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Frame's Types of Inequalities and Stratification

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ABSTRACT

In this paper we examine some inequalities of Frame's type on the interval $(0, \pi/2)$. By observing this domain we simply obtain the results using the appropriate families of stratified functions and MTP - Mixed Trigonometric Polynomials. Additionally, from those families we specify a minimax approximant as a function with some optimal properties.

RESUMEN

En este artículo examinamos algunas desigualdades de tipo Frame en el intervalo $(0, \pi/2)$. Observando este dominio simplemente obtenemos los resultados usando las familias apropiadas de funciones estratificadas y PTM - Polinomios Trigonométricos Mezclados. Adicionalmente, a partir de esas familias, especificamos un aproximante minimax como una función con algunas propiedades optimales.

Keywords and Phrases: Frame's type inequalities, stratified families of functions, mixed trigonometric polynomial functions.

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1 Introduction

This paper deals with some inequalities that are discussed in [10, 19], see also the monograph [11, part 3.4.20]. In [1, 13] is stated the Cusa-Huygens approximation:

$$x \approx \frac{3\sin x}{2+\cos x}, \quad \text{for} \quad x \in (0,\pi),$$

which in the paper [9] is specified using families of stratified functions on the domain $(0, \pi/2)$. L. Zhu in [19] gives the following two inequalities:

$$x - \frac{3\sin x}{2+\cos x} > \frac{1}{180}x^5, \quad \text{for} \quad x \in (0,\pi) \tag{1.1}$$

and

$$x - \frac{3\sin x}{2+\cos x}\left(1 + \frac{(1-\cos x)^2}{9(3+2\cos x)}\right) > \frac{1}{2100}x^7, \quad \text{for} \quad x \in (0,\pi), \tag{1.2}$$

and names them Frame's inequalities. In the monograph *Analytic Inequalities* by D.S. Mitrinović [11, part 3.4.20.] inequalities (1.1) and (1.2) appear with the wrong relation, which L. Zhu corrects in [19].

Based on inequality (1.1) the following assertion is proved in the paper [10].

Theorem 1.1. *The following inequalities are true:*

$$\frac{1}{180}\,x^5 \le x - \frac{3\sin x}{2+\cos x} \le \frac{1}{m_1}\,x^5\,, \tag{1.3}$$

where $x \in [0,\pi]$ and $m_1 = 92.96406\ldots = 1/f(x_0)$. The value $f(x_0)$ is determined for the function

$$f(x) = \left(x - \frac{3\sin x}{2+\cos x}\right) \Big/ x^5 : (0,\pi) \longrightarrow \mathbb{R}$$

at the point $x_0 = 2.83982\ldots$ at which the function reaches its maximum $f(x_0)$ on the interval $(0,\pi)$. The equality in (1.3) *holds for both sides when $x = 0$ and holds for the right hand side when $x = x_0$.*

Inequality (1.3) is used to estimate the Cusa-Huygens function $\varphi(x) = x - \dfrac{3\sin x}{2+\cos x}$ over $(0,\pi)$ [10].

The motivation for this paper is to improve the previous results, by finding the minimax approximant for unconsidered values of parameters. We will observe the shorter interval $(0, \pi/2)$, for a more precise estimate in the origin's neighbourhood. The used approach combines the concept of stratification [9] with a method for proving MTP inequalities [8]. This way we can simply prove the known results, and also establish novel ones. Analogously, this procedure can be applied to consider other types of MTP inequalities. In addition, it is possible to apply this approach in solving concrete practical problems such as in [5] and [12].

This paper is organized as follows. The required theoretical background is presented in section 2. In subsection 2.1 are given definitions of stratification and the minimax approximant, as well as Nike theorem in two forms. In subsection 2.2 is explained the used method for proving MTP inequalities. In section 3 are analyzed two inequalities of Frame's type using stratification and MTP method. In subsection 3.1 are given improved results regarding the inequality (1.1). In subsection 3.2 are given improved results regarding the inequality (1.2), obtained analogously to subsection 3.1. Section 4 concludes the paper.

2 Preliminaries

2.1 Stratification and Nike theorem

In this subsection we state relevant concepts and assertions from the paper [9].

The functions $\varphi_p(x)$, where $x \in (a,b) \subseteq \mathbb{R}$ and $p \in \mathbb{D} \subseteq \mathbb{R}^+$, are *increasingly stratified* if $p_1 > p_2 \iff \varphi_{p_1}(x) > \varphi_{p_2}(x)$ holds for each $x \in (a,b)$, and conversely, *decreasingly stratified* if $p_1 > p_2 \iff \varphi_{p_1}(x) < \varphi_{p_2}(x)$ holds for each $x \in (a,b)$ $(p_1, p_2 \in \mathbb{D})$.

Our aim is to determine the maximal subset $I \subseteq \mathbb{D}$ such that, for $p \in I$, we have $\varphi_p(x) > 0$ for each $x \in (a,b)$. Likewise, we want to determine the maximal subset $J \subseteq \mathbb{D}$ such that, for $p \in J$, we have $\varphi_p(x) < 0$ for each $x \in (a,b)$. We will assume that $\mathbb{D} = \mathbb{R}^+$, $I \cup J \subsetneqq \mathbb{D}$, $I \neq \emptyset$ and $J \neq \emptyset$. In that case, it is important to examine the sign of the function $\varphi_p(x)$ in terms of the parameter $p \in \mathbb{D} \setminus (I \cup J)$, for $x \in (a,b)$.

The value $\sup\limits_{x\in(a,b)} |\varphi_p(x)|$ is called *the approximation error* on the interval (a,b) and denoted by

$$d^{(p)} = \sup_{x \in (a,b)} |\varphi_p(x)|, \tag{2.1}$$

for $p \in \mathbb{D}$. Our aim is to determine the unique value of the parameter $p = p_0 \in \mathbb{D}$ for which the infimum of the error $d^{(p)}$ is attained:

$$d_0 = \inf_{p \in \mathbb{R}^+} \sup_{x \in (a,b)} |\varphi_p(x)|. \tag{2.2}$$

For such a value p_0 of the parameter p, the function $\varphi_{p_0}(x)$ is called *the minimax approximant* on (a,b).

If the family $\varphi_p(x)$ allows us to consider $x \in [a,b]$ and $p \in \mathbb{D} = [c,d] \subset \mathbb{R}^+$, then we have

$$d_0 = \min_{p\in[c,d]} \max_{x\in[a,b]} |\varphi_p(x)|.$$

The following assertions are proved in [9].

Theorem 2.1 ([9])**.** *Let $\varphi_p(x)$ be a family of functions that are continuous with respect to $x \in (a,b)$ for each $p \in \mathbb{R}^+$ and increasingly (decreasingly) stratified for $p \in \mathbb{R}^+$, and let $c, d \in \mathbb{R}^+$, where $c < d$. If:*

(a) $\varphi_c(x) < 0$ ($\varphi_c(x) > 0$) and $\varphi_d(x) > 0$ ($\varphi_d(x) < 0$) for each $x \in (a,b)$, and at the endpoints $\varphi_c(a+) = \varphi_d(a+) = 0$, $\varphi_c(b-) = 0$ ($\varphi_d(b-) = 0$) and $\varphi_d(b-) \in \mathbb{R}^+$ ($\varphi_c(b-) \in \mathbb{R}^+$) hold;

(b) the functions $\varphi_p(x)$ are continuous with respect to $p \in (c,d)$ for each $x \in (a,b)$ and $\varphi_p(b-)$ are also continuous with respect to $p \in (c,d)$;

(c) for each $p \in (c,d)$, there is a right neighbourhood of the point a in which $\varphi_p(x) < 0$;

(d) for each $p \in (c,d)$ the function $\varphi_p(x)$ has exactly one extremum at $t^{(p)}$ on (a,b), which is minimum;

then there is exactly one solution p_0, for $p \in \mathbb{R}^+$, to the following equation:

$$|\varphi_p(t^{(p)})| = \varphi_p(b-),$$

and for $d_0 = |\varphi_{p_0}(t^{(p_0)})| = \varphi_{p_0}(b-)$ we have

$$d_0 = \inf_{p\,\in\,\mathbb{R}^+} \sup_{x\,\in\,(a,\,b)} |\varphi_p(x)|.$$

Theorem 2.2 (Nike theorem, [7, 9])**.** *Let $\varphi_p(x) : (a,b) \longrightarrow \mathbb{R}$ be at least m times differentiable function, for some $m \geq 2$, $m \in \mathbb{N}$, which satisfies the following conditions:*

(a) $f^{(m)} > 0$ for $x \in (0,c)$;

(b) there is a right neighbourhood of zero in which the following inequalities hold:

$$f < 0, f' < 0, \ldots, f^{(m-1)} < 0;$$

(c) there is a left neighbourhood of the point c in which the following inequalities hold:

$$f > 0, f' > 0, \ldots, f^{(m-1)} > 0.$$

Then the function f has exactly one root $x_0 \in (0,c)$ and $f(x) < 0$ for $x \in (0,x_0)$ and $f(x) > 0$ for $x \in (x_0,c)$. Additionally, the function f has exactly one local minimum on the interval $(0,c)$. More precisely, there is exactly one point $t \in (0,x_0) \subset (0,c)$ such that $f(t) < 0$ is the smallest value of the function f on the interval $(0,x_0) \subset (0,c)$.

Theorem 2.3 (Nike theorem, II form, [9]). *Let $\varphi_p(x) : (a,b) \longrightarrow \mathbb{R}$ be at least m times differentiable function, for some $m \geq 2$, $m \in \mathbb{N}$, which satisfies the following conditions:*

(a) $f^{(m)}$ has exactly one root x_m on $(0,c)$ such that $f^{(m)} > 0$ on $(0,x_m)$ and $f^{(m)} < 0$ on (x_m,c);

(b) there is a right neighbourhood of zero in which the following inequalities hold:

$$f < 0, f' < 0, \ldots, f^{(m-1)} < 0;$$

(c) there is a left neighbourhood of the point c in which the following inequalities hold:

$$f > 0, f' > 0, \ldots, f^{(m-1)} > 0.$$

Then the function f has exactly one root $x_0 \in (0,c)$ and $f(x) < 0$ for $x \in (0,x_0)$ and $f(x) > 0$ for $x \in (x_0,c)$. Additionally, the function f has exactly one local minimum on the interval $(0,c)$. More precisely, there is exactly one point $t \in (0,x_0) \subset (0,c)$ such that $f(t) < 0$ is the smallest value of the function f on the interval $(0,x_0) \subset (0,c)$.

2.2 A method for proving MTP inequalities

In this subsection we present relevant assertions from the paper [8] for proving inequalities of the form

$$f(x) = \sum_{i=1}^{n} \alpha_i x^{p_i} \cos^{q_i} x \sin^{r_i} x > 0\,, \tag{2.3}$$

where $x \in (\delta_1, \delta_2)$, $\delta_1 \leq 0 \leq \delta_2$ and $\delta_1 < \delta_2$, where $\alpha_i \in \mathbb{R} \setminus \{0\}$, $p_i, q_i, r_i \in \mathbb{N}_0$ and $n \in \mathbb{N}$. The function $f(x)$ we denote as MTP - Mixed Trigonometric Polynomial [4], and the corresponding inequality (2.3) we denote as MTP inequality.

Let the function $f(x)$ be approximated by Taylor polynomial $T_k(x)$ of degree k in the neighbourhood of some point a. If there is $\eta > 0$ such that on the interval $x \in (a-\eta, a+\eta)$, it holds that $T_k(x) \geq f(x)$, then $T_k(x)$ denotes *the upward approximation* of the function $f(x)$ in the neighbourhood of the point a. In this case, the polynomial $T_k(x)$ is denoted by $\overline{T}_k^{f,a}(x)$, or short $\overline{T}_k(x)$. Analogously, if there is $\eta > 0$ such that on the interval $x \in (a-\eta, a+\eta)$, it holds that $T_k(x) \leq f(x)$, then $T_k(x)$ denotes *the downward approximation* of the function $f(x)$ in the neighbourhood of the point a. In this case, the polynomial $T_k(x)$ we also denote by $\underline{T}_k^{f,a}(x)$, or short $\underline{T}_k(x)$.

The following assertions are proved in [8].

Lemma 2.4. *(a) For the polynomial*

$$T_n(t) = \sum_{i=0}^{(n-1)/2} \frac{(-1)^i t^{2i+1}}{(2i+1)!}\,,$$

where $n = 4k+1,\ k \in \mathbb{N}_0$, *it holds:*

$$\overline{T}_n(t) \geq \overline{T}_{n+4}(t) \geq \sin t, \quad \forall t \in \left[0, \sqrt{(n+3)(n+4)}\right]$$

$$\underline{T}_n(t) \leq \underline{T}_{n+4}(t) \leq \sin t, \quad \forall t \in \left[-\sqrt{(n+3)(n+4)}, 0\right].$$

For $t = 0$ *the inequalities turn into equalities. For* $t = \pm\sqrt{(n+3)(n+4)}$ *the equalities* $\overline{T}_n(t) = \overline{T}_{n+4}(t)$ *and* $\underline{T}_n(t) = \underline{T}_{n+4}(t)$ *hold, respectively.*

(b) For the polynomial

$$T_n(t) = \sum_{i=0}^{(n-1)/2} \frac{(-1)^i t^{2i+1}}{(2i+1)!}\,,$$

where $n = 4k+3,\ k \in \mathbb{N}_0$, *it holds:*

$$\underline{T}_n(t) \leq \underline{T}_{n+4}(t) \leq \sin t, \quad \forall t \in \left[0, \sqrt{(n+3)(n+4)}\right],$$

$$\overline{T}_n(t) \geq \overline{T}_{n+4}(t) \geq \sin t, \quad \forall t \in \left[-\sqrt{(n+3)(n+4)}, 0\right].$$

For $t = 0$ *the inequalities turn into equalities. For* $t = \pm\sqrt{(n+3)(n+4)}$ *the equalities* $\underline{T}_n(t) = \underline{T}_{n+4}(t)$ *and* $\overline{T}_n(t) = \overline{T}_{n+4}(t)$ *hold, respectively.*

(c) For the polynomial

$$T_n(t) = \sum_{i=0}^{n/2} \frac{(-1)^i t^{2i}}{(2i)!}\,,$$

where $n = 4k,\ k \in \mathbb{N}_0$, *it holds:*

$$\overline{T}_n(t) \geq \overline{T}_{n+4}(t) \geq \cos t, \quad \forall t \in \left[-\sqrt{(n+3)(n+4)}, \sqrt{(n+3)(n+4)}\right].$$

For $t = 0$ *the inequalities turn into equalities. For* $t = \pm\sqrt{(n+3)(n+4)}$ *the equality* $\overline{T}_n(t) = \overline{T}_{n+4}(t)$ *holds.*

(d) For the polynomial

$$T_n(t) = \sum_{i=0}^{n/2} \frac{(-1)^i t^{2i}}{(2i)!},$$

where $n = 4k+2$, $k \in \mathbb{N}_0$, it holds:

$$\underline{T}_n(t) \le \underline{T}_{n+4}(t) \le \cos t, \quad \forall t \in \left[-\sqrt{(n+3)(n+4)}, \sqrt{(n+3)(n+4)}\right].$$

For $t = 0$ the inequalities turn into equalities. For $t = \pm\sqrt{(n+3)(n+4)}$ the equality $\underline{T}_n(t) = \underline{T}_{n+4}(t)$ holds.

The main idea of the method described in [8] is to, for a given MTP function $f(x)$ defined on $(0, \pi/2)$, find a polynomial $P(x)$ using Lemma 1, such that $f(x) > P(x)$ and $P(x) > 0$ when $x \in (0, \pi/2)$. If such polynomial exists, then $f(x) > 0$ for $x \in (0, \pi/2)$.

For example, all results from the paper [20] can be proved by reduction to the appropriate MTP inequalities with the application of this method.

3 Main results

3.1 Improved results for inequality (1.1)

In this subsection we prove the results regarding the family of functions

$$\varphi_p(x) = x - \frac{3\sin x}{2+\cos x} - px^5, \quad \left(x \in (0, \pi/2) \quad \text{and} \quad p \in \mathbb{R}^+\right),$$

with the aim of improving the results for Frame's inequality (1.1) on the interval $(0, \pi/2)$. The following assertions are true.

Lemma 3.1. *The family of functions*

$$\varphi_p(x) = x - \frac{3\sin x}{2+\cos x} - p\,x^5, \quad \textit{for} \quad x \in (0, \pi/2)$$

is decreasingly stratified with respect to parameter $p \in \mathbb{R}^+$.

Proof. It holds that $\dfrac{\partial \varphi_p(x)}{\partial p} = -x^5 < 0$, for each $x \in (0, \pi/2)$. □

Proposition 3.2. *Let*

$$A = \frac{1}{180} = 0.00\overline{5} \quad \textit{and} \quad B = \frac{16\,(\pi-3)}{\pi^5} = 0.00740306\ldots$$

Then for $x \in (0, \pi/2)$, it holds:

$$\varphi_A(x) > 0 \quad \textit{and} \quad \varphi_B(x) < 0.$$

Proof. Let us write $\varphi_A(x)$ in the form:

$$\varphi_A(x) = x - \frac{3\sin x}{2+\cos x} - \frac{x^5}{180} = \frac{f_A(x)}{180(2+\cos x)},$$

where

$$f_A(x) = -540\sin x + (-x^5 + 180x)\cos x + 2(-x^5 + 180x)$$

is a MTP function defined on $[0, \pi/2]$.

Since $180(2+\cos x) > 0$ for each $x \in (0, \pi/2)$, it is sufficient to prove that $f_A(x) > 0$ for $x \in (0, \pi/2)$. We will use a method given in subsection 2.2.

The following inequalities are true based on Lemma 2.4:

$$\sin t < \overline{T}_5^{\sin,0}(t) \quad \text{for} \quad t \in (0, \sqrt{72}) = (0, 8.485\ldots)$$

and

$$\cos t > \underline{T}_6^{\cos,0}(t) \quad \text{for} \quad t \in (0, \sqrt{90}) = (0, 9.4868\ldots).$$

For each $x \in (0, \pi/2)$ it holds:

$$f_A(x) > P_{11}(x) = \underbrace{-540}_{<\,0}\overline{T}_5^{\sin,0}(x) + \underbrace{(-x^5 + 180x)}_{>\,0}\underline{T}_6^{\cos,0}(x) + \underbrace{2(-x^5 + 180x)}_{>\,0}.$$

The polynomial $P_{11}(x)$ can be written in the following way:

$$P_{11}(x) = \frac{1}{720}x^{11} - \frac{1}{24}x^9 + \frac{1}{4}x^7 = \frac{x^7}{720}\left(x^4 - 30x^2 + 180\right) = \frac{x^7}{720}P_4(x).$$

The first positive root of the biquadratic equation $P_4(x) = 0$ is $x_1 = \sqrt{15 - 3\sqrt{5}} = 2.879\ldots > \pi/2$. Since $P_4(x_1/2) = P_4(1.439) = 122.108 > 0$, it follows that $P_4(x) > 0$ for $x \in (0, \pi/2)$. Furthermore, $f_A(x) > P_{11}(x) > 0$ for $x \in (0, \pi/2)$. Therefore, $\varphi_A(x) > 0$ for each $x \in (0, \pi/2)$.

We prove $\varphi_B(x) < 0$ in a similar way. Let us write $\varphi_B(x)$ in the form:

$$\varphi_B(x) = x - \frac{3\sin x}{2+\cos x} - \frac{16(\pi-3)x^5}{\pi^5} = \frac{f_B(x)}{\pi^5(2+\cos x)}.$$

Since $\pi^5(2+\cos x) > 0$ for each $x \in (0, \pi/2)$, the requested inequality is equivalent to $f_B(x) < 0$ for $x \in (0, \pi/2)$, where

$$f_B(x) = -3\pi^5\sin x + (16(3-\pi)x^5 + \pi^5 x)\cos x + 2(16(3-\pi)x^5 + \pi^5 x)$$

is a MTP function defined on $[0, \pi/2]$. Let us notice that $f_B(0) = f_B(\pi/2) = 0$. For that reason, we consider two cases:

(1) $x \in (0, 1.199)$: We have $16(3-\pi)x^5 + \pi^5 x = x\left(16(3-\pi)x^4 + \pi^5\right) > 0$ on $(0, 1.199)$. The following inequalities are true based on Lemma 2.4:

$$\sin t > \underline{T}_7^{\sin,0}(t) \quad \text{for} \quad t \in (0, \sqrt{110}) = (0, 10.488\ldots)$$

and

$$\cos t < \overline{T}_4^{\cos,0}(t) \quad \text{for} \quad t \in (0, \sqrt{56}) = (0, 7.483\ldots).$$

For each $x \in (0, 1.199)$ it holds:

$$f_B(x) < Q_9(x) = \underbrace{-3\pi^5}_{<\,0}\,\underline{T}_7^{\sin,0}(x) + \underbrace{(16(3-\pi)x^5 + \pi^5 x)}_{>\,0}\,\overline{T}_4^{\cos,0}(x) + \underbrace{2(16(3-\pi)x^5 + \pi^5 x)}_{>\,0}\,.$$

The polynomial $Q_9(x)$ can be written in the following way:

$$\begin{aligned} Q_9(x) &= \frac{x^5}{1680}\Big(-1120(\pi-3)x^4 + (\pi^5 + 13440\pi - 40320)x^2 + 28\pi^5 - 80640\pi + 241920\Big) \\ &= \frac{x^5}{1680} Q_4(x)\,. \end{aligned}$$

The first positive root of the biquadratic equation $Q_4(x) = 0$ is $x_1 = 1.1993\ldots > 1.199$. Since $Q_4(x_1/2) = Q_4(0.599\ldots) = -2075.583\ldots < 0$, it follows that $Q_4(x) < 0$ on $(0, 1.199)$. Furthermore, $f_B(x) < Q_9(x) < 0$. Therefore, $\varphi_B(x) < 0$ for $x \in (0, 1.199)$.

(2) $x \in [1.199, \pi/2)$: Let us define a function

$$g_B(x) = f_B\left(\frac{\pi}{2} - x\right) = -3\pi^5\,\cos x + r(x)\,\sin x + 2\,r(x)\,,$$

where $r(x)$ is the polynomial

$$r(x) = \left(\frac{\pi}{2} - x\right)\left(16(3-\pi)\left(\frac{\pi}{2} - x\right)^4 + \pi^5\right),$$

for $x \in [1.199, \pi/2)$. It is easy to show that $r(x) > 0$ for each $x \in [1.199, \pi/2)$.

Here we prove the inequality $f_B(x) < 0$ for $x \in [1.199, \pi/2)$, which is equivalent to the MTP inequality $g_B(x) < 0$ for $x \in (0, c]$, where $c = \pi/2 - 1.199 = 0.371796\ldots$

The following inequalities are true based on Lemma 2.4:

$$\sin t < \overline{T}_5^{\sin,0}(t) \quad \text{for} \quad t \in (0, \sqrt{72}) = (0, 8.485\ldots)$$

and

$$\cos t > \underline{T}_2^{\cos,0}(t) \quad \text{for} \quad t \in (0, \sqrt{30}) = (0, 5.477\ldots).$$

For each $x \in (0, c]$, it holds:

$$g_B(x) < \underbrace{-3\pi^5}_{<\,0}\underline{T}_2^{\cos,0}(x) + \underbrace{r(x)}_{>\,0}\overline{T}_5^{\sin,0}(x) + \underbrace{2r(x)}_{>\,0} = x\,R(x)\,,$$

where $R(x)$ is the polynomial

$$\begin{aligned} R(x) = & \left(\frac{2\pi}{15} - \frac{2}{5}\right)x^9 + \left(-\frac{\pi^2}{3} + \pi\right)x^8 + \left(\frac{\pi^3}{3} - \pi^2 - \frac{8\pi}{3} + 8\right)x^7 \\ & + \left(-\frac{\pi^4}{6} + \frac{\pi^3}{2} + \frac{20\pi^2}{3} - 20\pi\right)x^6 + \left(\frac{\pi^5}{30} - \frac{\pi^4}{8} - \frac{20\pi^3}{3} + 20\pi^2 + 16\pi - 48\right)x^5 \\ & + \left(\frac{\pi^5}{80} + \frac{10\pi^4}{3} - 10\pi^3 - 40\pi^2 + 152\pi - 96\right)x^4 \\ & + \left(-\frac{2\pi^5}{3} + \frac{5\pi^4}{2} + 40\pi^3 - 200\pi^2 + 240\pi\right)x^3 \\ & + \left(-\frac{\pi^5}{4} - 20\pi^4 + 140\pi^3 - 240\pi^2\right)x^2 + \left(\frac{11\pi^5}{2} - 55\pi^4 + 120\pi^3\right)x + \left(\frac{19\pi^5}{2} - 30\pi^4\right). \end{aligned}$$

It is sufficient to prove that $R(x) < 0$ for $x \in (0, c]$. Let us denote the coefficients of the polynomial $R(x)$ respectively by $a_9, \ldots, a_0$:

$$\begin{aligned} R(x) &= a_9x^9 + a_8x^8 + a_7x^7 + a_6x^6 + a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 \\ &= (a_9x + a_8)x^8 + (a_7x^2 + a_6x + a_5)x^5 + (a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0) \\ &= (a_9x + a_8)x^8 + (a_7x^2 + a_6x + a_5)x^5 + S(x). \end{aligned}$$

It holds:

$$a_9x + a_8 = \left(\frac{2\pi}{15} - \frac{2}{5}\right)x + \left(-\frac{\pi^2}{3} + \pi\right) < 0$$

and

$$\begin{aligned} a_7x^2 + a_6x + a_5 = & -\left(\pi^2 + \frac{8\pi}{3}\right)x^2 - \left(\frac{\pi^4}{6} - \frac{\pi^3}{2} - \frac{20\pi^2}{3} + 20\pi\right)x \\ & + \left(\frac{11\pi^5}{2} - 55\pi^4 + 120\pi^3\right) < 0, \end{aligned}$$

for each $x \in (0, c]$. Let us prove that

$$\begin{aligned} S(x) &= a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 \\ &= \Big(\frac{\pi^5}{80} + \frac{10\pi^4}{3} - 10\pi^3 - 40\pi^2 + 152\pi - 96\Big)x^4 \end{aligned}$$

$$+\Big(-\frac{2\pi^5}{3}+\frac{5\pi^4}{2}+40\pi^3-200\pi^2+240\pi\Big)x^3+\Big(-\frac{\pi^5}{4}-20\pi^4+140\pi^3-240\pi^2\Big)x^2$$
$$+\Big(\frac{11\pi^5}{2}-55\pi^4+120\pi^3\Big)x+\Big(\frac{19\pi^5}{2}-30\pi^4\Big)<0$$

for each $x\in(0,c]$. The third derivative of the polynomial $S(x)$ is

$$S'''(x)=\left(\frac{3}{10}\pi^5+80\pi^4-240\pi^3-960\pi^2+3648\pi-2304\right)x-4\pi^5+15\pi^4$$
$$+240\pi^3-1200\pi^2+1440\pi.$$

It holds that $S'''(x)>0$ for $x\in(0,c]$. Thus, $S''(x)$ is a monotonically increasing function for $x\in(0,c]$. Furthermore, $S''(x)$ is a quadratic function with roots $x_1=-6.034\ldots$ and $x_2=0.279\ldots$ This implies that $S'(x)$ has exactly one extremum on $(0,c]$ which is minimum at the point x_2. Since we have $S'(x_2)=31.480\ldots>0$ at the point of minimum, it follows that $S'(x)>0$ for each $x\in(0,c]$. Thus, the function $S(x)$ is monotonically increasing for each $x\in(0,c]$. Since $S(c)=-1.933\ldots<0$, it follows that $S(x)<0$ for each $x\in(0,c]$.

Therefore:

$$\begin{aligned}R(x)<0\,,\quad\text{for}\quad x\in(0,c]&\Longrightarrow g_B(x)<0\,,\quad\text{for}\quad x\in(0,c]\\&\Longrightarrow f_B(x)<0,\quad\text{for}\quad x\in[1.199,\pi/2)\\&\Longrightarrow \varphi_B(x)<0\,,\quad\text{for}\quad x\in[1.199,\pi/2).\end{aligned}$$

This completes the proof that $\varphi_B(x)<0$ for each $x\in(0,\pi/2)$. □

Proposition 3.3. *Let*

$$A=\frac{1}{180}=0.00\overline{5}\quad\textit{and}\quad B=\frac{16\,(\pi-3)}{\pi^5}=0.00740306\ldots$$

(i) If $p\in(0,A]$, then

$$x\in(0,\pi/2)\implies x-\frac{3\sin x}{2+\cos x}>A\,x^5\ge p\,x^5\,.$$

(ii) If $p\in(A,B)$, then $\varphi_p(x)=x-\dfrac{3\sin x}{2+\cos x}-p\,x^5$ has a unique root $x_0^{(p)}$ on $(0,\pi/2)$. Also,

$$x\in(0,x_0^{(p)})\implies x-\frac{3\sin x}{2+\cos x}<p\,x^5$$

and

$$x\in(x_0^{(p)},\pi/2)\implies x-\frac{3\sin x}{2+\cos x}>p\,x^5\,.$$

Every function $\varphi_p(x)$ has exactly one minimum $t_0^{(p)}\in(0,x_0^{(p)})$, for $p\in(A,B)$.

(iii) *If* $p \in [B, \infty)$, *then*

$$x \in (0, \pi/2) \implies x - \frac{3 \sin x}{2 + \cos x} < B\, x^5 \leq p\, x^5 \,.$$

(iv) *There is exactly one solution to the equation*

$$|\varphi_p(t_0^{(p)})| = \varphi_p\,(\pi/2-)$$

with respect to parameter $p \in (A, B)$, *determined numerically as*

$$p_0 = 0.0072274\ldots$$

For the value

$$d_0 = \varphi_{p_0}\,(\pi/2-) = 0.0016797\ldots$$

it holds:

$$d_0 = \min_{p \in [0,\infty)} \max_{x \in [0,\pi/2]} |\varphi_p(x)|.$$

(v) *For the value* $p_0 = 0.0072274\ldots$ *the minimax approximant of the family* $\varphi_p(x)$ *is*

$$\varphi_{p_0}(x) = x - \frac{3 \sin x}{2 + \cos x} - p_0\, x^5,$$

which determines the appropriate minimax approximation

$$x - \frac{3 \sin x}{2 + \cos x} \approx 0.0072274\, x^5.$$

Proof. It has been shown in Proposition 3.2 that the inequalities $\varphi_A(x) > 0$ and $\varphi_B(x) < 0$ hold for each $x \in (0, \pi/2)$. Since the family of functions $\varphi_p(x)$ is decreasingly stratified, it follows that $\varphi_p(x) \geq \varphi_A(x) > 0$ for $p \in (0, A)$ and $\varphi_p(x) \leq \varphi_B(x) < 0$ for $p \in (B, \infty)$, for each $x \in (0, \pi/2)$. That proves the assertions (i) and (iii).

In order to prove the assertion (ii), we will use the Theorem 2.3 (Nike theorem, II form). Namely, for $p \in (A, B)$, the functions $\varphi_p(x)$ satisfy the conditions of Theorem 2.3:

(a) For $m = 6$, we have

$$\varphi_p^{(vi)}(x) = \frac{d^6 \varphi_p}{d\, x^6} = \frac{6 \sin x}{(2 + \cos x)^7} h(x)\,, \tag{3.1}$$

where $h(x)$ is the following MTP function:

$$h(x) = -(\cos^5 x - 98 \cos^4 x + 886 \cos^3 x - 892 \cos^2 x - 1216 \cos x + 104)\,.$$

Since $\dfrac{6\sin x}{(2+\cos x)^7} > 0$ for each $x \in (0,\pi/2)$, functions $\varphi_p^{(vi)}(x)$ and $h(x)$ have the same roots and sign on $(0,\pi/2)$.

By introducing the substitute $t = \cos x$, we get

$$H(t) = h(\arccos t) = -(t^5 - 98t^4 + 886t^3 - 892t^2 - 1216t + 104)\,.$$

It can be shown by numerical methods that $H(t)$ has a root $t_1 = 0.081088\ldots$ Since $H(t)$ is a polynomial with rational coefficients on the interval with rational endpoints $(0,1)$, using Sturm's algorithm [3, 14], we can conclude that $H(t)$ has exactly one root $t_1 = 0.081088\ldots$ on the interval $(0,1)$. Thus, $h(x)$ also has exactly one root $x_1 = \arccos t_1 = 1.489619\ldots$ on the interval $(0,\pi/2)$.

Let us notice again that $h(x)$ has only one root $x_1 = 1.489619\ldots$ on $(0,\pi/2)$. Since $h(1) = 681.964\ldots > 0$ and $h(1.5) = -13.831\ldots < 0$, it follows that

$$h(x) > 0 \quad \text{on} \quad (0,x_1) \quad \text{and} \quad h(x) < 0 \quad \text{on} \quad (x_1,\pi/2).$$

Considering (3.1), the previous conclusion is equivalent to

$$\varphi_p^{(vi)}(x) > 0 \quad \text{on} \quad (0,x_1) \quad \text{and} \quad \varphi_p^{(vi)}(x) < 0 \quad \text{on} \quad (x_1,\pi/2),$$

which satisfies the first condition of Theorem 2.3.

(b) Taylor approximations of functions $\varphi_p(x)$ around $x = 0$ are:

$$\varphi_p(x) = \left(\frac{1}{180} - p\right)x^5 + \frac{1}{1512}x^7 + O(x^9)\,.$$

Since we consider $p \in (A,B) = \left(\frac{1}{180}, \frac{16(\pi-3)}{\pi^5}\right)$, the coefficient next to x^5 in the approximation is negative, so we conclude that there is a right neighbourhood $\mathcal{U}_0$ of the point 0 such that

$$\varphi_p(x), \varphi_p'(x), \varphi_p''(x), \varphi_p'''(x), \varphi_p^{(iv)}(x), \varphi_p^{(v)}(x) < 0, \quad x \in \mathcal{U}_0.$$

(c) Taylor approximations of functions $\varphi_p(x)$ around $x = \dfrac{\pi}{2}$ are:

$$\begin{aligned}
\varphi_p(x) \quad = \quad & \left(-\frac{\pi^5 p}{32} + \frac{\pi-3}{2}\right) + \left(-\frac{5\pi^4 p}{16} + \frac{1}{4}\right)\left(x - \frac{\pi}{2}\right) + \left(-\frac{5\pi^3 p}{4} + \frac{3}{8}\right)\left(x - \frac{\pi}{2}\right)^2 \\
& + \left(-\frac{5\pi^2 p}{2} + \frac{5}{16}\right)\left(x - \frac{\pi}{2}\right)^3 + \left(-\frac{5\pi p}{2} + \frac{5}{32}\right)\left(x - \frac{\pi}{2}\right)^4 \\
& + \left(-p + \frac{13}{320}\right)\left(x - \frac{\pi}{2}\right)^5 - \frac{13}{1920}\left(x - \frac{\pi}{2}\right)^6 + O\left(\left(x - \frac{\pi}{2}\right)^7\right)\,.
\end{aligned}$$

Since we consider $p \in (A, B)$, it is easy to show that in the approximation all coefficients next to $\left(x - \frac{\pi}{2}\right)^n$, $0 \leq n \leq 5$, are positive, so we conclude that there is a left neighbourhood $\mathcal{U}_{\pi/2}$ of the point $\frac{\pi}{2}$ such that

$$\varphi_p(x), \varphi'_p(x), \varphi''_p(x), \varphi'''_p(x), \varphi^{(iv)}_p(x), \varphi^{(v)}_p(x) > 0, \quad x \in \mathcal{U}_{\pi/2}.$$

Since the conditions of Theorem 2.3 are satisfied, the function $\varphi_p(x)$ has exactly one extremum $t^{(p)}$, which is minimum, on $(0, \pi/2)$ $\big($and one root $x_0^{(p)}$ on $(0, \pi/2)\big)$, and it holds that $\varphi_p(x) < 0$ for $x \in (0, x_0^{(p)})$ and $\varphi_p(x) > 0$ for $x \in (x_0^{(p)}, \pi/2)$. That proves the assertion (ii).

(iv), (v): The family of functions $\varphi_p(x)$, for values $p \in (A, B)$, satisfies the conditions of Theorem 2.1, which means that the minimax approximant exists. The minimax approximant and its error (infimum of the approximation error) can be numerically determined using Maple software. Let $f(x, p) := \varphi_p(x)$. Based on Maple code

```
fsolve({diff(f(x,p),x)=0,abs(f(x,p)=f(Pi/2,p)},{x=0..Pi/2,p=A..B});
```

we get numerical values

$$\{p = 0.007227413, x = 1.272430755\}.$$

For the value $p_0 = 0.0072274\ldots$ we obtain the minimax approximant of the family

$$\varphi_{p_0}(x) = x - \frac{3 \sin x}{2 + \cos x} - p_0\, x^5$$

and numerical value of the minimax error

$$d_0 = f(\pi/2, p_0) = 0.0016797\ldots.$$

This completes the proof. $\square$

The following statement holds based on previous conclusions.

Proposition 3.4. *For each $0 < x < \pi/2$, it holds:*

$$\frac{1}{180}\, x^5 < x - \frac{3 \sin x}{2 + \cos x} < \frac{16\,(\pi - 3)}{\pi^5}\, x^5\,, \tag{3.2}$$

where the constants $A = \dfrac{1}{180} = 0.00\overline{5}$ and $B = \dfrac{16\,(\pi - 3)}{\pi^5} = 0.00740306\ldots$ are the best possible.

3.2 Improved results for inequality (1.2)

In this subsection we present the appropriate results for the family of functions

$$\varphi_p(x) = x - \frac{3\sin x}{2+\cos x}\left(1 + \frac{(1-\cos x)^2}{9(3+2\cos x)}\right) - p\,x^7, \quad x \in (0,\pi/2) \quad \text{and} \quad p \in \mathbb{R}^+,$$

with the aim of improving the results for the Frame's inequality (1.2) on the interval $(0,\pi/2)$. The following statements are proved analogously to statements from the previous subsection.

Lemma 3.5. *The family of functions:*

$$\varphi_p(x) = x - \frac{3\sin x}{2+\cos x}\left(1 + \frac{(1-\cos x)^2}{9(3+2\cos x)}\right) - p\,x^7, \quad \text{for} \quad x \in (0,\pi/2)$$

is decreasingly stratified with respect to parameter $p \in \mathbb{R}^+$.

Proposition 3.6. *Let:*

$$A = \frac{1}{2100} = 0.000\overline{476190} \quad \text{and} \quad B = \frac{64(9\pi-28)}{9\pi^7} = 0.0006459\ldots$$

Then for $x \in (0,\pi/2)$, it holds:

$$\varphi_A(x) > 0 \quad \text{and} \quad \varphi_B(x) < 0\,.$$

Proposition 3.7. *Let:*

$$A = \frac{1}{2100} = 0.000\overline{476190} \quad \text{and} \quad B = \frac{64(9\pi-28)}{9\pi^7} = 0.0006459\ldots$$

(i) If $p \in (0,A]$, then

$$x \in (0,\pi/2) \implies x - \frac{3\sin x}{2+\cos x}\left(1 + \frac{(1-\cos x)^2}{9(3+2\cos x)}\right) > A\,x^7 \geq p\,x^7\,.$$

(ii) If $p \in (A,B)$, then $\varphi_p(x) = x - \frac{3\sin x}{2+\cos x}\left(1 + \frac{(1-\cos x)^2}{9(3+2\cos x)}\right) - p\,x^7$ has a unique root $x_0^{(p)}$ on $(0,\pi/2)$. Also:

$$x \in (0,x_0^{(p)}) \implies x - \frac{3\sin x}{2+\cos x}\left(1 + \frac{(1-\cos x)^2}{9(3+2\cos x)}\right) < p\,x^7$$

and

$$x \in (x_0^{(p)},\pi/2) \implies x - \frac{3\sin x}{2+\cos x}\left(1 + \frac{(1-\cos x)^2}{9(3+2\cos x)}\right) > p\,x^7\,.$$

Every function $\varphi_p(x)$ has exactly one minimum $t_0^{(p)} \in (0,x_0^{(p)})$, for $p \in (A,B)$.

(iii) If $p \in [B, \infty)$, then:

$$x \in (0, \pi/2) \implies x - \frac{3\sin x}{2+\cos x}\left(1 + \frac{(1-\cos x)^2}{9(3+2\cos x)}\right) < B\, x^7 \leq p\, x^7\,.$$

(iv) There is exactly one solution to the equation:

$$|\varphi_p(t_0^{(p)})| = \varphi_p\,(\pi/2-)$$

with respect to parameter $p \in (A, B)$, determined numerically as:

$$p_0 = 0.000632762\ldots$$

For the value:

$$d_0 = \varphi_{p_0}\,(\pi/2-) = 0.000310091\ldots$$

it holds:

$$d_0 = \min_{p\in[0,\infty)}\ \max_{x\in[0,\pi/2]} |\varphi_p(x)|.$$

(v) For the value $p_0 = 0.000632762\ldots$ the minimax approximant of the family $\varphi_p(x)$ is:

$$\varphi_{p_0}(x) = x - \frac{3\sin x}{2+\cos x}\left(1 + \frac{(1-\cos x)^2}{9(3+2\cos x)}\right) - p_0\, x^7,$$

which determines the appropriate minimax approximation:

$$x - \frac{3\sin x}{2+\cos x}\left(1 + \frac{(1-\cos x)^2}{9(3+2\cos x)}\right) \approx 0.000632762x^7.$$

The following statement holds based on previous conclusions.

Proposition 3.8. *For each $0 < x < \pi/2$, it holds:*

$$\frac{1}{2100}\, x^7 < x - \frac{3\sin x}{2+\cos x}\left(1 + \frac{(1-\cos x)^2}{9(3+2\cos x)}\right) < \frac{64(9\pi-28)}{9\pi^7}\, x^7, \tag{3.3}$$

where the constants $A = \dfrac{1}{2100} = 0.000\overline{476190}$ and $B = \dfrac{64(9\pi-28)}{9\pi^7} = 0.0006459\ldots$ are the best possible.

4 Conclusion

Inequalities that we study in this paper are mainly used to estimate the precision of the Cusa-Huygens approximation. The Cusa-Huygens inequality and the estimate of the quality of approximation may be relevant to concrete applications such as [5, 12], see also the monograph [2]. The known results related to Frame's inequalities are obtained for special cases of parameters only. In this paper, we achieve the previous results based on the concept of stratification, and also expand the conclusions for unconsidered values of parameters. In analogy with this approach over families of stratified functions, it is possible to examine other types of inequalities and get new results in the Theory of Analytic Inequalities.

It should be noted that one part of the given method is limited to MTP inequalities (subsection 2.2). The aim of future research is to consider other classes of inequalities in a similar way, by combining different methods with the concept of stratification. In that regard, we refer the reader to papers [6, 15–18] for understanding the latest progress in the field.

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