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Quotient rings satisfying some identities

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ABSTRACT

This paper investigates the commutativity of the quotient ring $\mathcal{R}/P$, where $\mathcal{R}$ is an associative ring with a prime ideal P, and the possibility of forms of derivations satisfying certain algebraic identities on $\mathcal{R}$. We provide some results for strong commutativity-preserving derivations of prime rings.

RESUMEN

Este artículo investiga la conmutatividad del anillo cociente $\mathcal{R}/P$, donde $\mathcal{R}$ es un anillo asociativo con un ideal primo P, y la posibilidad de formas de derivaciones que satisfacen ciertas identidades algebraicas en $\mathcal{R}$. Entregamos algunos resultados para derivaciones de anillos primos que preservan la conmutatividad fuerte.

Keywords and Phrases: Derivations, prime ideals, prime rings.

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1 Introduction

In all that follows, $\mathcal{R}$ always denotes an associative ring with center $Z(\mathcal{R})$ and $\mathcal{C}$ is the extended centroid of $\mathcal{R}$ (we refer the reader to [3] for more information about these objects). As usual, the symbols $[s,t]$ and $s\circ t$ denote the commutator $st-ts$ and the anticommutator $st+ts$, respectively. Recall that a ring $\mathcal{R}$ is prime if $x\mathcal{R}y=\{0\}$ implies $x=0$ or $y=0$, and $\mathcal{R}$ is semiprime if $x\mathcal{R}x=\{0\}$ implies $x=0$.

A map $\mathcal{D}:\mathcal{R}\to\mathcal{R}$ is called a multiplicative derivation if $\mathcal{D}(xy)=\mathcal{D}(x)y+x\mathcal{D}(y)$ for all $x,y\in\mathcal{R}$, if $\mathcal{D}$ is also additive, we say that $\mathcal{D}$ is a derivation of $\mathcal{R}$.

The study of commutativity preserving mappings has been an active research area in matrix theory, operator theory, and ring theory (see [8, 19] for references). According to [5], let $\mathcal{S}$ be a subset of $\mathcal{R}$. A map $\varphi:\mathcal{R}\to\mathcal{R}$ is said to be strongly commutativity preserving (SCP) on $\mathcal{S}$ if $[\varphi(x),\varphi(y)]=[x,y]$ for all $x,y\in\mathcal{S}$. In [4], Bell and Daif investigated commutativity in rings admitting a derivation which is SCP on a nonzero right ideal. In particular, they proved that if a semiprime ring $\mathcal{R}$ admits a derivation $\mathcal{D}$ satisfying $[\mathcal{D}(x),\mathcal{D}(y)]=[x,y]$ for all x,y in a right ideal I of $\mathcal{R}$, then $I\subseteq Z(\mathcal{R})$ (see [9] for more information). In particular, $\mathcal{R}$ is commutative if $I=\mathcal{R}$. Later, Deng and Ashraf [10] proved that if there exists a derivation $\mathcal{D}$ of a semiprime ring $\mathcal{R}$ and a map $\varphi:I\to\mathcal{R}$ defined on a non-zero ideal I of $\mathcal{R}$ such that $[\varphi(x),\mathcal{D}(y)]=[x,y]$ for all $x,y\in I$, then $\mathcal{R}$ contains a non-zero central ideal. In particular, they showed that $\mathcal{R}$ is commutative if $I=\mathcal{R}$. Recently, this result was extended to Lie ideals and symmetric elements of prime rings by Lin and Liu in [12] and [13]. There is also a growing literature on strong commutativity preserving (SCP) maps and derivations (for references see [4, 8, 16], etc.) In [1], Ali *et al.* showed that if $\mathcal{R}$ is a semiprime ring and f is an endomorphism that is a strong commutativity preserving (simple, SCP) map on a non-zero ideal U of $\mathcal{R}$, then f commutes on U. In [18], Samman proved that an epimorphism of a semiprime ring is strongly commutativity preserving if and only if it is centralizing. Derivations and SCP mappings have been extensively studied in the context of operator algebras, prime rings, and semiprime rings. Many related generalizations of these results can be found in the literature (see for example [8, 11, 14, 15, 17]).

In this paper, we discuss the notion of a derivation that satisfies one of the following conditions:

(i) $[\mathcal{D}(x),\mathcal{D}(y)]+H([x,y])\in P$, for all $x,y\in\mathcal{R}$,

(ii) $\mathcal{D}(x)\circ\mathcal{D}(y)+H(x\circ y)\in P$, for all $x,y\in\mathcal{R}$,

(iii) $[\mathcal{D}(x),F(y)]+H(x\circ y)\in P$, for all $x,y\in\mathcal{R}$,

(iv) $\mathcal{D}(x)\circ\mathcal{D}(y)+H([x,y])\in P$, for all $x,y\in\mathcal{R}$,

where P is a prime ideal of $\mathcal{R}$, $\mathcal{D}$ is a derivation, and H is a multiplier of $\mathcal{R}$.

2 Results

In this section, we discuss some well-known results in the rings theory, which will be used in the following sections.

(*i*) $[x, yz] = y[x, z] + [x, y]z$.

(*ii*) $[xy, z] = [x, z]y + x[y, z]$.

(*iii*) $xy \circ z = (x \circ z)y + x[y, z] = x(y \circ z) - [x, z]y$.

(*iv*) $x \circ yz = y(x \circ z) + [x, y]z = (x \circ y)z + y[z, x]$.

Lemma 2.1 ([2], Lemma 2.1)**.** *Let $\mathcal{R}$ be a ring, P be a prime ideal of $\mathcal{R}$, and $\mathcal{D}$ a derivation of $\mathcal{R}$. If $[\mathcal{D}(x), x] \in P$ for all $x \in \mathcal{R}$, then $\mathcal{D}(\mathcal{R}) \subseteq P$ or $\mathcal{R}/P$ is commutative.*

Lemma 2.2 ([6])**.** *Let $\mathcal{R}$ be a prime ring. If functions $\mathcal{F} : \mathcal{R} \to \mathcal{R}$ and $\mathcal{G} : \mathcal{R} \to \mathcal{R}$ are such that $\mathcal{F}(x)y\mathcal{G}(z) = \mathcal{G}(x)y\mathcal{F}(z)$ for all $x, y, z \in \mathcal{R}$, and $\mathcal{F} \neq 0$, then there exists λ in the extended centroid of $\mathcal{R}$ such that $\mathcal{G}(x) = \lambda\mathcal{F}(x)$ for all $x \in \mathcal{R}$.*

The following two Lemmas are also used to prove our theorems. The primary goal is to establish a connection between the commutativity of rings $\mathcal{R}/\mathcal{P}$ and the behavior of their derivations.

Lemma 2.3. *Let $\mathcal{R}$ be a ring and P be a prime ideal of $\mathcal{R}$. If $\mathcal{R}$ admits a derivation $\mathcal{D}$ such that $\mathcal{R}$ satisfies one of the following assertions:*

(*i*) $[x, \mathcal{D}(y)] \in P$ *for all* $x, y \in \mathcal{R}$,

(*ii*) $x \circ \mathcal{D}(y) \in P$ *for all* $x, y \in \mathcal{R}$,

then $\mathcal{D}(\mathcal{R}) \subseteq P$ or $\mathcal{R}/P$ is commutative.

Proof. (*i*) Suppose that

$$[x, \mathcal{D}(y)] \in P \quad \text{for all} \quad x, y \in \mathcal{R}. \tag{2.1}$$

Replacing y by yt in (2.1), we obtain

$$\mathcal{D}(y)[x, t] + [x, \mathcal{D}(y)]t + y[x, \mathcal{D}(t)] + [x, y]\mathcal{D}(t) \in P \quad \text{for all} \quad x, y, t \in \mathcal{R}.$$

Using (2.1), we get

$$\mathcal{D}(y)[x, t] + [x, y]\mathcal{D}(t) + y[x, \mathcal{D}(t)] \in P \quad \text{for all} \quad x, y, t \in \mathcal{R}. \tag{2.2}$$

For $x = t$ in (2.2), it follows that

$$[t, y]\mathcal{D}(t) + y[t, \mathcal{D}(t)] \in P \quad \text{for all} \quad y, t \in \mathcal{R}. \tag{2.3}$$

Taking ry instead of y in (2.3) and using (2.3), we conclude that

$$[t, r]y\mathcal{D}(t) \in P \quad \text{for all} \quad r, y, t \in \mathcal{R}.$$

Equivalently,

$$[t, r]\mathcal{R}\mathcal{D}(t) \subseteq P \quad \text{for all} \quad r, t \in \mathcal{R}.$$

By primeness of P, we arrive at

$$[t, r] \in P \quad \text{or} \quad \mathcal{D}(t) \in P \quad \text{for all} \quad r, t \in \mathcal{R}. \tag{2.4}$$

If there exists $t_0 \in \mathcal{R}$ such that $\mathcal{D}(t_0) \in P$, then (2.3) implies that $y[t_0, \mathcal{D}(t_0)] \in P$ for all $y \in \mathcal{R}$ which implies that $[t_0, \mathcal{D}(t_0)]y[t_0, \mathcal{D}(t_0)] \in P$ for all $y \in \mathcal{R}$. Since P is prime, then $[t_0, \mathcal{D}(t_0)] \in P$. So, (2.4) becomes $[t, \mathcal{D}(t)] \in P$ for all $t \in P$, in this case Lemma 2.1 forces that $\mathcal{D}(\mathcal{R}) \subseteq P$ or $\mathcal{R}/P$ is commutative.

(ii) Using the same techniques as those used in the proof of (i) with minor modifications, we can easily arrive at our result. $\square$

Lemma 2.4. *Let $\mathcal{R}$ be a ring and P be a prime ideal of $\mathcal{R}$. If $\mathcal{R}$ admits a derivation D such that $\mathcal{R}$ satisfies any of the following assertions:*

(i) $[x, \mathcal{D}(x)] \in P$ *for all* $x \in \mathcal{R}$,

(ii) $x \circ \mathcal{D}(x) \in P$ *for all* $x \in \mathcal{R}$,

then $\mathcal{D}(\mathcal{R}) \subseteq P$ or $\mathcal{R}/P$ is commutative.

Proof. (i) Assuming that

$$[x, \mathcal{D}(x)] \in P \quad \text{for all} \quad x \in \mathcal{R}. \tag{2.5}$$

Linearizing Eq. (2.5), we obtain

$$[x, \mathcal{D}(y)] + [y, \mathcal{D}(x)] \in P \quad \text{for all} \quad x, y \in \mathcal{R}. \tag{2.6}$$

Replacing y by yx in (2.6), and using it with (2.5) we obtain

$$y[x, \mathcal{D}(x)] + [x, y]\mathcal{D}(x) \in P \quad \text{for all} \quad x, y \in \mathcal{R}. \tag{2.7}$$

Putting yz instead of y in (2.7), where $z \in \mathcal{R}$ and using it again, we get

$$[x, y]z\mathcal{D}(x) \in P \quad \text{for all} \quad x, y, z \in \mathcal{R}.$$

Since P is prime ideal of $\mathcal{R}$, we arrive at

$$[x, y] \in P \quad \text{or} \quad \mathcal{D}(x) \in P \quad \text{for all} \quad x, y \in \mathcal{R}. \tag{2.8}$$

Suppose that $\mathcal{D}(\mathcal{R}) \nsubseteq P$. There exists $x \in \mathcal{R}$ such that $\mathcal{D}(x) \notin P$. By (2.8), we get $[x, y] \in P$ for all $y \in \mathcal{R}$ which implies that $\bar{x} \in Z(\mathcal{R}/P)$. Let $z \in \mathcal{R}$ such that $\bar{z} \notin Z(\mathcal{R}/P)$. Then, there exists $y_0 \in \mathcal{R}$ such that $[z, y_0] \notin P$. Therefore, from (2.8), we find that $\mathcal{D}(z) \in P$. On the other hand, since $\mathcal{D}(x) \notin P$, we can derive $\mathcal{D}(x + z) \notin P$. Using (2.8) again, the last expression gives $[x + z, y] \in P$ for all $y \in \mathcal{R}$, which forces that $[z, y] \in P$ for all $y \in \mathcal{R}$, a contradiction.

(ii) Suppose that

$$x \circ \mathcal{D}(x) \in P \quad \text{for all} \quad x \in \mathcal{R}. \tag{2.9}$$

Linearizing (2.9), we get

$$x \circ \mathcal{D}(y) + y \circ \mathcal{D}(x) \in P \quad \text{for all} \quad x, y \in \mathcal{R}. \tag{2.10}$$

Substituting yx for y in (2.10), and using it again, we find that

$$y(x \circ \mathcal{D}(x)) + [x, y]\mathcal{D}(x) + y[x, \mathcal{D}(x)] \in P \quad \text{for all} \quad x, y \in \mathcal{R}. \tag{2.11}$$

Replacing y by yz in (2.11), where $z \in \mathcal{R}$ and using it again, we obtain

$$[x, y]z\mathcal{D}(x) \in P \quad \text{for all} \quad x, y, z \in \mathcal{R}. \tag{2.12}$$

Continuing with the same techniques as used in (i), and we get the required result. □

Corollary 2.5. *Let $\mathcal{R}$ be a prime ring. If $\mathcal{R}$ admits a nonzero derivation $\mathcal{D}$, then the following assertions are equivalent:*

(i) $[x, \mathcal{D}(x)] = 0$ *for all* $x \in \mathcal{R}$.

(ii) $\mathcal{R}$ *is commutative.*

Corollary 2.6. *Let $\mathcal{R}$ be a prime ring. If $\mathcal{R}$ admits a nonzero derivation $\mathcal{D}$ then the following assertions are equivalent:*

(i) $x \circ \mathcal{D}(x) = 0$ *for all* $x \in \mathcal{R}$.

(*ii*) *$\mathcal{R}$ is commutative of characteristic equal* 2.

Proof. By Lemma 2.4 we get that $\mathcal{R}$ is commutative. In this case, our identity becomes $2xD(x) = 0$ for all $x, y \in \mathcal{R}$. Linearizing the last equation, we find $2xD(y) + 2yD(x) = 0$ for all $x, y \in \mathcal{R}$. Replacing y by yx, we obtain $2xD(yx) = 0$ for all $x, y \in \mathcal{R}$. This implies that $2xD(y)x = 0$ for all $x, y \in \mathcal{R}$. Replacing yt with y and using the last expression, we get $2x\mathcal{R}y\mathcal{R}D(t)\mathcal{R}x = \{0\}$ for all $x, y \in \mathcal{R}$. Since $D \neq 0$, we conclude that $2x = 0$ for all $x \in \mathcal{R}$. □

Theorem 2.7. *Let $\mathcal{R}$ be a ring and P a prime ideal of $\mathcal{R}$. Suppose that $\mathcal{R}$ admits a multiplier H and a derivation $\mathcal{D}$ of $\mathcal{R}$ such that $\mathcal{D}(P) \subseteq P$. If $[\mathcal{D}(x), \mathcal{D}(y)] + H([x, y]) \in P$ for all $x, y \in \mathcal{R}$, then one of the following assertions holds:*

(*i*) $H(\mathcal{R}) \subseteq P$.

(*ii*) *There exists $\lambda \in C$ such that $\mathcal{D} - \lambda$ maps $\mathcal{R}$ into P with $(\lambda^2 + H)([x, y]) \in P$ for all $x, y \in \mathcal{R}$.*

(*iii*) *$\mathcal{R}/P$ is a commutative ring.*

Proof. Suppose that $\mathcal{R}/P$ is not a commutative ring and

$$[\mathcal{D}(x), \mathcal{D}(y)] + H([x, y]) \in P \quad \text{for all} \quad x, y \in \mathcal{R}. \tag{2.13}$$

Replacing x by xt in (2.13) and using it, we conclude that

$$\mathcal{D}(x)[t, \mathcal{D}(y)] + x[\mathcal{D}(t), \mathcal{D}(y)] + [x, \mathcal{D}(y)]\mathcal{D}(t) + H(x)[t, y] \in P \quad \text{for all} \quad x, y, t \in R. \tag{2.14}$$

Substituting ux for x in (2.14), we find that

$$\mathcal{D}(u)x[t, \mathcal{D}(y)] + u\mathcal{D}(x)[t, \mathcal{D}(y)] + ux[\mathcal{D}(t), \mathcal{D}(y)] + u[x, \mathcal{D}(y)]\mathcal{D}(t) + [u, \mathcal{D}(y)]x\mathcal{D}(t) + uH(x)[t, y] \in P. \tag{2.15}$$

Left-multiplying (2.14) by u and comparing it with (2.15), we get

$$(\mathcal{D}(u)x - u\mathcal{D}(x))[t, \mathcal{D}(y)] + u\mathcal{D}(x)[t, \mathcal{D}(y)] + [u, \mathcal{D}(y)]x\mathcal{D}(t) \in P \quad \text{for all} \quad x, y, u, t \in \mathcal{R}. \tag{2.16}$$

Taking $t = \mathcal{D}(y)$ in (2.16), we obtain $[u, \mathcal{D}(y)]\mathcal{R}\mathcal{D}(\mathcal{D}(y)) \subseteq P$ for all $u, y \in \mathcal{R}$.

By primeness of P, it follows that for each y in $\mathcal{R}$ either $[u, \mathcal{D}(y)] \in P$ for all $u \in \mathcal{R}$ or $\mathcal{D}(\mathcal{D}(y)) \in P$. Let $A = \{y \in \mathcal{R} \mid [u, \mathcal{D}(y)] \in P$ for all $u \in \mathcal{R}\}$ and $B = \{y \in \mathcal{R} \mid \mathcal{D}(\mathcal{D}(y)) \in P\}$. Clearly, A and B are additive subgroups of $\mathcal{R}$ such that $A \cup B = \mathcal{R}$. The fact that a group cannot be a union of two of its proper subgroups, forces us to conclude that either $\mathcal{R} = A$ or $\mathcal{R} = B$.

Assume that $\mathcal{R} = A$. Then by Lemma 2.3 (i) and our hypothesis, we get $\mathcal{D}(\mathcal{R}) \subseteq P$. In the latter case, from our assumption we get

$$[u, \mathcal{D}(yw)] = \mathcal{D}(y)[u, w] + [u, \mathcal{D}(y)]w + [u, y\mathcal{D}(w)] \in P \quad \text{for all} \quad y, u, w \in \mathcal{R}.$$

Since $[u, \mathcal{D}(yw)] \in P$ and $[u, \mathcal{D}(y)] \in P$ for all $w, u, y \in P$, it is easy to notice that $\mathcal{D}(y)[u, w] \in P$ for all $y, u, w \in \mathcal{R}$. From this, we can easily arrive at $\mathcal{D}(y)\mathcal{R}[u, w] \subseteq P$ for all $u, w, y \in \mathcal{R}$. Hence, it follows that $\mathcal{D}(\mathcal{R}) \subseteq P$. From our initial hypothesis (2.13), we get

$$H([x, y]) \in P \quad \text{for all} \quad x, y \in \mathcal{R}. \tag{2.17}$$

In (2.17), replacing x by xt and using it again, we find that $[x, y]H(t) \in P$ for all $x, y, t \in \mathcal{R}$. Replacing y by yr, where $r \in \mathcal{R}$, we get $[x, y]rH(t) \in P$ for all $x, y, r, t \in \mathcal{R}$, which implies that by the primeness of P that $H(\mathcal{R}) \subseteq P$.

Next, we consider the case $\mathcal{R} = B$, it follows that $\mathcal{D}(\mathcal{D}(y)) \in P$ for all $y \in \mathcal{R}$. It implies that for each $x, y \in \mathcal{R}$, we have $\mathcal{D}([\mathcal{D}(x), \mathcal{D}(y)]) \in P$. Applying d to equation (2.13) and using the condition $d(P) \subseteq P$, we get

$$\mathcal{D}(H([x, y])) \in P \quad \text{for all} \quad x, y \in \mathcal{R}. \tag{2.18}$$

Replacing x by xy in (2.18) and using it, we find that

$$H([x, y])\mathcal{D}(y) \in P \quad \text{for all} \quad x, y \in \mathcal{R}.$$

Replacing x by xt and using it, we find $H([x, y])t\mathcal{D}(y) \in P$ for all $x, y, t \in \mathcal{R}$. Therefore, either $H([\mathcal{R}, \mathcal{R}]) \subseteq P$ or $\mathcal{D}(\mathcal{R}) \subseteq P$. If $H([\mathcal{R}, \mathcal{R}]) \subseteq P$, then as in (2.17) we have $H(\mathcal{R}) \subseteq P$. Let us suppose that $\mathcal{D}(\mathcal{R}) \subseteq P$, from (2.16) we have $(\mathcal{D}(u)x - u\mathcal{D}(x))[t, \mathcal{D}(y)] \in P$ for all $x, y, t, u \in \mathcal{R}$, which means that $(\mathcal{D}(u)x - u\mathcal{D}(x)) \in P$ for all $x, u \in \mathcal{R}$ or $[t, \mathcal{D}(y)] \in P$ for all $y, t \in \mathcal{R}$ (the second case is already discussed above). So, we assume that $\mathcal{D}(u)x - u\mathcal{D}(x) \in P$ for all $u, x \in \mathcal{R}$. Replacing u by uy, we get

$$\overline{\mathcal{D}(u)yI_{\mathcal{R}}(x)} = \overline{I_{\mathcal{R}}(u)y\mathcal{D}(x)} \quad \text{for all} \quad x, y, u \in \mathcal{R},$$

where $I_{\mathcal{R}}$ is the identity map of $\mathcal{R}$.

Using Lemma 2.2, there exists $\overline{\lambda} \in \overline{C}$ such that $\overline{\mathcal{D}(x)} = \overline{\lambda x}$ for all $x \in \mathcal{R}$. It implies that $\mathcal{D}(x) - \lambda x \in P$ for all $x \in \mathcal{R}$. Hence,

$$[\{\mathcal{D} - \lambda\}(x), \{\mathcal{D} + \lambda\}(y)] \in P.$$

In view of our hypothesis, we get $(\lambda^2 + H)([x, y]) \in P$ for all $x, y \in \mathcal{R}$. $\square$

In the same way, we can get the following result:

Theorem 2.8. *Let $\mathcal{R}$ be a ring and P a prime ideal of $\mathcal{R}$. If $\mathcal{R}$ admits a multiplier H and a derivation $\mathcal{D}$ with $\mathcal{D}(P) \subseteq P$, such that any one of the following assertions holds:*

(a) $\mathcal{D}(x) \circ \mathcal{D}(y) + H([x,y]) \in P$ *for all* $x, y \in \mathcal{R}$,

(b) $\mathcal{D}(x) \circ \mathcal{D}(y) + H(x \circ y) \in P$ *for all* $x, y \in \mathcal{R}$,

(c) $[\mathcal{D}(x), \mathcal{D}(y)] + H(x \circ y) \in P$ *for all* $x, y \in \mathcal{R}$,

then one of the following holds:

(i) $H(\mathcal{R}) \subseteq P$.

(ii) *There exists* $\lambda \in C$ *such that* $\mathcal{D} - \lambda$ *maps* $\mathcal{R}$ *into* P *with* $(\lambda^2 + H)([x,y]) \in P$ *for all* $x, y \in \mathcal{R}$.

(iii) $\mathcal{R}/P$ *is a commutative ring.*

Proof. (a) By our assumption

$$\mathcal{D}(x) \circ \mathcal{D}(y) + H([x,y]) \in P \quad \text{for all} \quad x, y \in \mathcal{R}.$$

Replacing x by xt in the above expression and using it, we conclude that

$$\mathcal{D}(x)[t, \mathcal{D}(y)] + (x \circ \mathcal{D}(y))\mathcal{D}(t) + x[\mathcal{D}(t), \mathcal{D}(y)] + H(x)[t,y] \in P. \tag{2.19}$$

Substituting ux for x in (2.19), we find that

$$\begin{aligned} \mathcal{D}(u)x[t, \mathcal{D}(y)] + u\mathcal{D}(x)[t, \mathcal{D}(y)] + u(x \circ \mathcal{D}(y))\mathcal{D}(t) - [u, \mathcal{D}(y)]x\mathcal{D}(t) \\ + ux[\mathcal{D}(t), \mathcal{D}(y)] + uH(x)[t,y] \in P \quad \text{for all} \quad x, y, t, u \in \mathcal{R}. \end{aligned} \tag{2.20}$$

From the Left multiplying (2.19) by u and comparing with (2.20), we get

$$(\mathcal{D}(u)x - u\mathcal{D}(x))[t, \mathcal{D}(y)] + u\mathcal{D}(x)[t, \mathcal{D}(y)] - [u, \mathcal{D}(y)]x\mathcal{D}(t) \in P \quad \text{for all} \quad x, y, t, u \in \mathcal{R}. \tag{2.21}$$

We process using the same approach as in Theorem 2.7, and finally, we arrive at our result. We can reach the conclusions of (b) and (c) by using similar techniques as before, with the necessary variations of (c). □

It is easy to prove that the maps $I_{\mathcal{R}}$ and $-I_{\mathcal{R}}$ are multipliers of $\mathcal{R}$. We get the following results by replacing H with $\mp I_{\mathcal{R}}$:

Corollary 2.9. *Let $\mathcal{R}$ be a ring and P be a proper prime ideal of $\mathcal{R}$. If $\mathcal{R}$ admits a derivation $\mathcal{D}$ with $\mathcal{D}(P) \subseteq P$, such that $\mathcal{R}$ satisfies one of the following assertions:*

(a) $[\mathcal{D}(x), \mathcal{D}(y)] \pm [x, y] \in P$ *for all* $x, y \in \mathcal{R}$,

(b) $\mathcal{D}(x) \circ \mathcal{D}(y) \pm [x, y] \in P$ *for all* $x, y \in \mathcal{R}$,

(c) $\mathcal{D}(x) \circ \mathcal{D}(y) \pm (x \circ y) \in P$ *for all* $x, y \in \mathcal{R}$,

(d) $[\mathcal{D}(x), \mathcal{D}(y)] \pm (x \circ y) \in P$ *for all* $x, y \in \mathcal{R}$,

then one of the following holds:

(i) *there exists* $\lambda \in C$ *such that* $\mathcal{D} - \lambda$ *maps* $\mathcal{R}$ *into* P *with* $(\lambda^2 \mp I)([x, y]) \in P$ *for all* $x, y \in \mathcal{R}$;

(ii) $\mathcal{R}/P$ *is a commutative ring.*

Replacing H by $-I_{\mathcal{R}}$ in the Theorem 2.7 and P by $\{0\}$, we get the following corollary:

Corollary 2.10. *If $\mathcal{R}$ is a prime ring admitting a strong commutativity preserving (SCP) derivation $\mathcal{D}$, then one of the following assertions holds:*

(1) *There exists* $\lambda \in C$ *such that* $\mathcal{D}(x) = \lambda x$ *for all* $x \in \mathcal{R}$ *with* $\lambda^2 = 1$;

(2) $\mathcal{R}$ *is a commutative ring.*

Replacing H by $-I_{\mathcal{R}}$ in the Theorem 2.8 and P by $\{0\}$, we obtain the following corollary:

Corollary 2.11. *Let $\mathcal{R}$ be a prime ring. If $\mathcal{R}$ admits a derivation $\mathcal{D}$, such that any one of the following assertions hold:*

(a) $\mathcal{D}(x) \circ \mathcal{D}(y) = [x, y]$ *for all* $x, y \in \mathcal{R}$,

(b) $\mathcal{D}(x) \circ \mathcal{D}(y) = x \circ y$ *for all* $x, y \in \mathcal{R}$,

(c) $[\mathcal{D}(x), \mathcal{D}(y)] = x \circ y$ *for all* $x, y \in \mathcal{R}$,

then one of the following holds:

(i) *There exists* $\lambda \in C$ *such that* $\mathcal{D}(x) = \lambda x$ *for all* $x \in \mathcal{R}$ *with* $\lambda^2 = 1$;

(ii) $\mathcal{R}$ *is a commutative ring.*

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