more details) the following long exact sequence

$$\cdots\to W_{n+1}\xrightarrow{b_{n+1}}\Gamma_n\to H_n(\mathbb{L}(W))\xrightarrow{h_n}W_n\xrightarrow{b_n}\cdots \tag{2.2}$$

called the Whitehead exact sequence of $(\mathbb{L}(W),\delta)$, where

$$\Gamma_n=\ker(j_n: H_n(\mathbb{L}(W_{\leq n}))\to W_n),\quad \forall n. \tag{2.3}$$

Remark 2.1. *If $(\mathbb{L}(W),\delta)$ is the Quillen model of a space X, then by the properties of this model, the DGL $(\mathbb{L}(W_{\leq n}),\delta)$ can be chosen as the Quillen model of the $(n+1)$-skeleton X^{n+1}. Thus, we derive the following identification*

$$\Gamma_{n+1}(X)\cong\Gamma_n,\quad \forall n\geq 1 \tag{2.4}$$

where

$$\Gamma_{n+1}(X)=\ker(\pi_{n+1}(X^{n+1})\otimes\mathbb{Q}\longrightarrow\pi_{n+1}(X^{n+1};X^n)\otimes\mathbb{Q}). \tag{2.5}$$

2.2 Whitehead exact sequence of a Sullivan algebra

Likewise, let $(\Lambda V, \partial)$ be a Sullivan algebra. In [1, 4, 5], it is shown that with $(\Lambda V, \partial)$, we can associate the following long exact sequence

$$\cdots \to V^n \xrightarrow{b^n} H^{n+1}(\Lambda V^{\leq n-1}) \longrightarrow H^{n+1}(\Lambda V) \longrightarrow V^{n+1} \xrightarrow{b^{n+1}} \cdots \tag{2.6}$$

called the Whitehead exact sequence of $(\Lambda V, \partial)$. Recall that the linear map b^n is defined by setting $b^n(v) = [\partial(v)]$. Here $[\partial(v)]$ denotes the cohomology class of $\partial(v) \in \Lambda V^{\leq n-1}$.

Remark 2.2. *If $(\Lambda V, \partial)$ is the Sullivan minimal of a given space X, then by virtue of the properties of this model, $(\Lambda V^{\leq n-1}, \partial)$ can be chosen as the Sullivan model of the $(n-1)$-Postnikov section $X^{[n-1]}$. Thus, we derive the following identification*

$$H^{n+1}(X^{[n-1]}; \mathbb{Q}) \cong H^{n+1}(\Lambda V^{\leq n-1}), \quad \forall n \geq 2. \tag{2.7}$$

Proposition 2.3. *If $(\Lambda V, \partial)$ is the Sullivan model of a space X and $(\mathbb{L}(W), \delta)$ its Quillen model, then we have*

$$\Gamma_n = H^{n+2}(\Lambda V^{\leq n}), \quad \forall n \geq 2. \tag{2.8}$$

where Γ_n is defined in (2.3).

Proof. Applying the exact functor $\mathrm{Hom}(\cdot, \mathbb{Q})$ to the exact sequence (2.2) we obtain

$$\cdots \leftarrow \mathrm{Hom}(W_{n+1}, \mathbb{Q}) \leftarrow \mathrm{Hom}(\Gamma_n, \mathbb{Q}) \leftarrow \mathrm{Hom}(H_n(\mathbb{L}(W)), \mathbb{Q}) \leftarrow \mathrm{Hom}(W_n, \mathbb{Q}) \xleftarrow{b_n} \cdots \tag{2.9}$$

Taking into account that by virtues of the Quillen and Sullivan models we have

- Any vector space involved in this paper is of finite dimension which implies that it has the same dimension as its dual.
- $\mathrm{Hom}(W_n, \mathbb{Q}) \cong H^{n+1}(\Lambda V) \cong H^{n+1}(X; \mathbb{Q})$ for all $n \geq 1$.
- $\mathrm{Hom}(H_n(\mathbb{L}(W)); \mathbb{Q}) \cong V^{n+1} \cong \pi_{n+1}(X) \otimes \mathbb{Q}$ for all $n \geq 1$.
- The two maps $H^{n+1}(\Lambda V) \to V^{n+1}$ and $\mathrm{Hom}(W_n, \mathbb{Q}) \to \mathrm{Hom}(H_n(\mathbb{L}(W)), \mathbb{Q})$ appearing in (2.6) and (2.9) are the same linear map because they can be identified with the following linear map

$$\mathrm{Hom}(H_{n+1}(X; \mathbb{Q}); \mathbb{Q}) \cong H^{n+1}(X; \mathbb{Q}) \to \mathrm{Hom}(\pi_{n+1}(X) \otimes \mathbb{Q}; \mathbb{Q}),$$

  which is the dual of the Hurewicz homomorphism $\pi_{n+1}(X) \otimes \mathbb{Q} \to H_{n+1}(X; \mathbb{Q})$. Here we use the well-known universal coefficient theorem.

Finally, by comparing the sequences (2.6), (2.9) we get (2.8). □

Corollary 2.4. *If X is a given space, then*

$$\Gamma_{n+1}(X) \cong H^{n+2}(X^{[n]};\mathbb{Q}), \quad \textit{as vector spaces}, \quad \forall n \geq 1. \tag{2.10}$$

Proof. It suffices to apply the identifications (2.4), (2.7) and Proposition 2.3 to the Sullivan model and the Quillen model of the space X. □

3 The main result

As it is stated in the introduction, a space X is called rationally elliptic if both the graded vector spaces $H^*(X;\mathbb{Q})$ and $\pi_*(X)\otimes\mathbb{Q}$ are finite dimensional.

Proposition 3.1 ([8, Proposition 32.10])**.** *If X is a rationally elliptic space and $(\Lambda V,\partial)$ its Sullivan model, then* $\dim H^{\text{even}}(\Lambda V) \geq \dim H^{\text{odd}}(\Lambda V)$. *Furthermore, the following statements are equivalent*

(1) *X is an F_0-space.*

(2) $\dim V^{\text{even}} = \dim V^{\text{odd}}$ *and $(\Lambda V,\partial)$ is pure, i.e., $\partial(V^{\text{even}}) = 0$ and $\partial(V^{\text{odd}}) \subseteq \Lambda V^{\text{even}}$.*

Using the identification (1.1) and (1.2), we can translate the above Proposition in terms of the Model of the Quillen. Thus, we have the following result.

Proposition 3.2. *If $(\mathbb{L}(W),\delta)$ is the Quillen model of a rationally elliptic space X, then* $\dim W_{\text{odd}} \geq \dim W_{\text{even}}$. *Moreover, the following statements are equivalent*

(1) *X is an F_0-space.*

(2) $W_{\text{even}} = 0$.

(3) $H_{\text{even}}(\mathbb{L}(W)) = H_{\text{even}}(\mathbb{L}(W))$.

Subsequently, we need the following obvious remark.

Remark 3.3. *Let $(\mathbb{L}(W),\delta)$ be the Quillen model of a rationally elliptic space X.*

(1) *If $W_{\text{odd}} = 0$, then X is rationally trivial. Indeed, Since X is a rationally elliptic space, using Proposition 3.1, it follows that* $\dim W_{\text{odd}} \geq \dim W_{\text{even}}$. *Hence, if $W_{\text{odd}} = 0$, then $W = W_{\text{odd}} \oplus W_{\text{even}} = 0$ implying that X is rationally trivial.*

(2) *If X is a (non-trivial) rationally elliptic space such that $\pi_{\text{even}}(X) \otimes \mathbb{Q} = 0$, then X cannot be an F_0-space. Indeed, if so, then we must have*

$$\dim\, \pi_{\text{even}}(X) \otimes \mathbb{Q} = \dim\, \pi_{\text{odd}}(X) \otimes \mathbb{Q}.$$

Therefore, $\dim\, \pi_*(X) \otimes \mathbb{Q} = \dim\, \pi_{\text{even}}(X) \otimes \mathbb{Q} + \dim\, \pi_{\text{odd}}(X) \otimes \mathbb{Q} = 0$. *As a result, X is rationally trivial.*

Proposition 3.4. *Let $(\mathbb{L}(W), \delta)$ be the Quillen model of a rationally elliptic space such that $H_{\text{odd}}(\mathbb{L}(W)) \neq 0$. If $\Gamma_{\text{odd}} = 0$, then $W_{\text{even}} = 0$.*

Proof. Assume by contradiction that $W_{\text{even}} \neq 0$ and let $w_0 \in W_{\text{even}}$ such that

$$|w_0| = \max\{|w|, \quad w \in W_{\text{even}}\}. \tag{3.1}$$

Let us consider the Whitehead exact sequence (2.2) of $(\mathbb{L}(W), \delta)$. Since $\Gamma_{\text{odd}} = 0$, it follows that $b_{|w_0|}(w_0) = 0$ and from the relation (2.1) there exists a decomposable element in $q_0 \in \mathbb{L}(W)$ such that $\delta(w_0 + q_0) = 0$.

Next, as $H_{\text{odd}}(\mathbb{L}(W)) \neq 0$, there exists a non-trivial homology class $\{w + y\} \in H_{2m+1}(\mathbb{L}(W))$, where $w \in W_{2m+1}$ and y is a decomposable element in $\mathbb{L}_{2m+1}(W)$, for a certain $m \in \mathbb{N}$.

Therefore, the bracket $[w_0 + q_0, w + y]$ is a decomposable cycle of degree $|w_0| + 2m - 1$ providing a homology class in the vector space

$$\Gamma_{|w_0|+2m+1} \subset H_{|w_0|+2m+1}(\mathbb{L}(W_{\leq |w_0|+2m+1})).$$

It is worth noting that as $|w_0|$ is even, then $|w_0| + 2m + 1$ is odd and by taking into account the relation (3.1), the cycle $[w_0 + q_0, w + y]$ cannot be a boundary in $\Gamma_{|w_0|+2m+1}$ implying that $\Gamma_{\text{odd}} \neq 0$. Contradiction. □

Corollary 3.5. *Let X be a rationally elliptic space such that $\pi_{\text{even}}(X) \otimes \mathbb{Q} \neq 0$ and let $\Gamma_*(X)$ as in* (2.5)*. If $\Gamma_{\text{even}}(X) = 0$, then X is an F_0-space.*

Proof. Working algebraically, let $(\mathbb{L}(W), \delta)$ be the Quillen model of X. Since $\Gamma_{\text{even}}(X) = 0$, the identifications (2.4) implies that $\Gamma_{\text{odd}} = 0$. Next, by applying Proposition 3.4, it follows that $W_{\text{even}} = 0$ and by the identifications (1.2), we deduce that $H^{\text{even}}(X; \mathbb{Q}) = 0$. Hence, X is an F_0-space. □

Corollary 3.5 implies the following result which gives a characterization of an F_0-space X in terms of the homotopy groups of its skeletons.

Corollary 3.6. *Let X be a rationally elliptic space such that $\pi_{\mathrm{even}}(X) \otimes \mathbb{Q} \neq 0$. If*

$$\pi_{2n}(X^{2n}) \otimes \mathbb{Q} = 0, \quad \forall n \geq 1, \tag{3.2}$$

then X is an F_0-space.

Proof. First, according to (2.5), we know that $\Gamma_{2n}(X) \subset \pi_{2n}(X^{2n}) \otimes \mathbb{Q}$ for all $n \geq 1$. Therefore, the relation (3.2) implies that $\Gamma_{\mathrm{even}}(X) = 0$. Then, it suffices to apply Corollary 3.5. □

The next result gives characterization of an F_0-space X in terms of the rational cohomology of its Postnikov sections.

Corollary 3.7. *Let X be a rationally elliptic space such that $\pi_{\mathrm{even}}(X) \otimes \mathbb{Q} \neq 0$. If*

$$H^{2n+1}(X^{[2n-1]}; \mathbb{Q}) = 0, \quad \forall n \geq 1,$$

then X is an F_0-space.

Proof. First, by Corollary 2.4, if $H^{2n+1}(X^{[2n-1]}; \mathbb{Q}) = 0$ for all n, then $\Gamma_{\mathrm{even}}(X) = 0$. Next, it suffices to apply Corollary 3.6. □

Proposition 3.8. *If X is an F_0-space, then $\Gamma_{2n}(X) = 0, \quad \forall n \geq 1$.*

Proof. Let $(\Lambda V, \partial)$ be the Sullivan model of X. By (2.6), the Whitehead exact sequence of $(\Lambda V, \partial)$ can be written as

$$\cdots \to V^{2n} \xrightarrow{b^{2n}} H^{2n+1}(\Lambda V^{\leq 2n-1}) \longrightarrow H^{2n+1}(\Lambda V) \longrightarrow V^{2n+1} \xrightarrow{b^{2n+1}} \cdots$$

As X is an F_0-space, then by Proposition 3.1, the Sullivan model $(\Lambda V, \partial)$ of X satisfies $H^{\mathrm{odd}}(\Lambda V) = 0$ and $\partial(V^{\mathrm{even}}) = 0$, it follows that the maps $b^{\mathrm{even}} = 0$. Consequently, $H^{2n+1}(\Lambda V^{\leq 2n-1}) = 0$ for every $n \geq 1$. Hence, the result follows from the formula (2.10). □

Proof of Theorem 1.1. It follows from Corollaries 3.5, 3.7 and Proposition 3.8 after taking Remark 3.3 into account. □

Conflict of interest

The author has not disclosed any competing interests.

References

[1] M. Benkhalifa, "On the group of self-homotopy equivalences of an elliptic space," *Proc. Amer. Math. Soc.*, vol. 148, no. 6, pp. 2695–2706, 2020, doi: 10.1090/proc/14900.

[2] M. Benkhalifa, "The effect of cell-attachment on the group of self-equivalences of an elliptic space," *Michigan Math. J.*, vol. 71, no. 3, pp. 611–617, 2022, doi: 10.1307/mmj/20195840.

[3] M. Benkhalifa, "On the Euler-Poincaré characteristics of a simply connected rationally elliptic CW-complex," *J. Homotopy Relat. Struct.*, vol. 17, no. 2, pp. 163–174, 2022, doi: 10.1007/s40062-022-00301-2.

[4] M. Benkhalifa, "The group of self-homotopy equivalences of a rational space cannot be a free abelian group," *J. Math. Soc. Japan*, vol. 75, no. 1, pp. 113–117, 2023, doi: 10.2969/jmsj/87158715.

[5] M. Benkhalifa, "On the characterization of F_0-spaces," *Commun. Korean Math. Soc.*, vol. 38, no. 2, pp. 643–648, 2023, doi: 10.4134/CKMS.c220179.

[6] M. Benkhalifa, "On the group of self-homotopy equivalence of a formal F_0-space," *Bollettino dell'Unione Matematica Italiana*, vol. 16, no. 3, pp. 641–647, 2023, doi: 10.1007/s40574-023-00354-y.

[7] M. Benkhalifa, "On the group of self-homotopy equivalences of an almost formal space," *Quaest. Math.*, vol. 46, no. 5, pp. 855–862, 2023, doi: 10.2989/16073606.2022.2044405.

[8] Y. Félix, S. Halperin, and J.-C. Thomas, *Rational homotopy theory*, ser. Graduate Texts in Mathematics. New York, USA: Springer-Verlag, 2001, vol. 205, doi: 10.1007/978-1-4613-0105-9.