Double asymptotic inequalities for the generalized Wallis ratio

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ABSTRACT

Asymptotic estimates for the generalized Wallis ratio

\[ W^*(x) := \frac{\Gamma\left(x + \frac{1}{2}\right)}{\sqrt{\pi} \cdot \Gamma\left(x + 1\right)} \]

are presented for \( x \in \mathbb{R}^+ \) on the basis of Stirling’s approximation formula for the \( \Gamma \) function. For example, for an integer \( p \geq 2 \) and a real \( x > -\frac{1}{2} \) we have the following double asymptotic inequality

\[ A(p, x) < W^*(x) < B(p, x) \]

where

\[ A(p, x) := W_p(x) \left( 1 - \frac{1}{8(x+p)} + \frac{1}{128(x+p)^2} + \frac{1}{256(x+p)^3} \right), \]

\[ B(p, x) := W_p(x) \left( 1 - \frac{1}{8(x+p)} + \frac{1}{128(x+p)^2} + \frac{1}{192(x+p)^3} \right), \]

\[ W_p(x) := \frac{1}{\sqrt{\pi(x+p)}} \cdot \frac{(x+1)^{(p)}}{(x+\frac{1}{2})^{(p)}} \]

with \( y^{(p)} \equiv y(y+1) \cdots (y+p-1) \), the Pochhammer rising (upper) factorial of order \( p \).

RESUMEN

Se presentan estimaciones asímptóticas para la razón generalizada de Wallis

\[ W^*(x) := \frac{\Gamma\left(x + \frac{1}{2}\right)}{\sqrt{\pi} \cdot \Gamma\left(x + 1\right)} \]

para \( x \in \mathbb{R}^+ \) sobre la base de la fórmula de aproximación de Stirling para la función \( \Gamma \). Por ejemplo, para un entero \( p \geq 2 \) y un real \( x > -\frac{1}{2} \), tenemos la siguiente desigualdad doble asímptótica

\[ A(p, x) < W^*(x) < B(p, x) \]

donde

\[ A(p, x) := W_p(x) \left( 1 - \frac{1}{8(x+p)} + \frac{1}{128(x+p)^2} + \frac{1}{256(x+p)^3} \right), \]

\[ B(p, x) := W_p(x) \left( 1 - \frac{1}{8(x+p)} + \frac{1}{128(x+p)^2} + \frac{1}{192(x+p)^3} \right), \]

\[ W_p(x) := \frac{1}{\sqrt{\pi(x+p)}} \cdot \frac{(x+1)^{(p)}}{(x+\frac{1}{2})^{(p)}} \]

con \( y^{(p)} \equiv y(y+1) \cdots (y+p-1) \), el factorial ascendiente de Pochhammer (superior) de orden \( p \).

Keywords and Phrases: Approximation, asymptotic, estimate, generalized Wallis’ ratio, double inequality.

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1 Introduction

In pure and applied mathematics, e.g. in number theory, probability, combinatorics, statistics, and also in several exact sciences as, for example in statistical physics and quantum mechanics, we often encounter the Wallis ratios \( w_n \),

\[
\begin{align*}
    w_n := & \frac{1 \cdot 3 \cdots (2n-1)}{2 \cdot 4 \cdots (2n)} = 4^{-n} \frac{(2n)!}{(n!)^2} = 4^{-n} \binom{2n}{n} \\
    = & \frac{2^n \prod_{k=1}^{n} (k - \frac{1}{2})}{2^n \cdot n!} = \frac{\Gamma(n + \frac{1}{2})}{\sqrt{\pi} \Gamma(n+1)} = \frac{\Gamma(n + \frac{1}{2})}{\sqrt{\pi} \Gamma(n+1)} \quad (n \in \mathbb{N}).
\end{align*}
\]

The sequence \( n \mapsto W_n := \frac{1}{2n+1} \left( \prod_{k=1}^{n} \frac{2k}{2k-1} \right)^2 \), called the Wallis sequence, is closely connected to the sequence of the Wallis ratios \( w_n \) by the identity \( W_n = w_n^{-2}/(2n+1) \). The Wallis sequence was intensively studied by several mathematicians, see e.g. [9–11,14,19].

According to (1.1), the continuous version \( W^*(x) \) of the Wallis ratio is defined as

\[
W^*(x) := \frac{1}{\sqrt{\pi}} \cdot \frac{\Gamma(x + \frac{1}{2})}{\Gamma(x+1)} \quad (x > -\frac{1}{2}).
\]

Thus, we have \( W^*(0) = 1 \) and, referring to [11,19], we have also

\[
W^*(x) = \frac{2}{\pi} \cdot H(2x),
\]

where \( H(x) \) is the “Wallis-cos-sin” function, defined as

\[
H(x) := \int_{0}^{\pi/2} (\cos t)^x \, dt = \int_{0}^{\pi/2} (\sin t)^x \, dt \quad (x \geq -1).
\]

Here, for \( x > -1 \), we have the derivatives

\[
H'(x) = \int_{0}^{\pi/2} (\ln \cos t) (\cos t)^x \, dt < 0, \quad H''(x) = \int_{0}^{\pi/2} (\ln \cos t)^2 (\cos t)^x \, dt > 0.
\]

Consequently, using (1.3), we conclude that \( W^*(x) \) is strictly decreasing and convex on the open interval \((-\frac{1}{2}, \infty)\).

Referring to (1.2), we have

\[
W^*(x) = \frac{1}{\sqrt{\pi}} \cdot Q_{\Gamma}(x, \frac{1}{2}, 1) \quad (x > -\frac{1}{2}),
\]

where the ratio \( Q_{\Gamma}(x, a, b) \) is defined as

\[
Q_{\Gamma}(x, a, b) := \frac{\Gamma(x + a)}{\Gamma(x + b)}, \quad \text{for } x > -\max\{a, b\}.
\]
The ratio $Q_{\Gamma}(x, a, b)$ was studied by many researchers, see e.g. the papers [2, 3, 5–7, 12, 13, 15–18, 20–27, 29, 30, 32]. Just recently several accurate estimates of $Q_{\Gamma}(x, a, b)$ were presented in [16], as for example in the following proposition.

**Proposition 1** ([16, Theorem 1]). For $a, b \in [0, 1]$, $r \in \mathbb{N} \cup \{0\}$ and $x \in \mathbb{R}^+$ we have

$$Q_{\Gamma}(x, a, b) = \left(1 + \frac{a}{x}\right)^x \left(1 + \frac{b}{x}\right)^{-x} \left(\frac{x+a}{x+b}\right)^{a-1/2} \exp(b-a)$$

$$\cdot \exp\left(\sum_{i=1}^{r} \frac{B_{2i}}{2i(2i-1)} \left((x+a)^{1-2i} - (x+b)^{1-2i}\right) + \delta_r(x, a, b)\right),$$

where

$$|\delta_r(x, a, b)| < \Delta_r(x, a, b) := \frac{|B_{2r+2}|}{(2r+1)(2r+2)(x+\min\{a, b\})^{2r+1}} \quad (1.8)$$

and the symbol $B_k$ denotes the $k$-th Bernoulli coefficient [1, 23.1.2].

Thus, for $a = \frac{1}{2}$ and $b = 1$, the Proposition produces the formula

$$W^*(x) = \frac{1}{\sqrt{\pi(x+1)}} \cdot \left(1 + \frac{1}{2x}\right)^x \left(1 + \frac{1}{x}\right)^{-x} \sqrt{e}$$

$$\cdot \exp\left(\sum_{i=1}^{r} \frac{B_{2i}}{2i(2i-1)} \left((x+1/2)^{1-2i} - (x+1)^{1-2i}\right)\right) \cdot \exp\left(\delta_r(x, \frac{1}{2}, 1)\right), \quad (1.9)$$

where

$$|\delta_r(x, \frac{1}{2}, 1)| < \frac{|B_{2r+2}|}{(2r+1)(2r+2)(x+\frac{1}{2})^{2r+1}}, \quad (1.10)$$

for integers $r \geq 0$ and $x > 0$ with $r$ being a parameter that affects the magnitude of the error term $\delta_r(x, 1/2, 1)$.

In this paper we will introduce a formula that is more compact than that given by (1.9)–(1.10). Our results are close to some formulas given in [4] and [31], where the main role is played by complete monotonicity of suitable functions. Unfortunately, using these articles, our results cannot be achieved easily/quickly. In this paper, we offer a simple and fast derivation using the Stirling approximation formula for the gamma function.

**Remark 1.1.** In 2011, the Wallis quotient function $W(x, s, t) := \frac{\Gamma(x+t)}{\Gamma(x+s)}$ was introduced$^3$ in [2]. In this paper and also in the subsequent articles [3, 7], the authors investigate the qualitative profile of $W(x, s, t)$ using asymptotic expansions. Quantitative estimates were mostly not given there. However, for us, the quantitative estimates are essential.

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$^1$Instead of the symbol $Q_{\Gamma}$ there was used in [2, 3, 7] the letter $W$: $Q_{\Gamma}(x, a, b) = W(x, a, b)$.

$^2$Consider that $\sum_{i=1}^{n} x_i = 0$, by definition.

$^3$Clearly, $W^*(x) = W(x, 1, \frac{1}{2})$. 
2 Background

Using the definition (1.2) and the equality $\Gamma(y + 1) = y\Gamma(y)$, valid for $y \in \mathbb{R}^+$, by induction we note the identity

\[ W^*(x) = \frac{(x + 1)^{(p)}}{(x + \frac{1}{2})^{(p)}} W^*(x + p), \]  

(2.1)

valid for an integer $p \geq 0$ and real $x > -\frac{1}{2}$, where $y^{(p)}$ denotes the Pochhammer rising (upper) factorial, defined as

\[ y^{(0)} := 1, \quad y^{(p)} := \prod_{i=0}^{p-1} (y + i) = y(y + 1) \cdots (y + p - 1) \quad (for\ p \geq 1). \]

Using the duplication formula [1, 6.1.18], we have, for $x > 0$,

\[ 2x\Gamma(2x) = 2x \cdot (2\pi)^{-1/2} 2^{2x-1/2} \Gamma(x)\Gamma(x + \frac{1}{2}) = \pi^{-1/2} 2^{2x} \Gamma(x + 1)\Gamma(x + \frac{1}{2}). \]

Hence, using (1.2), we obtain, for $x > 0$,

\[ W^*(x) = \frac{1}{\sqrt{\pi}} \frac{\Gamma(x + \frac{1}{2})}{\Gamma(x + 1)} = 2^{-2x} \frac{2x\Gamma(2x)}{(\Gamma(x + 1))^2} = 2^{-2x} \frac{2x\Gamma(2x)}{(x\Gamma(x))^2}. \]  

(2.2)

The continuous version of Stirling’s factorial formula of order $r \geq 0$, for $x \in \mathbb{R}^+$, can be given in the following way [8, Sect. 9.5]

\[ x\Gamma(x) = \left(\frac{x}{e}\right)^x \sqrt{2\pi x} \cdot \exp \left( s_r(x) + d_r(x) \right), \]  

(2.3)

where

\[ s_0(x) \equiv 0 \quad \text{and} \quad s_r(x) = \sum_{i=1}^{r} \frac{c_i}{x^{2i-1}} \quad \text{for} \ r \geq 1, \]  

(2.4)

\[ c_i = \frac{B_{2i}}{2i(2i - 1)} \quad \text{for} \ i \geq 1, \]  

(2.5)

and, for some $\vartheta_r(x) \in (0, 1)$,

\[ d_r(x) = \vartheta_r(x) \cdot \frac{c_{r+1}}{x^{2r+1}}. \]  

(2.6)

Here $B_2, B_4, B_6, \ldots$ are the Bernoulli coefficients, alternating in sign as

\[ B_{2i} = (-1)^{i+1}|B_{2i}| \quad \text{for} \ i \geq 1, \]  

(2.7)

thanks to [1, 23.1.15, p. 805]. For example, using Mathematica [28],

\[ B_2 = \frac{1}{6}, \quad B_4 = B_8 = \frac{-1}{30}, \quad B_6 = \frac{1}{42}, \quad B_{10} = \frac{5}{66}, \quad B_{12} = \frac{-691}{2730}, \quad B_{14} = \frac{7}{6}, \]
with the estimates \(|B_{12}| < \frac{1}{7}, |B_{16}| < 7, B_{18} < 55, |B_{20}| < 530, B_{22} < 6200.\)

3 Result

According to (2.2) and (2.3), we calculate, for \(x > 0,\)

\[
W^*(x) = 2^{-2x} \frac{2x \Gamma(2x)}{x \left(\Gamma(x)\right)^2} \\
= 2^{-2x} \left(\frac{2x}{e}\right)^{2x} \sqrt{2\pi \cdot 2x} \cdot \exp \left(s_r(2x) + d_r(2x)\right) \cdot \left[\left(\frac{e}{x}\right)^x \frac{1}{\sqrt{2\pi x}} \cdot \exp \left(-s_r(x) - d_r(x)\right)\right]^2 \\
= \frac{1}{\sqrt{\pi x}} \exp \left(s_r(2x) - 2s_r(x) + d_r(2x) - 2d_r(x)\right). \tag{3.1}
\]

Referring to (3.1) and (2.3)–(2.6), we derive the following lemma.

**Lemma 3.1.** For any \(r \in \mathbb{N} \cup \{0\}\) and \(x \in \mathbb{R}^+\) we have\(^4\)

\[
W^*(x) = \frac{1}{\sqrt{\pi x}} \cdot \exp \left(- \sum_{i=1}^{r} \frac{(1 - 4^{-i})B_{2i}}{i(2i - 1)x^{2i-1}}\right) \cdot \exp \left(\delta_r(x)\right), \tag{3.2}
\]

where

\[
|\delta_r(x)| < \frac{|B_{2r+2}|}{(r + 1)(2r + 1)x^{2r+1}}. \tag{3.3}
\]

**Proof.** According to (2.4)–(2.5), we have

\[
s_r(2x) - 2s_r(x) = \sum_{i=1}^{r} \frac{c_i}{(2x)^{2i-1}} - 2 \sum_{i=1}^{r} \frac{c_i}{x^{2i-1}} = - \sum_{i=1}^{r} \frac{c_i}{x^{2i-1}} \left(2 - 2 \cdot 4^{-i}\right) = - \sum_{i=1}^{r} \frac{(1 - 4^{-i})B_{2i}}{i(2i - 1)x^{2i-1}}.
\]

Similarly, referring to (2.5)–(2.6), we have the error

\[
\delta_r(x) := d_r(2x) - 2d_r(x) = \vartheta_r^*(x) \cdot \frac{c_{r+1}}{(2x)^{2r+1}} - 2\vartheta_r(x) \cdot \frac{c_{r+1}}{x^{2r+1}} = \frac{c_{r+1}}{x^{2r+1}} \left(\frac{\vartheta_r^*(x)}{2^{2r+1}} - 2\vartheta_r(x)\right),
\]

for some \(\vartheta_r(x), \vartheta_r^*(x) \in (0, 1).\) Thus, using (2.5), we get, for \(x > 0,\)

\[
|\delta_r(x)| < \frac{|B_{2r+2}|}{(2r + 2)(2r + 1)x^{2r+1}} \cdot 2. \tag*{Q.E.D.}
\]

**Remark 3.2.** The formula for \(W^*(x),\) given in (3.2)–(3.3), is more compact, but slightly less accurate, than the formula, given in (1.9)–(1.10), where \(x = 0\) is a regular point as opposed to (3.2)–(3.3), where this point is seemingly singular.

\(^4\)Consider that \(\sum_{i=1}^{0} x_i = 0,\) by definition.
Thanks to (3.3), the absolute value of $\delta_r(x)$ is small for large $x$ and any $r \geq 0$. But, for small $x > 0$, the formula in Lemma 3.1 becomes useless. This problem can be avoided by replacing $x$ in Lemma 3.1 by $x + p$, for $p$ large, $p \in \mathbb{N}$. In fact, using (2.1) and replacing $x$ by $x' = x + p$ in Lemma 3.1, immediately follows the next theorem, with $\delta_{p,r}(x) = \delta_r(x + p, a, b)$.

**Theorem 3.3.** For integers $p, r \geq 1$ and for $x > -\frac{1}{2}$, the ratio $W^*(x)$ can be expressed in the form

$$W^*(x) = W^*_{p,r}(x) \cdot \exp \left( \frac{\delta_{p,r}^*(x)}{1 + \frac{(1 - 4^{-i})B_{2i}}{i(2i - 1)(x + p)^{2i - 1}}} \right),$$

(3.4)

where

$$W^*_{p,r}(x) := \frac{1}{\sqrt{\pi (x + p)}} \cdot \frac{(x + 1)^{(p)}}{(x + \frac{1}{2})^{(p)}} \cdot \exp \left( -\frac{1}{8(x + 3)} + \frac{1}{192(x + 3)^3} \right) - \frac{1}{640(x + 3)^5} + \frac{17}{14336(x + 3)^7} - \frac{31}{18432(x + 3)^9} \cdot \exp \left( \delta_{p,r}^*(x) \right),$$

(3.5)

and

$$|\delta_{p,r}^*(x)| < \frac{|B_{2r+2}|}{(r + 1)(2r + 1)(x + p)^{2r+1}}.$$  

(3.6)

Here, $p$ and $r$ are parameters that affect the magnitude of the error term $\delta_{p,r}^*(x)$.

**Example 3.4.** Setting $p = 3$ and $r = 5$ in Theorem 3.3, we obtain

$$W^*(x) := \frac{(x + 1)(x + 2)}{(x + \frac{1}{2})(x + \frac{3}{2})} \cdot \frac{\sqrt{x + 3}}{\pi} \cdot \exp \left( -\frac{1}{8(x + 3)} + \frac{1}{192(x + 3)^3} \right) - \frac{1}{640(x + 3)^5} + \frac{17}{14336(x + 3)^7} - \frac{31}{18432(x + 3)^9} \cdot \exp \left( \delta_{3,5}^*(x) \right),$$

where $|\delta_{3,5}^*(x)| < \frac{1}{260(x + 3)^7}$, for all $x > -\frac{1}{2}$. Consequently, $|\delta_{3,5}^*(x)| < 2 \cdot 10^{-7}$ for $x \in (-\frac{1}{2}, 0]$, $|\delta_{3,5}^*(x)| < 3 \cdot 10^{-8}$, for $x \in [0, 1]$, and $|\delta_{3,5}^*(x)| < 10^{-9}$, for $x \geq 1$.

A direct, immediate consequence of Theorem 3.3 is the sequence of asymptotic expansions given in the following corollary.

**Corollary 3.5.** For any integer $p \geq 1$ we have the asymptotic expansion

$$\ln \left( W^*(x) \right) \sim \ln \left( \frac{(x + 1)^{(p)}}{(x + \frac{1}{2})^{(p)}} \right) - \frac{1}{2} \ln \left( \pi(x + p) \right) - \sum_{i=1}^{\infty} \frac{(1 - 4^{-i})B_{2i}}{i(2i - 1)(x + p)^{2i-1}},$$

as $x \to \infty$.

**Theorem 3.6.** For an integer $p \geq 2$ and real $x > -\frac{1}{2}$ there holds the following double asymptotic inequality

$$A(p, x) < W^*(x) < B(p, x),$$

(3.7)
where

\[ A(p, x) := W_p^*(x) \left( 1 - \frac{1}{8(x + p)} + \frac{1}{128(x + p)^2} + \frac{1}{379(x + p)^3} \right), \]  

(3.8)

\[ B(p, x) := W_p^*(x) \left( 1 - \frac{1}{8(x + p)} + \frac{1}{128(x + p)^2} + \frac{1}{191(x + p)^3} \right), \]

(3.9)

\[ W_p^*(x) := W_{p,0}^*(x) = \frac{1}{\sqrt[2]{\pi(x + p)}} \cdot \frac{(x + 1)^{p}}{(x + \frac{1}{2})^{p}}. \]

(3.10)

Proof. We use Theorem 3.3 with \( r = 2 \), when \( |\delta_{p,2}^*(x)| < \frac{1}{630(x+p)^5} \) and thus we estimate

\[ y_-(p, x) < -\sum_{i=1}^{2} \frac{(1 - 4^{-i})B_{2i}}{i(2i - 1)(x + p)^{2i-1}} + \delta_{p,2}^*(x) < y_+(p, x) < 0, \]

(3.11)

for \( p \in \mathbb{N} \) and \( x \in \mathbb{R}^+ \), where

\[ y_-(p, x) := -\frac{1}{8(x + p)} + \frac{1}{192(x + p)^3} - \frac{1}{630(x + p)^5}, \]

(3.12)

\[ y_+(p, x) := -\frac{1}{8(x + p)} + \frac{1}{192(x + p)^3} + \frac{1}{630(x + p)^5}. \]

(3.13)

Furthermore, by Taylor’s formula of orders 3 and 2 we have, for \( y < 0 \),

\[ 1 + y + \frac{y^2}{2} + \frac{y^3}{6} < e^y < 1 + y + \frac{y^2}{2}. \]

Thus, referring to (3.11)–(3.13), we have, for \( p \in \mathbb{N} \),

\[ \exp (y_-(p, x)) > 1 + y_-(p, x) + \frac{1}{2} y_-(p, x) + \frac{1}{6} y_-(p, x)^3, \]

(3.14)

\[ \exp (y_+(p, x)) < 1 + y_+(p, x) + \frac{1}{2} y_+(p, x). \]

(3.15)

Now, due to (3.12), we estimate, for \( x > -\frac{1}{2} \) and \( x + p > -\frac{1}{2} + 2 > 1 \), as follows:

\[ 1 + y_-(p, x) + \frac{1}{2} y_-(p, x) + \frac{1}{6} y_-(p, x)^3 \]

\[ = 1 - \frac{1}{8(x + p)} + \frac{1}{128(x + p)^2} + \frac{5}{1024(x + p)^4} - \frac{1}{24576(x + p)^6} + \frac{1}{219984(x + p)^8} \]

\[ + \frac{1}{630(x + p)^5} > 1 - \frac{1}{8(x + p)} + \frac{1}{128(x + p)^2} + \frac{5}{1024(x + p)^4} - \frac{1}{24576(x + p)^6} - \frac{1}{589824(x + p)^8} \]

\[ - \frac{1}{630(x + p)^5} > 1 - \frac{1}{8(x + p)} + \frac{1}{128(x + p)^2} + \frac{1}{379(x + p)^4}, \]

(3.16)
and

\[ 1 + y_+(p, x) + \frac{1}{2} y_+^2(p, x) = 1 - \frac{1}{8(x+p)} + \frac{1}{128(x+p)^2} + \frac{1}{192(x+p)^3} - \frac{1}{1536(x+p)^4} + \frac{1}{73728(x+p)^5} \]

\[ < 1 - \frac{1}{8(x+p)} + \frac{1}{128(x+p)^2} + \frac{1}{192(x+p)^3} \cdot \quad (3.17) \]

Using Theorem 3.3, (3.11), (3.14)–(3.15) and (3.16)–(3.17) we note the double inequality (3.7). \( \Box \)

**Example 3.7.** We have

\[ A(2, -\frac{49}{100}) = 32.25 \ldots < W(-\frac{49}{100}) = 32.27 \ldots < B(2, -\frac{49}{100}) = 32.28 \ldots \]

However, \( A(1, -\frac{49}{100}) = 32.42 \ldots > W(-\frac{49}{100}) = 32.27 \ldots \)

**Example 3.8.** We have

\[ B(2, 49\frac{1}{100}) - A(2, 49\frac{1}{100}) < 3 \cdot 10^{-2}, B(2, 0) - A(2, 0) < 4 \cdot 10^{-4} \text{ and } B(2, \pi) - A(2, \pi) < 6 \cdot 10^{-6} \]

**Example 3.9.** We have exactly

\[ W(3) = w_3 = \frac{5}{16} = 0.3125 \]

and, thanks to Theorem 3.6, we estimate \( 0.3124996 < A(9, 3) < W(3) < B(9, 3) < 0.3125001 \).

Figure 1 illustrates the estimate (3.7) by plotting\(^5\) the graphs of the functions \( x \mapsto A(2, x) \), \( x \mapsto W(x) \) and \( x \mapsto B(2, x) \), where all graphs practically coincide.

![Figure 1: The graphs of the functions](image)

**Corollary 3.10.** For an integer \( p \geq 2 \) and \( x > -\frac{1}{2} \) the approximation \( W^*(x) \approx A(p, x) \) has the relative error

\[ \rho(p, x) := \frac{W^*(x) - A(p, x)}{W^*(x)} \]

estimated as

\[ 0 < \rho(p, x) < \frac{B(p, x) - A(p, x)}{A(p, x)} < \frac{1}{330(x+p)^3} \cdot \]

**Proof.** Thanks to Theorem 3.6 we have

\[ 0 < \rho(p, x) < \frac{B(p, x) - A(p, x)}{A(p, x)} = \frac{B(p, x)}{A(p, x)} - 1 = \frac{S + \Delta_2}{S + \Delta_1} - 1, \]

\(^5\)All figures and more demanding computations made in this paper were produced using Mathematica [28].
where

\[ S = 1 - \frac{1}{8(x + p)} + \frac{1}{128(x + p)^2} \]  

(3.18)

and

\[ \Delta_1 = \frac{1}{379}(x + p)^{-3}, \quad \Delta_2 = \frac{1}{191}(x + p)^{-3}. \]  

(3.19)

Thus,

\[ 0 < \rho(p, x) < \left(1 + \frac{\Delta_2 - \Delta_1}{S + \Delta_1}\right) - 1 < \frac{\Delta_2 - \Delta_1}{S}, \]

where the assumptions \( x > -\frac{1}{2} \) and \( p \geq 2 \) imply the estimate \( x + p > 1 \), which, due to (3.18), implies the inequalities

\[ S \geq 1 - \frac{1}{8(x + p)} + \frac{1}{128(x + p)^2} > 1 - \frac{1}{7(x + p)} \geq \frac{6}{7}. \]

Consequently, thanks to (3.19),

\[ \frac{\Delta_2 - \Delta_1}{S} < \frac{7}{6} \left(\frac{1}{191} - \frac{1}{379}\right) \frac{1}{(x + p)^3} < \frac{1}{330(x + p)^3}. \]
References


