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On uniqueness of L-functions in terms of zeros of strong uniqueness polynomial

Abhijit Banerjee[1]
Arpita Kundu[1,✉]

[1] *Department of Mathematics, University of Kalyani, West Bengal 741235, India.*
abanerjeekal@gmail.com
arpitakundu.math.ku@gmail.com[✉]

ABSTRACT

In this article, we have mainly focused on the uniqueness problem of an L-function $\mathcal{L}$ with an L-function or a meromorphic function f under the condition of sharing the sets, generated from the zero set of some strong uniqueness polynomials. We have introduced two new definitions, which extend two existing important definitions of URSM and UPM in the literature and the same have been used to prove one of our main results. As an application of the result, we have exhibited a much improved and extended version of a recent result of Khoai-An-Phuong [13]. Our remaining results are about the uniqueness of L-function under weighted sharing of sets generated from the zeros of a suitable strong uniqueness polynomial, which improve and extend some results in [12].

RESUMEN

En este arículo nos hemos enfocado principalmente en el problema de unicidad de una L-función $\mathcal{L}$ con una L-función o una función meromorfa f bajo la condición de compartir los conjuntos, generados a partir del conjunto de ceros de algunos polinomios de unicidad fuerte. Hemos introducido dos definiciones nuevas, que extienden dos importantes definiciones existentes en la literatura de URSM y UPM, y las mismas han sido usadas para probar uno de nuestros resultados principales. Como una aplicación del resultado, exhibimos una versión mejorada y extendida de un resultado reciente de Khoai-An-Phuong [13]. Nuestros resultados restantes son sobre la unicidad de una L-función bajo la condición de compartir conjuntos pesados generados a partir de los ceros de un polinomio de unicidad fuerte apropiado, que mejora y extiende algunos resultados en [12].

Keywords and Phrases: Meromorphic function, strong uniqueness polynomial, uniqueness, shared sets, $\mathcal{L}$ function.

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1 Introduction

Riemann hypothesis can be generalized by replacing Riemann's zeta function by the formally similar, but much more general L-functions. Considering $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ as a prototype in 1989, Selberg defined a rather general class $\mathcal{S}$ of convergent Dirichlet series $L(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}$ which are absolutely convergent for $Re(s) > 1$. In the meantime, this so-called Selberg class L-function became important object of research as it plays a pivotal role in analytic number theory. An L-function in $\mathcal{S}$ need to satisfy the following axioms (see [18]):

(i) Ramanujan hypothesis: $a(n) \ll n^{\epsilon}$ for every $\epsilon > 0$.

(ii) Analytic continuation: There is a non-negative integer k such that $(s-1)^k \mathcal{L}(s)$ is an entire function of finite order.

(iii) Functional equation: $\mathcal{L}$ satisfies a functional equation of type

$$\Lambda_{\mathcal{L}}(s) = \omega \overline{\Lambda_{\mathcal{L}}(1-\overline{s})},$$

where

$$\Lambda_{\mathcal{L}}(s) = \mathcal{L}(s) Q^s \prod_{j=1}^{K} \Gamma(\lambda_j s + \nu_j)$$

with positive real numbers Q, λ_j and complex numbers ν_j, ω with $Re\ \nu_j \geq 0$ and $|\omega| = 1$.

(iv) Euler product hypothesis: $\mathcal{L}$ can be written over prime as

$$\mathcal{L}(s) = \prod_{p} \exp\left(\sum_{k=1}^{\infty} b(p^k)/p^{ks} \right)$$

with suitable coefficients $b(p^k)$ satisfying $b(p^k) \ll p^{k\theta}$ for some $\theta < 1/2$, where the product is taken over all prime numbers p. The degree $d_{\mathcal{L}}$ of an L-function $\mathcal{L}$ is defined to be

$$d_{\mathcal{L}} = 2 \sum_{j=1}^{K} \lambda_j,$$

where λ_j and K are respectively the positive real number and the positive integer as in axiom (iii) above.

In this paper we are going to discuss some results under the periphery of value distribution of L-functions in $\mathcal{S}$. Throughout this paper by an L-function we will mean an L-function of non-zero degree with the normalized condition $a(1) = 1$. On the other hand, by meromorphic function f we mean meromorphic function in the whole complex plane $\mathbb{C}$. Let $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ and $\overline{\mathbb{N}} = \mathbb{N} \cup \{0\}$, where $\mathbb{C}$ and $\mathbb{N}$ denote the set of all complex numbers and natural numbers respectively and by $\mathbb{Z}$ we denote the set of all integers. Before entering into the detail literature, let us assume $\mathcal{M}(\mathbb{C})$ as the field of meromorphic functions over $\mathbb{C}$ and assume f, g be two non-constant meromorphic functions in $\mathcal{M}(\mathbb{C})$. The proofs of the theorems of the paper are heavily depending on Nevanlinna theory and we assume that the readers are familiar with the standard notations like the characteristic function $T(r,f)$, the proximity function $m(r,f)$, counting (reduced counting) function $N(r,f)$ ($\overline{N}(r,f)$) that are also explained in [9, 20]. By $S(r,f)$ we mean any quantity that satisfies $S(r,f) = O(\log(rT(r,f)))$ when $r \to \infty$, except possibly on a set of finite Lebesgue measure. When f has finite order, then $S(r,f) = O(\log r)$ for all r. For any $f \in \mathcal{M}(\mathbb{C})$, the order of f is defined as

$$\rho(f) := \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r}.$$

2 Definitions

Before proceeding further, we require the following definitions.

Definition 2.1. *For some $a \in \mathbb{C} \cup \{\infty\}$, we define $E_f(S) = \cup_{a\in S}\{z : f(z) - a = 0\}$, where each point is counted according to its multiplicity. If we do not count the multiplicity then the set $\cup_{a\in S}\{z : f(z) - a = 0\}$ is denoted by $\overline{E}_f(S)$. If $E_f(S) = E_g(S)$, then we say f and g share the set S Counting Multiplicity (CM). On the other hand, if $\overline{E}_f(S) = \overline{E}_g(S)$ then we say f and g share the set S Ignoring Multiplicity (IM).*

The following definition is more generalized than Definition 2.1 and somehow been inspired from the idea in [11].

Definition 2.2. *Let S_1, $S_2 \subset \mathbb{C}$ and if $E_f(S_1) = E_g(S_2)$ ($\overline{E}_f(S_1) = \overline{E}_g(S_2)$) holds then we say that f, g have the same inverse image with respect to the sets S_1 and S_2 respectively, counting multiplicity (ignoring multiplicity) and abbreviated it as IICM $\{(S_1)(f), (S_2)(g)\}$ (IIIM$\{(S_1)(f), (S_2)(g)\}$).*

Definition 2.3 ([14])**.** *Let k be a positive integer, $b \in \mathbb{C}$ and $E_k(0; f-b)$ be the set of all zeros of $f - b$, where a zero of multiplicity p is counted p times if $p \le k$, and $k+1$ times if $p > k$. If $E_k(0; f-b) = E_k(0; g-b)$, we say that $f-b$, $g-b$ share the 0 with weight k and we write it as $f-b$ and $g-b$ share $(0,k)$ or f and g share (b,k). For $S \subset \mathbb{C}\cup\{\infty\}$, we define $E_f(S,k) = \cup_{a\in S} E_k(a; f)$, where k is a non-negative integer or infinity. Clearly $E_f(S) = E_f(S,\infty)$. In particular, $E_f(S,k) = E_g(S,k)$ and $E_f(\{a\},k) = E_g(\{a\},k)$ implies f and g share the set S and the value a with weight k.*

Definition 2.4 ([14])**.** *Let $b \in \mathbb{C}$, by $N(r,b; f \mid\ge s)$ ($N(r,b; f \mid\le s)$) we denote the counting function of those zeros of $f-b$ of multiplicity $\ge s$ ($\le s$). Also $\overline{N}(r,b; f \mid\ge s)$ ($\overline{N}(r,b; f \mid\le s)$) are defined analogously.*

Definition 2.5 ([21]). *If for some set $S \subset \mathbb{C}$, $E_f(S) = E_g(S)$ implies $f = g$, then we will say S unique range set of meromorphic function and denote it as URSM.*

Definition 2.6. *If for two sets $S_1, S_2 \subset \mathbb{C}$, $E_f(S_1) = E_g(S_2)$ implies $f = g$, then we will say $\{S_1, S_2\}$ belong to the extended class unique range set of meromorphic function and we denote it by ECURSM. Similarly we can define extended class unique range set of L-function and denote it as ECURSL.*

Definition 2.7 ([4]). *A set $S \subset \mathbb{C}$ is called a unique range set for meromorphic (entire) functions with weight k if for any two non-constant meromorphic (entire) functions f and g, $E_f(S,k) = E_g(S,k)$ implies $f = g$. We write S is URSMk ($URSEk$) in short. In case of L-function it is reasonable to write it as $URSLk$.*

Definition 2.8 ([1]). *For a non-zero constant c, if $P(f) = cP(g)$ implies $f = g$ then P is called a strong uniqueness polynomial for meromorphic function and denote it by SUPM.*

Definition 2.9 ([15]). *A polynomial P is called a uniqueness polynomial for meromorphic functions if $P(f) = P(g)$ implies $f = g$ and we denote it as UPM.*

Definition 2.10. *Let P, Q be two polynomials of same degree. Now if $f = g$ for all f, g satisfying $P(f) = Q(g)$ then, then we call $\{P, Q\}$ belong to the the extended class of uniqueness polynomial of meromorphic function and denote it as ECUPM. Similarly we can define extended class of uniqueness polynomial of L-function and denote it as ECUPL.*

Definition 2.11 ([4]). *Let $P(z)$ be a polynomial of derivative index k, i.e., $P^{'}(z)$ has mutually k distinct zeros given by $d_1, d_2, \ldots, d_k$ with multiplicities $q_1, q_2, \ldots, q_k$ respectively. Then $P(z)$ is said to satisfy the critical injection property if $P(d_i) \neq P(d_j)$ for $i \neq j$, where $i, j \in \{1, 2, \ldots, k\}$.*

3 Background and main results

Recently the value distributions of L-functions have been studied exhaustively by many researchers ([10, 16, 19], etc.). The value distribution of L-function is all about the roots of $\mathcal{L}(s) = c$. In 2007, regarding uniqueness problem of two $\mathcal{L}$ functions, Steuding [19] proved that the number of shared values can be reduced significantly than that happens in case of ordinary meromorphic function. Below we invoke the result.

Theorem 3.1 ([19]). *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two non-constant L-functions and $c \in \mathbb{C}$. If $E_{\mathcal{L}_1}(c) = E_{\mathcal{L}_2}(c)$ holds, then $\mathcal{L}_1 = \mathcal{L}_2$.*

Since L-functions possess meromorphic continuations, it will be interesting to investigate under which conditions an L-function can share a set with an arbitrary meromorphic function. Inspired by the question of Gross [8] for meromorphic functions, Yuan-Li-Yi [22] proposed the analogous question for a meromorphic function f and an L-function $\mathcal{L}$ sharing one or two finite sets. Yuan-Li-Yi [22] answered the question by themselves by proving the following uniqueness result.

Theorem 3.2 ([22])**.** *Let f be a meromorphic function having finitely many poles in $\mathbb{C}$ and let $\mathcal{L} \in \mathcal{S}$ be a non-constant L-function. Let us consider the set $S = \{w : w^n + aw^m + b = 0\}$, where $(n, m) = 1$, $n > 2m + 4$. If $E_f(S) = E_{\mathcal{L}}(S)$, then we will have $f = \mathcal{L}$.*

Motivated by the results of [22], Khoai-An-Phuong [13] considered a different polynomial, whose zero set is not same with the set as in Theorem 3.2. Under the CM sharing of this set, they [13] obtained a uniqueness relation between an L-function and an arbitrary meromorphic function. In their paper, Khoai-An-Phuong ([13]) consider the polynomial.

$$P(z) = (m + n + 1) \left(\sum_{i=0}^{n} \binom{m}{i} \frac{(-1)^i}{m + n + 1 - i} z^{m+n+1-i} a^i \right) + 1, \tag{3.1}$$

and $(m + n + 1) \left(\sum_{i=0}^{n} \binom{m}{i} \frac{(-1)^i}{m+n+1-i} \right) a^{n+m+1} \neq -1, -2$. Then $P'(z) = (n + m + 1)z^n(z - a)^m$. In their recent paper, Khoai-An-Phuong ([13]) obtained the following result.

Theorem 3.3 ([13])**.** *Let f be a non-constant meromorphic function, $\mathcal{L}$ be an L-function, $P(z)$ be defined as in* (3.1) *and $S = \{z : P(z) = 0\}$. If $n \geq 2$, $m \geq 2$, $n + m \geq 8$, then $E_f(S) = E_{\mathcal{L}}(S)$ implies $f = \mathcal{L}$.*

Now from Theorem 3.3, the following questions are inevitable:

(1) The considered set in Theorem 3.3 is a particular one and it is clear by Example 6.1 given in the application part afterwards, that the set is actually a zero set of a particular SUPM. So it is obvious to explore, whether the set can be generalized by the set of zeros of an arbitrary SUPM.

(2) In Theorem 3.3, to obtain the uniqueness result between f and $\mathcal{L}$, the authors considered CM sharing of the set. So is it possible to relax the CM sharing of the set?

(3) The minimum cardinality of the set in Theorem 3.3 is nine. Is it possible to decrease the cardinality of the set?

In this article, inspired by Theorem 3.3, we have tried to explore and provide the best possible answer of the above questions. Before going to the result, let us consider the following polynomial,

$$P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_t z^t, \tag{3.2}$$

of simple zeros with $P'(z) = (z - d_1)^{q_1}(z - d_2)^{q_2} \cdots (z - d_l)^{q_k}$, satisfying the following properties:

(i) $P(z)$ is a critically injective polynomial of degree $t \geq 5$ with simple zeros and the derivative index of it is $k \geq 2$ and for $k = 2$, $\min\{q_1, q_2\} \geq 2$.

(ii) $P(z)$ be a SUPM.

Theorem 3.4. *Let f be a non-constant meromorphic function, $\mathcal{L}$ be a non-constant L-function, and $P(z)$ be defined as in (3.2) satisfying properties (i) and (ii) such that $S = \{z : P(z) = 0\}$. Now if $E_f(S, 2) = E_{\mathcal{L}}(S, 2)$ and $t \geq 2k + 4$, then we have $f = \mathcal{L}$.*

In the application part of this article in Example 6.1 , we have considered a more general version of polynomial (3.1) and by means of Example 6.1 , we have shown that our result Theorem 3.4 improves Theorem 3.3. Also in [12], the authors explored the things in a new direction. They found some sufficient conditions for a general polynomial to be a uniqueness polynomial for L-function and found a general unique range sets for L-functions as well. The following result extends the perimeter of unique range sets for L-functions.

Theorem 3.5 ([12])**.** *Let $P(z)$ be a uniqueness polynomial for L-functions. Suppose that $P(z)$ has no multiple zeros, and $P(1) \neq 0$. Then the set $S = \{z : P(z) = 0\}$, is a unique range set for L-functions, counting multiplicities.*

From the statement of Theorem 3.5, it will be interesting to ponder over the answer of the following question:

Question 3.1. *What happens in Theorem 3.5, if $P(1) = 0$?*

In the following theorem, we will deal with the answer of the above question. In fact, in view of Definition 2.6 and Definition 2.10, we will re-investigate Theorem 3.5 under a broader perspective, so that the same theorem will automatically be included in our result and at the same time the question will be answered. Now for the next theorem let us consider $Z^{-}(\mathcal{L})$ to denote the set of trivial zeros of L in the negative half plane, where each zero is counted according to its multiplicity.

Theorem 3.6. *Let $S_1 = \{z : P_o(z) = 0\}$ and $S_2 = \{z : Q_o(z) = 0\}$ where P_o be a uniqueness polynomial of L-function and $Q_o = k_1 P_o + k_2$ and having no multiple zeros. If*

(i) *$k_2 = 0$ and either $P_o(1) \neq 0$ or $P_o(0) \neq 0$ together with $Z^{-}(\mathcal{L}_1) = Z^{-}(\mathcal{L}_2)$,*

(ii) *$k_2 \neq 0$ and $Z^{-}(\mathcal{L}_1) = Z^{-}(\mathcal{L}_2)$, $P_o(1) \neq P_o(0)$. Also either $P_o(1)Q_o(1) \neq 0$ or $P_o(0)Q_o(0) \neq 0$; then $\{P_o, Q_o\}$ belong to ECUPL and $\{S_1, S_2\}$ belong to ECURSL.*

Clearly in the above theorem, when $k_2 = 0$ and $P_o(1) \neq 0$, then Theorem 3.6 is actually Theorem 3.5. Hence this result is an extension of Theorem 3.5.

Considering the IM sharing of set, in [12] the following result was obtained.

Theorem 3.7 ([12]). *Let $P(z)$ be a strong uniqueness polynomial for L-functions, and assume that $P(z)$ has no multiple zeros, and the degree q, the derivative index k of P satisfy inequality $q \geq 2k+6$. Then the zero set of $P(z)$ is a unique range set, ignoring multiplicities, for L-functions.*

As usual it will be interesting to further reduce the cardinality of the set. In the next theorem, we will show that with the help of weighted sharing of weight two the cardinality of the range set can significantly be reduced.

Theorem 3.8. *Let $P(z)$ be a strong uniqueness polynomial for L-functions with simple zeros, of degree t and of derivative index k such that $t \geq 2k+3$. Then the set $S = \{z : P(z) = 0\}$ is URSL2.*

4 Lemma

Next, we present some lemmas that will be needed in the sequel. Henceforth, we denote by H, the following function :

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G}\right),$$

Lemma 4.1 ([5]). *Let $F = P(f)$ and $G = P(g)$ be non-constant meromorphic functions where $P(z)$ is defined same as in (3.2). Also let F, G share $(0,m)$ Then,*

$$N_E^{1)}(r,0;F) \leq N(r,\infty;H) + S(r,F) + S(r,G).$$

Lemma 4.2. *Let F and G be defined same as in Lemma 4.1 and share $(0,m)$. Then,*

$$N(r,\infty;H) \leq \overline{N}_*(r,0;F,G) + \overline{N}(r,\infty;f) + \overline{N}(r,\infty;g) + \sum_{i=1}^{k} \overline{N}(r,\alpha_i;f) + \sum_{i=1}^{k} \overline{N}(r,\alpha_i;g)$$
$$+ \overline{N}_0(r,0;f') + \overline{N}_0(r,0;g') + S(r,f) + S(r,g),$$

where $\overline{N}_0(r,0;f')$ is the reduced counting function of those zeros of F' where $F\prod_{i=1}^{k}(f-\alpha_i) \neq 0$ and $\overline{N}_0(r,0;g')$ is similarly defined and α_i , $i = 1,2,\ldots,k$ are distinct zeros of $P'(z)$.

Proof. Here we are not giving the proof as the similar proof can be found in [14]. □

Lemma 4.3 ([3]). *Let F and G be non-constant meromorphic functions and let F, G share $(0,m)$. Then,*

$$\overline{N}(r,0;F) + \overline{N}(r,0;G) - N_E^{1)}(r,0;F) + \left(m - \frac{1}{2}\right)\overline{N}_*(r,0;F,G) \leq \frac{1}{2}[N(r,0;F) + N(r,0;G)].$$

Proof. Here we are not giving the proof as the similar proof can be found in [3]. □

Lemma 4.4 ([6]). *Let $P(z)$ be a polynomial defined as in (3.2) with property (i). Also assume f and g be two non-constant meromorphic functions such that,*

$$\frac{1}{P(f)} = \frac{c_0}{P(g)} + c_1,$$

$c_0 \neq 0$. *Then we will have* $c_1 = 0$.

Lemma 4.5 ([7]). *Let $P(z)$ be a polynomial defined as in (3.2) with property (i). Then $P(z)$ will be a UPM if and only if*

$$\sum_{1 \leq l < m \leq k} q_l q_m > \sum_{l=1}^{k} q_l.$$

In particular, the above inequality is always satisfied whenever $k \geq 4$. When $k = 3$ and $\max\{q_1, q_2, q_3\} \geq 2$ or when $k = 2$, $\min\{q_1, q_2\} \geq 2$ and $q_1 + q_2 \geq 5$, then also the above inequality holds.

Lemma 4.6 ([17]). *Let f be a non-constant meromorphic function and let*

$$R(f) = \frac{\sum\limits_{k=0}^{n} a_k f^k}{\sum\limits_{j=0}^{m} b_j f^j},$$

be an irreducible rational function in f with constant coefficients $\{a_k\}$ and $\{b_j\}$, where $a_n \neq 0$ and $b_m \neq 0$. Then

$$T(r, R(f)) = dT(r, f) + S(r, f),$$

where $d = \max\{n, m\}$.

Lemma 4.7 ([20]). *Let $f, g \in M(\mathbb{C})$ and let $\rho(f)$, $\rho(g)$ be the order of f and g, respectively. Then*

$$\rho(f \cdot g) \leq \max\{\rho(f), \rho(g)\}.$$

Lemma 4.8. *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two non-constant L-functions and for some $A > 0$, in $\sigma < -A$, $Z^-(\mathcal{L}_1) = Z^-(\mathcal{L}_2)$. Then we can find a infinite sequence of zeros in the same half plane of both $\mathcal{L}_i$, $i = 1, 2$.*

Proof. It is given that in $\sigma < -A$, $Z^-(\mathcal{L}_1) = Z^-(\mathcal{L}_2)$. From axiom (iii), let us assume

$$\begin{aligned} \mathcal{L}_i(s) &= \chi_i(s)\overline{\mathcal{L}_i(1-\overline{s})}, \quad \text{where} \\ \chi_i(s) &= \omega_i Q_i^{1-2s} \frac{\prod_{j=1}^{k_i} \Gamma(\lambda_{ij}(1-s) + \overline{\nu_{ij}})}{\prod_{j=1}^{k_i} \Gamma(\lambda_{ij} s + \nu_{ij})}, \quad \text{for } i = 1, 2. \end{aligned}$$

In particular, in $\sigma < -A$, the poles of $\prod_{j=1}^{k_1}\Gamma(\lambda_{1j}s+\nu_{1j})$ and $\prod_{j=1}^{k_2}\Gamma(\lambda_{2j}s+\nu_{2j})$ must match, also the poles of $\prod_{j=1}^{k_1}\Gamma(\lambda_{1j}(1-s)+\overline{\nu_{1j}})$ and $\prod_{j=1}^{k_2}\Gamma(\lambda_{2j}(1-s)+\overline{\nu_{2j}})$ must match in $\sigma > A$. Also in $-A<\sigma<0$, $\prod_{j=1}^{k_1}\Gamma(\lambda_{1j}s+\nu_{1j})$ and $\prod_{j=1}^{k_2}\Gamma(\lambda_{2j}s+\nu_{2j})$ can have finitely many poles. It follows that $\frac{\chi_1}{\chi_2}$ is a meromorphic function with finitely many poles and zeros. So here we can write it as $\frac{\chi_1(s)}{\chi_2(s)} = R(s)e^{as}$, where R is a rational function and a is a complex constant. Therefore here we have,

$$\mathcal{L}_1(s) = \chi_1(s)\overline{\mathcal{L}_1(1-\overline{s})},$$
$$\mathcal{L}_2(s) = \chi_1(s)R(s)e^{as}\overline{\mathcal{L}_2(1-\overline{s})}.$$

Then in some $\sigma < -B$, where $B \geq A$, it is possible to find a sequence $\{s_n(= -\frac{n+\nu_{1j}}{\lambda_{1j}})\}$ for some fixed j, of zeros of $\chi_1(s)$, which are also zeros of $\mathcal{L}_i(s)$ and $\mathcal{L}_i(1-\overline{s})$ never vanish in $\sigma > B$ for $i = 1, 2$. Also it can be seen that $Re(s_n) \to -\infty$ as $n \to \infty$. □

5 Proofs of the theorems

Proof of Theorem 3.4. Let us consider the following cases.

Case 1: First assume $H = 0$. Then integrating we have,

$$\frac{1}{P(\mathcal{L})} = \frac{c}{P(f)} + d, \tag{5.1}$$

where $c\ (\neq 0), d$ are constants. Clearly from Lemma 4.4 we have, $d = 0$. As from the hypothesis of the theorem we know $P(z)$ is a SUPM, from $P(f) = cP(\mathcal{L})$, we have $f = \mathcal{L}$.

Case 2: Next assume $H \neq 0$. Using the Second Fundamental Theorem we have,

$$\begin{aligned}(t-1)T(r,\mathcal{L}) &\leq \overline{N}(r,0;P(\mathcal{L})) + \overline{N}(r,\infty;\mathcal{L}) + S(r,\mathcal{L}) \\ &\leq \overline{N}(r,0;P(f)) + O(\log r) + S(r,\mathcal{L}) \\ &\leq nT(r,f) + S(r,\mathcal{L}).\end{aligned} \tag{5.2}$$

Similarly, we can have,

$$(t-2)T(r,f) \leq nT(r,\mathcal{L}) + S(r,f). \tag{5.3}$$

Clearly (5.2) and (5.3) we have, $\rho(f) = \rho(\mathcal{L}) = 1$ and hence $S(r,f) = S(r,\mathcal{L}) = S(r)$ (say).

Using the Second Fundamental theorem we have,

$$(t+k-1)(T(r,f)+T(r,\mathcal{L})) \leq \overline{N}(r,0;P(f))+\overline{N}(r,0;P(\mathcal{L}))+\overline{N}(r,\infty;f)+\overline{N}(r,\infty;\mathcal{L}) \\ +\sum_{i=1}^{k}(\overline{N}(r,\alpha_i;f))+\overline{N}(r,\alpha_i;\mathcal{L}))-N_0(r,0;f')-N_0(r,0;\mathcal{L}')+S(r).$$

i.e.,

$$(t-1)T(r,\mathcal{L})+(t-2)T(r,f) \leq \overline{N}(r,0;P(f))\overline{N}(r,0;P(\mathcal{L}))-N_0(r,0;f') \\ -N_0(r,0;\mathcal{L}')+S(r). \tag{5.4}$$

Using Lemmas 4.3, 4.1, 4.2 and 4.6, from (5.4) we have,

$$(t-1)T(r,\mathcal{L})+(t-2)T(r,f) \leq \frac{n}{2}\{T(r,f)+T(r,\mathcal{L})\}+\overline{N}(r,\infty;f)+\overline{N}(r,\infty;\mathcal{L}) \\ +\sum_{i=1}^{k}(\overline{N}(r,\alpha_i,f)+\overline{N}(r,\alpha_i;\mathcal{L}))+S(r),$$

i.e.,

$$\left(\frac{t}{2}-2\right)T(r,f)+\left(\frac{t}{2}-1\right)T(r,\mathcal{L})+S(r) \leq kT(r,\mathcal{L})+(k+1)T(r,f)+S(r),$$

$$(t-2k-6)T(r,f)+(t-2-2k)T(r,\mathcal{L}) \leq S(r). \tag{5.5}$$

Using (5.2) we have

$$(t-2k-6)\frac{t-1}{t}T(r,\mathcal{L})+(t-2-2k)T(r,\mathcal{L}) \leq S(r). \tag{5.6}$$

From (5.6) for $t \geq 2k+4$ we arrive at a contradiction. □

Proof of Theorem 3.8. Let us consider two non-constant L-functions $\mathcal{L}_1$ and $\mathcal{L}_2$ such that $E_{\mathcal{L}_1}(S,2) = E_{\mathcal{L}_2}(S,2)$ where S is the zero set of strong uniqueness polynomial for L-function. Also assume,

$$F = P(\mathcal{L}_1) \quad \text{and} \quad G = P(\mathcal{L}_2).$$

If $H = 0$, then from Case 1 of Theorem 3.4 we will have, $\mathcal{L}_1 = \mathcal{L}_2$. If $H \neq 0$, then proceeding similarly as done in (5.4), (5.5) we will have a contradiction for $t \geq 2k+3$. Hence finally we will have $\mathcal{L}_1 = \mathcal{L}_2$. □

Proof of Theorem 3.6. Let us assume for two non-constant L-functions, $\mathcal{L}_1$, $\mathcal{L}_2$; $E_{\mathcal{L}_1}(S_1) = E_{\mathcal{L}_2}(S_2)$. Clearly then we can set the auxiliary function

$$G = \frac{P_o(\mathcal{L}_1)}{Q_o(\mathcal{L}_2)} = (s-1)^l e^{p_1(s)}, \tag{5.7}$$

for some integer l and from Lemma 4.7 we will have $p_1(s) = as + b$, for some complex constants a, b. Now let us consider the following cases.

Case 1: First let $k_2 = 0$, *i.e.*, $Q_o = k_1 P_o$.

Subcase 1.1: $P_o(1) \neq 0$. Then,

$$G = \frac{(\mathcal{L}_1 - \alpha_1)(\mathcal{L}_1 - \alpha_2)\cdots(\mathcal{L}_1 - \alpha_t)}{(\mathcal{L}_2 - \alpha_1)(\mathcal{L}_2 - \alpha_2)\cdots(\mathcal{L}_2 - \alpha_t)} = k_1 (s-1)^l e^{as+b}, \tag{5.8}$$

from (5.8) taking limit $\sigma \to +\infty$ we have,

$$\lim_{\sigma\to+\infty} k_1 (s-1)^l e^{as+b} = 1,$$

which implies $a = l = 0$ and then simply $k_1 e^b = 1$. Finally we have, $\frac{P_o(\mathcal{L}_1)}{P_o(\mathcal{L}_2)} = 1$ and hence $\mathcal{L}_1 = \mathcal{L}_2$.

Subcase 1.2: Let us assume $P_o(1) = 0$ but $P_o(0) \neq 0$. Without loss of generality assume $\alpha_1 = 1$. Again $\mathcal{L}_i$ can be represented by a Dirichlet series, *i.e.*, $\mathcal{L}_i(s) = \sum_{n=1}^{\infty} \frac{a_i(n)}{n^s}$, $i = 1, 2$, absolutely convergent for $\sigma > 1$, where $a_i(1) = 1$. Also let n_1, n_2 be two integers such that $n_i = \min\{n\ (\geq 2) : a_i(n) \neq 0,\ i = 1, 2\}$. So,

$$\frac{\mathcal{L}_1 - 1}{\mathcal{L}_2 - 1} = \frac{\frac{1}{n_1^s}\left(a_1(n_1) + \sum_{n>n_1}^{\infty} a_1(n)(\frac{n_1}{n})^s\right)}{\frac{1}{n_2^s}\left(a_2(n_2) + \sum_{n>n_2}^{\infty} a_2(n)(\frac{n_2}{n})^s\right)} = \left(\frac{n_2}{n_1}\right)^s G_0(s), \tag{5.9}$$

where,

$$G_0(s) = \frac{a_1(n_1) + \sum_{n>n_1}^{\infty} a_1(n)(\frac{n_1}{n})^s}{a_2(n_2) + \sum_{n>n_2}^{\infty} a_2(n)(\frac{n_2}{n})^s}.$$

By the construction of $G_0(s)$ we note that $\lim_{\sigma\to+\infty} G_0(s) = \frac{a_1(n_1)}{a_2(n_2)}$. In view of (5.7), let us consider the following function

$$\begin{aligned} G_1 = G_0(s) \cdot \frac{(\mathcal{L}_1 - \alpha_2)\cdots(\mathcal{L}_1 - \alpha_t)}{(\mathcal{L}_2 - \alpha_2)\cdots(\mathcal{L}_2 - \alpha_t)} &= \frac{\mathcal{L}_1 - 1}{\mathcal{L}_2 - 1} \cdot q^s \cdot \frac{(\mathcal{L}_1 - \alpha_2)\cdots(\mathcal{L}_1 - \alpha_t)}{(\mathcal{L}_2 - \alpha_2)\cdots(\mathcal{L}_2 - \alpha_t)} \\ = q^s \frac{(\mathcal{L}_1 - 1)(\mathcal{L}_1 - \alpha_2)\cdots(\mathcal{L}_1 - \alpha_t)}{(\mathcal{L}_2 - 1)(\mathcal{L}_2 - \alpha_2)\cdots(\mathcal{L}_2 - \alpha_t)} &= q^s G = k_1 q^s (s-1)^l e^{as+b}, \end{aligned} \tag{5.10}$$

for some $q = \frac{n_1}{n_2} (\in \mathbb{Q}^+)$. We can write $q = e^{\log q} = e^{q'}$, then we can write it as,

$G_1 = k_1 q^s (s-1)^l e^{as+b} = k_1(s-1)^l e^{(q'+a)s+b} = k_1(s-1)^l e^{a's+b}$ where $a' = q' + a$. Let us consider $a' = a_1 + ia_2$ and $b = b_1 + ib_2$. With respect to the first equality of (5.10), taking limit $\sigma \to +\infty$, we have $\lim_{\sigma \to +\infty} G_1 = C_1$, for some constant $C_1 \in \mathbb{C}^*$. Next from the second and last equality of (5.10), taking limit $\sigma \to +\infty$, we have

$$\lim_{\sigma \to +\infty} \left| q^s \frac{(\mathcal{L}_1 - 1)}{(\mathcal{L}_2 - 1)} \cdot \frac{(\mathcal{L}_1 - \alpha_2) \cdots (\mathcal{L}_1 - \alpha_t)}{(\mathcal{L}_2 - \alpha_2) \cdots (\mathcal{L}_2 - \alpha_t)} \right| = |C_1| = \lim_{\sigma \to +\infty} |(s-1)^l e^{a's+\hat{b}}|$$
$$= \text{Constant} = \lim_{\sigma \to +\infty} |\sigma - 1 + it|^l e^{a_1\sigma - a_2 t + b_1}.$$

Therefore we must have $a_1 = 0 = l$, otherwise $\lim_{\sigma \to +\infty} |\sigma - 1 + it|^l e^{a_1\sigma - a_2 t + b_1} = \infty$ or 0 according as $a_1 >$ or < 0 and with the same argument it can be shown that $l = 0$. Also,

$$\lim_{\sigma \to +\infty} e^{-a_2 t + b_1} = |C_1|, \quad \forall t \in \mathbb{R},$$

implies $a_2 = 0$. Hence we have $a = a_1 + ia_2 = 0$ and $l = 0$. Therefore, $G_1 = k_1 e^b$ and from the last equality of (5.10), we get $G = q^{-s} k_1 e^b$, *i.e.*, from (5.8) we have

$$\frac{(\mathcal{L}_1 - 1)}{(\mathcal{L}_2 - 1)} \cdot \frac{(\mathcal{L}_1 - \alpha_2) \cdots (\mathcal{L}_1 - \alpha_t)}{(\mathcal{L}_2 - \alpha_2) \cdots (\mathcal{L}_2 - \alpha_t)} = q^{-s} k_1 e^b. \tag{5.11}$$

Now from Lemma 4.8, it is possible to find a sequence s_n of trivial zeros in $\sigma < -A$, whose real part diverges, *i.e.*, $Re(s_n) \to -\infty$, as $n \to \infty$. From (5.11) putting $s = s_n$ we have $q^{Re(-s_n)} |k_1| e^{Re(b)} = 1$, taking limit as $n \to \infty$ we will have $q^{Re(-s_n)} \to \infty$ or 0, according as $q > 1$ or < 1. So we must have $q = 1$ and hence $k_1 e^b = 1$. Therefore $P_o(\mathcal{L}_1) = P_o(\mathcal{L}_2) \implies \mathcal{L}_1 = \mathcal{L}_2$.

Case 2: Next let $k_2 \neq 0$. Then we can write G as,

$$G = \frac{P_o(\mathcal{L}_1)}{k_1 P_o(\mathcal{L}_2) + k_2} = \frac{(\mathcal{L}_1 - \alpha_1)(\mathcal{L}_1 - \alpha_2) \cdots (\mathcal{L}_1 - \alpha_t)}{(\mathcal{L}_2 - \beta_1)(\mathcal{L}_2 - \beta_2) \cdots (\mathcal{L}_2 - \beta_t)} = (s-1)^l e^{as+b}. \tag{5.12}$$

Subcase 2.1: Let us assume $P_o(1).Q_o(1) \neq 0$. From (5.11) taking $\sigma \to +\infty$, we will have, $G = C =$ non-zero constant. Hence we have, $P_o(\mathcal{L}_1) = k_1' P_o(\mathcal{L}_2) + k_2'$. In view of Lemma 4.8, Putting $s = s_n$ we have, $k_2' = P_o(0)(1 - k_1')$.

Subcase 2.1.1: First let $P_o(0) = 0$, then $k_2' = 0$. Using the fact $P_o(1) \neq 0$, with the same argument as in Subcase 1.1 we will have, $P_o(\mathcal{L}_1) = P_o(\mathcal{L}_2)$ and hence $\mathcal{L}_1 = \mathcal{L}_2$.

Subcase 2.1.2: Next let $P_o(0) \neq 0$. Then we have $P_o(\mathcal{L}_1) - P_o(0) = k_1'(P_o(\mathcal{L}_2) - P_o(0))$. Taking $\sigma \to +\infty$ and noting that $P_o(0) \neq P_o(1)$, we have, $k_1' = 1$ and hence $k_2' = 0$. And the from Subcase 1.1 we will have the result.

Subcase 2.2: Assume $P_o(1)Q_o(1) = 0$ but $P_o(0)Q_o(0) \neq 0$.

Subcase 2.2.1: Let us assume $P_o(1) = 0 = Q_o(1)$. Without loss of generality assume $\alpha_1 = \beta_1 = 1$. Then proceeding similarly as done in Subcase 1.2 we will have $\frac{P_o(\mathcal{L}1)}{Q_o(\mathcal{L}_2)} =$ constant. Noting that $P_o(0) \neq 0$, like Subcase 2.1 we can show that the constant is 1 and so we have $\mathcal{L}_1 = \mathcal{L}_2$.

Subcase 2.2.2: Next let $P_o(1) = 0$ but $Q_o(1) \neq 0$. Then let $\alpha_1 = 1$ and we can have,

$$\mathcal{L}_1 - 1 = \frac{1}{n_1^s}\left(a_1(n_1) + \sum_{n>n_1}^{\infty} a_1(n)\left(\frac{n_1}{n}\right)^s\right) = \frac{1}{n_1^s}G_1(s),$$

where $G_1(s) = n_1^s(\mathcal{L}_1 - 1) = a_1(n_1) + \sum_{n>n_1}^{\infty} a_1(n)(\frac{n_1}{n})^s$ and $\lim_{\sigma\to+\infty} G_1 = a_1(n_1)$.
Now, $G = \dfrac{(\mathcal{L}_1-1)(\mathcal{L}_1-\alpha_2)\cdots(\mathcal{L}_1-\alpha_n)}{(\mathcal{L}_2-\beta_1)(\mathcal{L}_2-\beta_2)\cdots(\mathcal{L}_2-\beta_n)} = (s-1)^l e^{as+b}$.
Let us set a function

$$\begin{aligned} G_2 &= G_1\frac{(\mathcal{L}_1-\alpha_2)\cdots(\mathcal{L}_1-\alpha_n)}{(\mathcal{L}_2-\beta_1)(\mathcal{L}_2-\beta_2)\cdots(\mathcal{L}_2-\beta_n)} \\ &= n_1^s\frac{(\mathcal{L}_1-1)(\mathcal{L}_1-\alpha_2)\cdots(\mathcal{L}_1-\alpha_n)}{(\mathcal{L}_2-\beta_1)(\mathcal{L}_2-\beta_2)\cdots(\mathcal{L}_2-\beta_n)} = n_1^s G = (s-1)^l n_1^s e^{as+b}. \end{aligned} \tag{5.13}$$

Therefore we can write, $G_2 = (s-1)^l e^{a''s}e^b$, where $a'' = a + \log n_1$. Next the first equality of (5.13) implies, $\lim_{\sigma\to+\infty} G_2 =$ Constant. But $\lim_{\sigma\to+\infty} |(s-1)^l e^{a''s+b}| = 0$ or ∞, according as $Re(a'') <$ or > 0, it follows that $Re(a'') = 0$. Since the limit is independent of t, we will have $Im(a'') = 0$. With similar arguments we will have $l = 0$. Therefore $a'' = 0 = l$ and we will have from the last equality of (5.13),

$$G_2 = e^b \implies G = n_1^{-s}e^b$$

i.e.,

$$\frac{(\mathcal{L}_1-1)(\mathcal{L}_1-\alpha_2)\cdots(\mathcal{L}_1-\alpha_n)}{(\mathcal{L}_2-\beta_1)(\mathcal{L}_2-\beta_2)\cdots(\mathcal{L}_2-\beta_n)} = n_1^{-s}e^b.$$

Proceeding similarly as in (5.11) we will have, $n_1 = 1$ and then we have $G = e^b =$ Constant. With the help of Subcase 2.1 we will have $\mathcal{L}_1 = \mathcal{L}_2$.

Subcase 2.2.3: Next let $P_o(1) \neq 0$ but $Q_o(1) = 0$, proceeding in a same way as done in Subcase 2.2.2 and then using Subcase 2.1 we will have $\mathcal{L}_1 = \mathcal{L}_2$.

Hence $\{P_o, Q_o\}$ belong to ECUPL and $\{S_1, S_2\}$ belong to ECURSL. □

6 Application

In this section, we show the application of Theorem 3.4. Not only that, next we are going to show that the much improved version of Theorem 3.3 falls under a special case of our Theorem 3.4. Below we explain this fact via the following example.

Example 6.1. We are going to consider a new polynomial of degree $m+n+1$ recently introduced in [2] as follows:

$$\begin{aligned} P(z) &= \sum_{j=0}^{n} \binom{n}{j} \frac{(-1)^j}{m+n+1-j} z^{m+n+1-j} a^j \\ &\quad + \sum_{i=1}^{m} \sum_{j=0}^{n} \binom{m}{i} \binom{n}{j} \frac{(-1)^{i+j}}{m+n+1-i-j} z^{m+n+1-i-j} a^j b^i - c = Q(z) - c, \end{aligned} \tag{6.1}$$

where a and b be distinct such that $a, b \in \mathbb{C}$, $c \neq 0$, $Q(a), Q(b)$ and $m \geq n+2$ and $n \geq 2$. It is easy to verify that,

$$P'(z) = (z-a)^n (z-b)^m.$$

Clearly from the choice of c, $P(z)$ has only simple zeros. First we will show that (6.1) is critically injective, strong uniqueness polynomial with derivative index 2 with $m \geq n+2$ and $n \geq 2$. From Remark 1 [2, p. 506] it can be shown that $P(z)$ is critically injective polynomial. Next, let us assume for some constant $A \neq 0$ and for two non-constant meromorphic functions f, g with finitely many poles we have

$$P(f) = AP(g). \tag{6.2}$$

By Lemma 4.4, we get,

$$T(r, f) = T(r, g) + O(1). \tag{6.3}$$

Also here, $\overline{N}(r, \infty; f) = S(r, f) = \overline{N}(r, \infty; g) = S(r, g)$.

Now, consider the cases,

Case 1: Suppose $A \neq 1$. Then

$$P(f) + c = A(P(g) + c) - c(A-1), \tag{6.4}$$

i.e.,

$$Q(f) = AQ(g) - c(A-1) \implies Q(f) - Q(b) = AQ(g) - (Q(b) + c(A-1)).$$

Recall that the only zeros of $Q'(z)$ are a and b. So only possible multiple zeros of $\phi(z) =$

$AQ(z) - (Q(b) + c(A-1))$ could be a and b. First assume b is one multiple zero of $\phi(z)$. Thus $\phi(b) = 0$, *i.e.*,

$$AQ(b) - (Q(b) + c(A-1)) = 0 \implies (Q(b) - c)(A-1) = 0 \implies Q(b) = c,$$

a contradiction as $Q(b) \neq c$.

Next assume a is the multiple zero of $\phi(z)$. It is easy to see that $\phi(z) = (z-a)^{n+1}\phi_1(z)$, where $\phi_1(a) \neq 0$ and all zeros of $\phi_1(z)$ are simple, namely μ_j, $j = 1, 2, \ldots, m$. Notice that $Q(z) - Q(b) = (z-b)^{m+1}\phi_2(z)$, where $\phi_2(b) \neq 0$ and all zeros of $\phi_2(z)$ are simple, namely ν_j, $j = 1, 2, \ldots, n$. From (6.4) we have,

$$\overline{N}(r, b; f) + \sum_{i=1}^{n} \overline{N}(r, \nu_j; f) = \overline{N}(r, a; g) + \sum_{i=1}^{m} \overline{N}(r, \mu_j; g). \tag{6.5}$$

Now using the Second Fundamental Theorem we have,

$$\begin{aligned} mT(r, g) &\leq \overline{N}(r, a; g) + \sum_{i=1}^{m} \overline{N}(r, \mu_j; g) + \overline{N}(r, \infty; g) + S(r, g) \\ &\leq \overline{N}(r, b; f) + \sum_{i=1}^{n} \overline{N}(r, \nu_j; f) + S(r, g) \\ &\leq (n+1)T(r, f) = (n+1)T(r, g) + S(r, g), \end{aligned}$$

this contradicts the fact $m \geq n + 2$.

Hence we see neither a nor b be the multiple zeros of $\phi(z)$ and hence all the zeros of $\phi(z)$ are simple, say δ_j, $j = 1, 2, \ldots, m + n + 1$. From (6.4) we have,

$$\begin{aligned} (m+n)T(r, g) &\leq \sum_{j=1}^{m+n+1} \overline{N}(r, \delta_j; g) + \overline{N}(r, \infty; g) + S(r, g) \\ &\leq \overline{N}(r, b; f) + \sum_{i=1}^{n} \overline{N}(r, \nu_j; f) + S(r, g) \\ &\leq (n+1)T(r, f) = (n+1)T(r, g) + S(r, g), \end{aligned}$$

a contradiction as $m \geq n + 2$ and $n \geq 2$.

Case 2: Assume $A = 1$.

$$P(f) = P(g).$$

Now the zeros of $P'(z)$ has multiplicities $m \geq 4$, $n \geq 2$ and $m + n \geq 6$. Hence from Lemma 4.5 we have from above, $f = g$. Now if we take $m = 5$, $n = 2$, then $P(z)$ becomes a polynomial of degree 8. So clearly from the above discussion if f be a meromorphic function and $\mathcal{L}$ be an L-function satisfying $E_f(S,2) = E_{\mathcal{L}}(S,2)$ such that the degree of $P(z)$ becomes $m+n+1 \geq 8$, then by Theorem 3.4, we have $f = \mathcal{L}$. As putting $b = 0$ in (6.1), we obtain the polynomial (3.1), for $m+n \geq 7$, the result in Theorem 3.3 holds as well. Clearly Theorem 3.4 is a three step improvements of Theorem 3.3:

(1) In *Theorem 3.4*, we have considered the zero set of an arbitrary SUPM satisfying properties (i) and (ii). By means of Example 6.1 we know that the polynomial (3.1) is itself a critically injective SUPM, so in terms of choice of SUPM, Theorem 3.4 is quite a generalization of Theorem 3.3.

(2) In the light of relaxation of sharing of the zero set Theorem 3.4 improves Theorem 3.3.

(3) Most importantly, it can be verified that the minimum cardinality of the considered set in Theorem 3.3 is nine, where as we have been able to show that in Theorem 3.3 still holds for the cardinality of the range set as $n+m+1 \geq 8$. That is the cardinality of the range set in Theorem 3.3 can further be diminished.

Again since $\mathcal{L}$ can be analytically continued as a meromorphic function with only one pole, then from the above discussion it can be observed that if $\mathcal{L}_1$ and $\mathcal{L}_2$ share the zero set S of the polynomial (6.1) with weight two, *i.e.*, $E_{\mathcal{L}_1}(S,2) = E_{\mathcal{L}_2}(S,2)$ where $n+m+1 \geq 7$, then according to Theorem 3.8 we will have $\mathcal{L}_1 = \mathcal{L}_2$.

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8 Conflict of interest

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