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Multiplicative maps on generalized n-matrix rings

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ABSTRACT

Let $\mathfrak{R}$ and $\mathfrak{R}'$ be two associative rings (not necessarily with identity elements). A bijective map φ of $\mathfrak{R}$ onto $\mathfrak{R}'$ is called an *m-multiplicative isomorphism* if $\varphi(x_1 \cdots x_m) = \varphi(x_1) \cdots \varphi(x_m)$ for all $x_1, \ldots, x_m \in \mathfrak{R}$. In this article, we establish a condition on generalized matrix rings, that assures that multiplicative maps are additive. And then, we apply our result for study of m-multiplicative isomorphisms and m-multiplicative derivations on generalized matrix rings.

RESUMEN

Sean $\mathfrak{R}$ y $\mathfrak{R}'$ dos anillos asociativos (no necesariamente con elementos identidad). Una aplicación biyectiva φ de $\mathfrak{R}$ en $\mathfrak{R}'$ se llama un *isomorfismo m-multiplicativo* si $\varphi(x_1 \cdots x_m) = \varphi(x_1) \cdots \varphi(x_m)$ para todos $x_1, \ldots, x_m \in \mathfrak{R}$. En este artículo, establecemos una condición en anillos de matrices generalizadas que asegura que las aplicaciones multiplicativas sean aditivas. Luego aplicamos nuestro resultado para estudiar isomorfismos m-multiplicativos y derivaciones m-multiplicativas de anillos de matrices generalizadas.

Keywords and Phrases: m-multiplicative maps, m-multiplicative derivations, generalized n-matrix rings, additivity.

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1 Introduction

Let $\mathfrak{R}$ and $\mathfrak{R}'$ be two associative rings (not necessarily with identity elements). We denote by $\mathfrak{Z}(\mathfrak{R})$ the center of $\mathfrak{R}$. A bijective map φ of $\mathfrak{R}$ onto $\mathfrak{R}'$ is called an *m-multiplicative isomorphism* if

$$\varphi(x_1 \cdots x_m) = \varphi(x_1) \cdots \varphi(x_m)$$

for all $x_1, \ldots, x_m \in \mathfrak{R}$. In particular, if $m = 2$ then φ is called a *multiplicative isomorphism*. Similarly, a map d of $\mathfrak{R}$ is called an *m-multiplicative derivation* if

$$d(x_1 \cdots x_m) = \sum_{i=1}^{m} x_1 \cdots d(x_i) \cdots x_m$$

for all $x_1, \ldots, x_m \in \mathfrak{R}$. If $d(xy) = d(x)y + xd(y)$ for all $x, y \in \mathfrak{R}$, we just say that d is a *multiplicative derivation* of $\mathfrak{R}$.

In last few decades, the multiplicative mappings on rings and algebras have been studied by many authors [1, 4–7, 10]. Martindale [7] established a condition on a ring such that multiplicative isomorphisms on this ring are all additive. In particular, every multiplicative isomorphism from a prime ring containing a nontrivial idempotent onto an arbitrary ring is additive. Lu [6] studied multiplicative isomorphisms of subalgebras of nest algebras which contain all finite rank operators but might contain no idempotents and proved that these multiplicative mappings are automatically additive and linear or conjugate linear. Further, Wang in [9, 10] considered the additivity of multiplicative maps on rings with idempotents and triangular rings respectively. Recently, in order to generalize the result in [10] the second author [3], defined a class of ring called triangular n-matrix ring and studied the additivity of multiplicative maps on that class of rings. In view of above discussed literature, in this article we discuss the additivity of multiplicative maps on a more general class of rings called generalized n-matrix rings.

We adopt and follow the same structure and demonstration presented in [3], in order to preserve the author ideas and to highlight the generalization of the triangular n-matrix results to the generalized n-matrix results.

Definition 1.1. *Let $\mathfrak{R}_1, \mathfrak{R}_2, \ldots, \mathfrak{R}_n$ be rings and $\mathfrak{M}_{ij}$ be an $(\mathfrak{R}_i, \mathfrak{R}_j)$-bimodule with $\mathfrak{M}_{ii} = \mathfrak{R}_i$ for all $i, j \in \{1, \ldots, n\}$. Let $\varphi_{ijk} : \mathfrak{M}_{ij} \otimes_{\mathfrak{R}_j} \mathfrak{M}_{jk} \longrightarrow \mathfrak{M}_{ik}$ be $(\mathfrak{R}_i, \mathfrak{R}_k)$-bimodule homomorphisms with $\varphi_{iij} : \mathfrak{R}_i \otimes_{\mathfrak{R}_i} \mathfrak{M}_{ij} \longrightarrow \mathfrak{M}_{ij}$ and $\varphi_{ijj} : \mathfrak{M}_{ij} \otimes_{\mathfrak{R}_j} \mathfrak{R}_j \longrightarrow \mathfrak{M}_{ij}$ the canonical isomorphisms for all $i, j, k \in \{1, \ldots, n\}$. Write $a \circ b = \varphi_{ijk}(a \otimes b)$ for $a \in \mathfrak{M}_{ij}$, $b \in \mathfrak{M}_{jk}$. Let*

$$\mathfrak{G} = \left\{ \begin{pmatrix} r_{11} & m_{12} & \ldots & m_{1n} \\ m_{21} & r_{22} & \ldots & m_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ m_{n1} & m_{n2} & \ldots & r_{nn} \end{pmatrix}_{n \times n} : \underbrace{r_{ii} \in \mathfrak{R}_i \ (= \mathfrak{M}_{ii}),\ m_{ij} \in \mathfrak{M}_{ij}}_{(i,j \in \{1,\ldots,n\})} \right\}$$

be the set of all $n \times n$ matrices (m_{ij}) with (i,j)-entry $m_{ij} \in \mathfrak{M}_{ij}$ for all $i, j \in \{1, \dots, n\}$. Observe that, with the obvious matrix operations of addition and multiplication, $\mathfrak{G}$ is a ring iff $a \circ (b \circ c) = (a \circ b) \circ c$ for all $a \in \mathfrak{M}_{ik}$, $b \in \mathfrak{M}_{kl}$ and $c \in \mathfrak{M}_{lj}$ for all $i, j, k, l \in \{1, \dots, n\}$. When $\mathfrak{G}$ is a ring, it is called a generalized n-matrix ring.

Note that if $n = 2$, we get the definition of generalized matrix ring. We denote by $\bigoplus_{i=1}^{n} r_{ii}$ the element

$$\begin{pmatrix} r_{11} & & & \\ & r_{22} & & \\ & & \ddots & \\ & & & r_{nn} \end{pmatrix}$$

in $\mathfrak{G}$.

Set

$$\mathfrak{G}_{ij} = \left\{ (m_{kt}) : \ m_{kt} = \begin{cases} m_{ij}, & \text{if} \quad (k,t) = (i,j) \\ 0, & \text{if} \quad (k,t) \neq (i,j) \end{cases}, \quad i, j \in \{1, \dots, n\} \right\}.$$

Then we can write $\mathfrak{G} = \bigoplus_{i,j \in \{1,\dots,n\}} \mathfrak{G}_{ij}$. Note that, this special structure allows us to use the argument given in [7] even if non-trivial idempotents exist. Henceforth the element a_{ij} belongs to $\mathfrak{G}_{ij}$ and the corresponding elements are in $\mathfrak{R}_1, \dots, \mathfrak{R}_n$ or $\mathfrak{M}_{ij}$. By a direct calculation $a_{ij} a_{kl} = 0$ if $j \neq k$. We define natural projections $\pi_i : \mathfrak{G} \longrightarrow \mathfrak{R}_i$ $(1 \leq i \leq n)$ by

$$\begin{pmatrix} r_{11} & m_{12} & \dots & m_{1n} \\ m_{21} & r_{22} & \dots & m_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ m_{n1} & m_{n2} & \cdots & r_{nn} \end{pmatrix} \longmapsto r_{ii}.$$

The following result is a characterization of the center of a generalized n-matrix ring. Henceforth, we will consider

(i) $\mathfrak{M}_{ij}$ is faithful as a left $\mathfrak{R}_i$-module and faithful as a right $\mathfrak{R}_j$-module with $i \neq j$,

(ii) if $m_{ij} \in \mathfrak{M}_{ij}$ is such that $\mathfrak{R}_i m_{ij} \mathfrak{R}_j = 0$ then $m_{ij} = 0$ with $i \neq j$.

We will call them *special conditions.*

Proposition 1.2. *Let $\mathfrak{G}$ be a generalized n-matrix ring. The center of $\mathfrak{G}$ is*

$$\mathfrak{Z}(\mathfrak{G}) = \left\{ \bigoplus_{i=1}^{n} r_{ii} \;\middle|\; r_{ii} m_{ij} = m_{ij} r_{jj} \text{ for all } m_{ij} \in \mathfrak{M}_{ij}, \ i \neq j \right\}.$$

Furthermore, $\mathfrak{Z}(\mathfrak{G})_{ii} \cong \pi_i(\mathfrak{Z}(\mathfrak{G})) \subseteq \mathfrak{Z}(\mathfrak{R}_i)$, and there exists a unique ring isomorphism τ_i^j from $\pi_i(\mathfrak{Z}(\mathfrak{G}))$ to $\pi_{i\mathfrak{R}_j}(\mathfrak{Z}(\mathfrak{G}))$, $i \neq j$, such that $r_{ii} m_{ij} = m_{ij} \tau_i^j(r_{ii})$ for all $m_{ij} \in \mathfrak{M}_{ij}$.

Proof. Let $S = \left\{ \bigoplus_{i=1}^{n} r_{ii} \;\middle|\; r_{ii} m_{ij} = m_{ij} r_{jj} \text{ for all } m_{ij} \in \mathfrak{M}_{ij}, \ i \neq j \right\}$. By a direct calculation we have that if $r_{ii} \in \mathfrak{Z}(\mathfrak{R}_i)$ and $r_{ii}m_{ij} = m_{ij}r_{jj}$ for every $m_{ij} \in \mathfrak{M}_{ij}$ with $i \neq j$, then $\bigoplus_{i=1}^{n} r_{ii} \in \mathfrak{Z}(\mathfrak{G})$; that is, $(\bigoplus_{i=1}^{n} \mathfrak{Z}(\mathfrak{R}_i)) \cap S \subseteq \mathfrak{Z}(\mathfrak{G})$. To prove that $S = \mathfrak{Z}(\mathfrak{G})$, we only need to show that $\mathfrak{Z}(\mathfrak{G}) \subseteq S$ and $S \subseteq \bigoplus_{i=1}^{n} \mathfrak{Z}(\mathfrak{R}_i)$.

Suppose that $x = \begin{pmatrix} r_{11} & m_{12} & \dots & m_{1n} \\ m_{21} & r_{22} & \dots & m_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ m_{n1} & m_{n2} & \dots & r_{nn} \end{pmatrix} \in \mathfrak{Z}(\mathfrak{G})$. Since $x\big(\bigoplus_{i=1}^{n} a_{ii}\big) = \big(\bigoplus_{i=1}^{n} a_{ii}\big)x$ for all $a_{ii} \in \mathfrak{R}_i$, we have $a_{ii}m_{ij} = m_{ij}a_{jj}$ for $i \neq j$. Making $a_{jj} = 0$ we conclude $a_{ii}m_{ij} = 0$ for all $a_{ii} \in \mathfrak{R}_i$ and so $m_{ij} = 0$ for all $i \neq j$ which implies that $x = \bigoplus_{i=1}^{n} r_{ii}$. Moreover, for any $m_{ij} \in \mathfrak{M}_{ij}$ as

$$x \begin{pmatrix} 0 & \dots & 0 & \dots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & & \vdots & & \vdots \\ 0 & \dots & 0 & \dots & m_{ij} & \dots & 0 \\ \vdots & & \vdots & \ddots & \vdots & & \vdots \\ 0 & \dots & 0 & \dots & 0 & \dots & 0 \\ \vdots & & \vdots & & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & \dots & 0 & \dots & 0 \end{pmatrix} = \begin{pmatrix} 0 & \dots & 0 & \dots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & & \vdots & & \vdots \\ 0 & \dots & 0 & \dots & m_{ij} & \dots & 0 \\ \vdots & & \vdots & \ddots & \vdots & & \vdots \\ 0 & \dots & 0 & \dots & 0 & \dots & 0 \\ \vdots & & \vdots & & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & \dots & 0 & \dots & 0 \end{pmatrix} x,$$

then $r_{ii}m_{ij} = m_{ij}r_{jj}$ for all $i \neq j$ which results in $\mathfrak{Z}(\mathfrak{G}) \subseteq S$. Now suppose $x = \bigoplus_{i=1}^{n} r_{ii} \in S$. Then for any $a_{ii} \in \mathfrak{R}_i$ $(i = 1, \dots, n-1)$, we have $(r_{ii}a_{ii} - a_{ii}r_{ii})m_{ij} = r_{ii}(a_{ii}m_{ij}) - a_{ii}(r_{ii}m_{ij}) = (a_{ii}m_{ij})r_{jj} - a_{ii}(m_{ij}r_{jj}) = 0$ for all $m_{ij} \in \mathfrak{M}_{ij}$ $(i \neq j)$ and hence $r_{ii}a_{ii} - a_{ii}r_{ii} = 0$ as $\mathfrak{M}_{ij}$ is a left faithful $\mathfrak{R}_i$-module.

The fact that $\pi_i(\mathfrak{Z}(\mathfrak{G})) \subseteq \mathfrak{Z}(\mathfrak{R}_i)$ for $i = 1, \dots, n$ are direct consequences of $\mathfrak{Z}(\mathfrak{G}) = S \subseteq \bigoplus_{i=1}^{n} \mathfrak{Z}(\mathfrak{R}_i)$. Now we prove the existence of the ring isomorphism $\tau_i^j : \pi_{\mathfrak{R}_i}(\mathfrak{Z}(\mathfrak{G})) \longrightarrow \pi_{\mathfrak{R}_j}(\mathfrak{Z}(\mathfrak{G}))$ for $i \neq j$. For this, let us consider a pair of indices (i, j) such that $i \neq j$. For any $r = \bigoplus_{k=1}^{n} r_{kk} \in \mathfrak{Z}(\mathfrak{G})$ let us define $\tau_i^j(r_{ii}) = r_{jj}$. The map is well defined because if $s = \bigoplus_{k=1}^{n} s_{kk} \in \mathfrak{Z}(\mathfrak{G})$ is such that $s_{ii} = r_{ii}$, then we have $m_{ij}r_{jj} = r_{ii}m_{ij} = s_{ii}m_{ij} = m_{ij}s_{jj}$ for all $m_{ij} \in \mathfrak{M}_{ij}$. Since $\mathfrak{M}_{ij}$ is a right faithful $\mathfrak{R}_j$-module, we conclude that $r_{jj} = s_{jj}$. Therefore, for any $r_{ii} \in \pi_{\mathfrak{R}_i}(\mathfrak{Z}(\mathfrak{G}))$, there exists a unique $r_{jj} \in \pi_{\mathfrak{R}_j}(\mathfrak{Z}(\mathfrak{G}))$, denoted by $\tau_i^j(r_{ii})$. It is easy to see that τ_i^j is bijective. Moreover, for any $r_{ii}, s_{ii} \in \pi_{\mathfrak{R}_i}(\mathfrak{Z}(\mathfrak{G}))$ we have $m_{ij}\tau_i^j(r_{ii}+s_{ii}) = (r_{ii}+s_{ii})m_{ij} = m_{ij}(r_{jj}+s_{jj}) = m_{ij}\big(\tau_i^j(r_{ii}) + \tau_i^j(s_{ii})\big)$ and $m_{ij}\tau_i^j(r_{ii}s_{ii}) = (r_{ii}s_{ii})m_{ij} = r_{ii}(s_{ii}m_{ij}) = (s_{ii}m_{ij})\tau_i^j(r_{ii}) = s_{ii}\big(m_{ij}\tau_i^j(r_{ii})\big) = m_{ij}\big(\tau_i^j(r_{ii})\tau_i^j(s_{ii})\big)$. Thus $\tau_i^j(r_{ii} + s_{ii}) = \tau_i^j(r_{ii}) + \tau_i^j(s_{ii})$ and $\tau_i^j(r_{ii}s_{ii}) = \tau_i^j(r_{ii})\tau_i^j(s_{ii})$ and so τ_i^j is a ring isomorphism. □

Proposition 1.3. *Let $\mathfrak{G}$ be a generalized n-matrix ring such that:*

(i) $a_{ii}\mathfrak{R}_i = 0$ *implies* $a_{ii} = 0$ *for* $a_{ii} \in \mathfrak{R}_i$;

(ii) $\mathfrak{R}_j b_{jj} = 0$ *implies* $b_{jj} = 0$ *for* $b_{jj} \in \mathfrak{R}_j$.

Then $u\mathfrak{G} = 0$ *or* $\mathfrak{G} u = 0$ *implies* $u = 0$ *for* $u \in \mathfrak{G}$.

Proof. First, let us observe that if $i \neq j$ and $\mathfrak{R}_i a_{ii} = 0$, then we have $\mathfrak{R}_i a_{ii} m_{ij} \mathfrak{R}_j = 0$, for all $m_{ij} \in \mathfrak{M}_{ij}$, which implies $a_{ii} m_{ij} = 0$ by condition *(ii)* of the special conditions. It follows that $a_{ii}\mathfrak{M}_{ij} = 0$ resulting in $a_{ii} = 0$. Hence, suppose $u = \bigoplus_{i,j\in\{1,\dots,n\}} u_{ij}$, with $u_{ij} \in \mathfrak{G}_{ij}$, satisfying $u\mathfrak{G} = 0$. Then $u_{kk}\mathfrak{R}_k = 0$ which yields $u_{kk} = 0$ for $k = 1, \dots, n-1$, by condition *(i)*. Now for $k = n$, $u_{nn}\mathfrak{R}_n = 0$, we have $\mathfrak{R}_i m_{in} u_{nn} \mathfrak{R}_n = 0$, for all $m_{in} \in \mathfrak{M}_{in}$, which implies $m_{in} u_{nn} = 0$ by condition *(ii)* of the special conditions. It follows that $\mathfrak{M}_{in} u_{nn} = 0$ which implies $u_{nn} = 0$. Thus $u_{ij}\mathfrak{R}_j = 0$ and then $u_{ij} = 0$ by condition *(ii)* of special conditions. Therefore $u = 0$. Similarly, we prove that if $\mathfrak{G} u = 0$ then $u = 0$. ☐

2 The main theorem

Follows our main result, where we are suppose that the special conditions hold. This generalizes the Theorem 2.1 in [3]. Our main result reads as follows.

Theorem 2.1. *Let $B : \mathfrak{G} \times \mathfrak{G} \longrightarrow \mathfrak{G}$ be a biadditive map such that:*

(i) $B(\mathfrak{G}_{pp}, \mathfrak{G}_{qq}) \subseteq \mathfrak{G}_{pp} \cap \mathfrak{G}_{qq}$; $B(\mathfrak{G}_{pp}, \mathfrak{G}_{rs}) \in \mathfrak{G}_{rs}$, $B(\mathfrak{G}_{ip}, \mathfrak{G}_{pq}) \in \mathfrak{G}_{iq}$ *and* $B(\mathfrak{G}_{rs}, \mathfrak{G}_{pp}) \in \mathfrak{G}_{rs}$; $B(\mathfrak{G}_{pq}, \mathfrak{G}_{rs}) = 0$;

(ii) *if* $B(\bigoplus_{1\le p\neq q\le n} c_{pq}, \mathfrak{G}_{nn}) = 0$ *or* $B(\bigoplus_{1\le r<n} \mathfrak{G}_{rr}, \bigoplus_{1\le p\neq q\le n} c_{pq}) = 0$, *then* $\bigoplus_{1\le p\neq q\le n} c_{pq} = 0$;

(iii) $B(\mathfrak{G}_{nn}, a_{nn}) = 0$ *implies* $a_{nn} = 0$ *and* $B(\bigoplus_{i=1}^{n} \mathfrak{G}_{ip}, a_{pq}) = 0$ *implies* $a_{pq} = 0$;

(iv) *if* $B(\bigoplus_{p=1}^{n} c_{pp}, \mathfrak{G}_{rs}) = B(\mathfrak{G}_{rs}, \bigoplus_{p=1}^{n} c_{pp}) = 0$ *for all* $1 \le r \neq s \le n$, *then* $\bigoplus_{p=1}^{n-1} c_{pp} \oplus (-c_{nn}) \in \mathfrak{Z}(\mathfrak{G})$;

(v) $B(c_{pp}, d_{pp}) = B(d_{pp}, c_{pp})$ *and* $B(c_{pp}, d_{pp}) d_{pn} d_{nn} = d_{pp} d_{pn} B(c_{nn}, d_{nn})$ *for all* $c = \bigoplus_{p=1}^{n} c_{pp} \in \mathfrak{Z}(\mathfrak{G})$;

(vi) $B\big(c_{rr}, B(c_{kl}, c_{nn})\big) = B\big(B(c_{rr}, c_{kl}), c_{nn}\big)$.

Suppose $f : \mathfrak{G} \times \mathfrak{G} \longrightarrow \mathfrak{G}$ is a map satisfying the following conditions:

(vii) $f(\mathfrak{G}, 0) = f(0, \mathfrak{G}) = 0$;

(viii) $B\big(f(x,y),z\big) = f\big(B(x,z),B(y,z)\big)$;

(ix) $B\big(z,f(x,y)\big) = f\big(B(z,x),B(z,y)\big)$

for all $x,y,z \in \mathfrak{G}$. *Then* $f = 0$.

Proof. Following the ideas of Ferreira in [3] we divide the proof into four cases. Then, let us consider arbitrary elements $x_{kl}, u_{kl}, a_{kl} \in \mathfrak{G}_{kl}$ $(k,l \in \{1,\dots,n\})$.

*First case***.** In this first case the reader should keep in mind that we want to show

$$f\left(\sum_{1\leq i<n} x_{ii}, \sum_{1\leq j\neq k\leq n} x_{jk}\right) = 0.$$

From the hypotheses of the theorem, we have

$$\begin{aligned} B\left(f\left(\sum_{1\leq i<n} x_{ii}, \sum_{1\leq j\neq k\leq n} x_{jk}\right), a_{nn}\right) &= f\left(B\left(\sum_{1\leq i<n} x_{ii}, a_{nn}\right), B\left(\sum_{1\leq j\neq k\leq n} x_{jk}, a_{nn}\right)\right) \\ &= f\left(0, B\left(\sum_{1\leq j\neq k\leq n} x_{jk}, a_{nn}\right)\right) \\ &= 0. \end{aligned}$$

In other words,

$$B\left(\sum_{1\leq p,q\leq n} f\left(\sum_{1\leq i<n} x_{ii}, \sum_{1\leq j\neq k\leq n} x_{jk}\right)_{pq}, a_{nn}\right) = 0.$$

Since by condition (i),

$$B\left(\sum_{1\leq p<n} f\left(\sum_{1\leq i<n} x_{ii}, \sum_{1\leq j\neq k\leq n} x_{jk}\right)_{pp}, a_{nn}\right) = 0,$$

$$B\left(\sum_{1\leq p\neq q\leq n} f\left(\sum_{1\leq i<n} x_{ii}, \sum_{1\leq j\neq k\leq n} x_{jk}\right)_{pq}, a_{nn}\right) \in \bigoplus_{1\leq p\neq q\leq n} \mathfrak{G}_{pq}$$

and

$$B\left(f\left(\sum_{1\leq i<n} x_{ii}, \sum_{1\leq j\neq k\leq n} x_{jk}\right)_{nn}, a_{nn}\right) \in \mathfrak{G}_{nn},$$

then

$$\sum_{1\leq p\neq q\leq n} f\left(\sum_{1\leq i<n} x_{ii}, \sum_{1\leq j\neq k\leq n} x_{jk}\right)_{pq} = 0 \quad \text{by condition } (ii).$$

Next, we have

$$\begin{aligned} B\left(a_{nn}, f\left(\sum_{1\leq i<n} x_{ii}, \sum_{1\leq j\neq k\leq n} x_{jk}\right)\right) &= f\left(B\left(a_{nn}, \sum_{1\leq i<n} x_{ii}\right), B\left(a_{nn}, \sum_{1\leq j\neq k\leq n} x_{jk}\right)\right) \\ &= f\left(0, B\left(a_{nn}, \sum_{1\leq j\neq k\leq n} x_{jk}\right)\right) \\ &= 0 \end{aligned}$$

which implies

$$\sum_{1\leq p,q\leq n} B\left(a_{nn}, f\left(\sum_{1\leq i<n} x_{ii}, \sum_{1\leq j\neq k\leq n} x_{jk}\right)_{pq}\right) = 0.$$

It follows that

$$B\left(a_{nn}, \sum_{1\leq p<n} f\left(\sum_{1\leq i<n} x_{ii}, \sum_{1\leq j\neq k\leq n} x_{jk}\right)_{pp}\right) = 0,$$

$$B\left(a_{nn}, \sum_{1\leq p\neq q\leq n} f\left(\sum_{1\leq i<n} x_{ii}, \sum_{1\leq j\neq k\leq n} x_{jk}\right)_{pq}\right) \in \bigoplus_{1\leq p\neq q\leq n} \mathfrak{G}_{pq}$$

and

$$B\left(a_{nn}, f\left(\sum_{1\leq i<n} x_{ii}, \sum_{1\leq j\neq k\leq n} x_{jk}\right)_{nn}\right) \in \mathfrak{G}_{nn}.$$

Hence,

$$B\left(a_{nn}, f\left(\sum_{1\leq i<n} x_{ii}, \sum_{1\leq j\neq k\leq n} x_{jk}\right)_{nn}\right) = 0$$

which yields

$$f\left(\sum_{1\leq i<n} x_{ii}, \sum_{1\leq j\neq k\leq n} x_{jk}\right)_{nn} = 0$$

by condition (iii). Yet, we have

$$\begin{aligned} B\left(\sum_{1\leq p<n} f\left(\sum_{1\leq i<n} x_{ii}, \sum_{1\leq j\neq k\leq n} x_{jk}\right)_{pp}, a_{rs}\right) &= B\left(f\left(\sum_{1\leq i<n} x_{ii}, \sum_{1\leq j\neq k\leq n} x_{jk}\right), a_{rs}\right) \\ &= f\left(B\left(\sum_{1\leq i<n} x_{ii}, a_{rs}\right), B\left(\sum_{1\leq j\neq k\leq n} x_{jk}, a_{rs}\right)\right) \\ &= f\left(B\left(\sum_{1\leq i<n} x_{ii}, a_{rs}\right), 0\right) \\ &= 0 \end{aligned}$$

and

$$
\begin{aligned}
B\left(a_{rs}, \sum_{1\le p<n} f\left(\sum_{1\le i<n} x_{ii}, \sum_{1\le j\ne k\le n} x_{jk}\right)_{pp}\right) &= B\left(a_{rs}, f\left(\sum_{1\le i<n} x_{ii}, \sum_{1\le j\ne k\le n} x_{jk}\right)\right)\\
&= f\left(B\left(a_{rs}, \sum_{1\le i<n} x_{ii}\right), B\left(a_{rs}, \sum_{1\le j\ne k\le n} x_{jk}\right)\right)\\
&= f\left(B\left(a_{rs}, \sum_{1\le i<n} x_{ii}\right), 0\right)\\
&= 0.
\end{aligned}
$$

It follows the condition (iv) that $\sum_{1\le p<n} f\left(\sum_{1\le i<n} x_{ii}, \sum_{1\le j\ne k\le n} x_{jk}\right)_{pp} + 0 \in \mathfrak{Z}(\mathfrak{G})$ and so

$$
\sum_{1\le p<n} f\left(\sum_{1\le i<n} x_{ii}, \sum_{1\le j\ne k\le n} x_{jk}\right)_{pp} = 0
$$

by Proposition 1.2. Consequently, we have $f\left(\sum_{1\le i<n} x_{ii}, \sum_{1\le j\ne k\le n} x_{jk}\right) = 0$.

*Second case***.** In the second case it must be borne in mind that we want to show

$$
f\left(\sum_{1\le i\ne j\le n} x_{ij}, \sum_{1\le k\ne l\le n} y_{kl}\right) = 0.
$$

From the hypotheses of the theorem, we have

$$
\begin{aligned}
B\left(\sum_{1\le p,q\le n} f\left(\sum_{1\le i\ne j\le n} x_{ij}, \sum_{1\le k\ne l\le n} y_{kl}\right)_{pq}, a_{rs}\right) &= B\left(f\left(\sum_{1\le i\ne j\le n} x_{ij}, \sum_{1\le k\ne l\le n} y_{kl}\right), a_{rs}\right)\\
&= f\left(B\left(\sum_{1\le i\ne j\le n} x_{ij}, a_{rs}\right), B\left(\sum_{1\le k\ne l\le n} y_{kl}, a_{rs}\right)\right)\\
&= f(0,0)\\
&= 0.
\end{aligned}
$$

Since

$$
B\left(\sum_{1\le p\ne q\le n} f\left(\sum_{1\le i\ne j\le n} x_{ij}, \sum_{1\le k\ne l\le n} y_{kl}\right)_{pq}, a_{rs}\right) = 0,
$$

then

$$
B\left(\sum_{1\le p\le n} f\left(\sum_{1\le i\ne j\le n} x_{ij}, \sum_{1\le k\ne l\le n} y_{kl}\right)_{pp}, a_{rs}\right) = 0.
$$

Smilarly, we prove that

$$B\left(a_{rs}, \sum_{1\leq p\leq n} f\left(\sum_{1\leq i\neq j\leq n} x_{ij}, \sum_{1\leq k\neq l\leq n} y_{kl}\right)_{pp}\right) = 0.$$

By condition (iv), it follows that

$$\sum_{1\leq p<n} f\left(\sum_{1\leq i\neq j\leq n} x_{ij}, \sum_{1\leq k\neq l\leq n} y_{kl}\right)_{pp} + \left(-f\left(\sum_{1\leq i\neq j\leq n} x_{ij}, \sum_{1\leq k\neq l\leq n} y_{kl}\right)_{nn}\right) \in \mathfrak{Z}(\mathfrak{G}). \quad (2.1)$$

Now, we observe that

$$\begin{aligned} B\left(f\left(\sum_{1\leq i\neq j\leq n} x_{ij}, \sum_{1\leq k\neq l\leq n} y_{kl}\right), a_{nn}\right) &= f\left(B\left(\sum_{1\leq i\neq j\leq n} x_{ij}, a_{nn}\right), B\left(\sum_{1\leq k\neq l\leq n} y_{kl}, a_{nn}\right)\right) \\ &= f\left(\sum_{1\leq i\neq j\leq n} B(x_{ij}, a_{nn}), \sum_{1\leq k\neq l\leq n} B(y_{kl}, a_{nn})\right). \end{aligned}$$

With (2.1), this implies that

$$\sum_{1\leq p<n} B\left(f\left(\sum_{1\leq i\neq j\leq n} x_{ij}, \sum_{1\leq k\neq l\leq n} y_{kl}\right), a_{nn}\right)_{pp} + \left(-B\left(f\left(\sum_{1\leq i\neq j\leq n} x_{ij}, \sum_{1\leq k\neq l\leq n} y_{kl}\right), a_{nn}\right)_{nn}\right) \in \mathfrak{Z}(\mathfrak{G}).$$

Since $B\left(f\left(\sum_{1\leq i\neq j\leq n} x_{ij}, \sum_{1\leq k\neq l\leq n} y_{kl}\right), a_{nn}\right) \in \bigoplus_{1\leq p\neq q\leq n} \mathfrak{G}_{pq} \bigoplus \mathfrak{G}_{nn}$ then

$$\sum_{1\leq p<n} B\left(f\left(\sum_{1\leq i\neq j\leq n} x_{ij}, \sum_{1\leq k\neq l\leq n} y_{kl}\right), a_{nn}\right)_{pp} = 0$$

which results in

$$eB\left(f\left(\sum_{1\leq i\neq j\leq n} x_{ij}, \sum_{1\leq k\neq l\leq n} y_{kl}\right), a_{nn}\right)_{nn} = 0$$

by Proposition 1.2. Hence $B\left(f\left(\sum_{1\leq i\neq j\leq n} x_{ij}, \sum_{1\leq k\neq l\leq n} y_{kl}\right), a_{nn}\right) \in \bigoplus_{1\leq p\neq q\leq n} \mathfrak{G}_{pq}$.

It follows that

$$\begin{aligned}
&B\left(a_{rr}, B\left(f\left(\sum_{1\leq i\neq j\leq n} x_{ij}, \sum_{1\leq k\neq l\leq n} y_{kl}\right), a_{nn}\right)\right)\\
&= B\left(a_{rr}, f\left(B\left(\sum_{1\leq i\neq j\leq n} x_{ij}, a_{nn}\right), B\left(\sum_{1\leq k\neq l\leq n} y_{kl}, a_{nn}\right)\right)\right)\\
&= f\left(B\left(a_{rr}, B\left(\sum_{1\leq i\neq j\leq n} x_{ij}, a_{nn}\right)\right), B\left(a_{rr}, B\left(\sum_{1\leq k\neq l\leq n} y_{kl}, a_{nn}\right)\right)\right)\\
&= f\left(B\left(a_{rr}, B\left(\sum_{1\leq i\neq j\leq n} x_{ij}, a_{nn}\right)\right), B\left(B\left(a_{rr}, \sum_{1\leq k\neq l\leq n} y_{kl}\right), a_{nn}\right)\right)\\
&= f\left(B\left(a_{rr}, a_{nn} + B\left(\sum_{1\leq i\neq j\leq n} x_{ij}, a_{nn}\right)\right)\right),\\
&\quad B\left(B\left(a_{rr}, \sum_{1\leq k\neq l\leq n} y_{kl}\right), a_{nn} + B\left(\sum_{1\leq i\neq j\leq n} x_{ij}, a_{nn}\right)\right)\bigg)\bigg)\\
&= B\left(f\left(a_{rr}, B\left(a_{rr}, \sum_{1\leq k\neq l\leq n} y_{kl}\right)\right), a_{nn} + B\left(\sum_{1\leq i\neq j\leq n} x_{ij}, a_{nn}\right)\right)\\
&= B\left(0, a_{nn} + B\left(\sum_{1\leq i\neq j\leq n} x_{ij}, a_{nn}\right)\right)\\
&= 0
\end{aligned}$$

by first case, for all $1 \leq r < n$.

So $B\left(f\left(\sum_{1\leq i\neq j\leq n} x_{ij}, \sum_{1\leq k\neq l\leq n} y_{kl}\right), a_{nn}\right) = 0$, by condition (ii). It follows that

$$\sum_{1\leq p\leq n} B\left(f\left(\sum_{1\leq i\neq j\leq n} x_{ij}, \sum_{1\leq k\neq l\leq n} y_{kl}\right)_{pp}, a_{nn}\right) + \sum_{1\leq p\neq q\leq n} B\left(f\left(\sum_{1\leq i\neq j\leq n} x_{ij}, \sum_{1\leq k\neq l\leq n} y_{kl}\right)_{pq}, a_{nn}\right) = 0$$

which yields

$$B\left(\sum_{1\leq p\neq q\leq n} f\left(\sum_{1\leq i\neq j\leq n} x_{ij}, \sum_{1\leq k\neq l\leq n} y_{kl}\right)_{pq}, a_{nn}\right) = 0$$

and so

$$\sum_{1\leq p\neq q\leq n} f\left(\sum_{1\leq i\neq j\leq n} x_{ij}, \sum_{1\leq k\neq l\leq n} y_{kl}\right)_{pq} = 0 \quad \text{by condition } (ii).$$

Hence,

$$B\left(a_{nn}, f\left(\sum_{1\leq i\neq j\leq n} x_{ij}, \sum_{1\leq k\neq l\leq n} y_{kl}\right)_{nn}\right) = B\left(a_{nn}, f\left(\sum_{1\leq i\neq j\leq n} x_{ij}, \sum_{1\leq k\neq l\leq n} y_{kl}\right)\right)$$
$$= f\left(B\left(a_{nn}, \sum_{1\leq i\neq j\leq n} x_{ij}\right), B\left(a_{nn}, \sum_{1\leq k\neq l\leq n} y_{kl}\right)\right)$$

and by (2.1) above we have

$$\sum_{1\leq p<n} B\left(a_{nn}, f\left(\sum_{1\leq i\neq j\leq n} x_{ij}, \sum_{1\leq k\neq l\leq n} y_{kl}\right)_{nn}\right)_{pp}$$
$$+\left(-B\left(a_{nn}, f\left(\sum_{1\leq i\neq j\leq n} x_{ij}, \sum_{1\leq k\neq l\leq n} y_{kl}\right)_{nn}\right)_{nn}\right) \in \mathfrak{Z}(\mathfrak{G}).$$

Since

$$B\left(a_{nn}, f\left(\sum_{1\leq i\neq j\leq n} x_{ij}, \sum_{1\leq k\neq l\leq n} y_{kl}\right)_{nn}\right) \in \mathfrak{G}_{nn}$$

then we have

$$\sum_{1\leq p<n} B\left(a_{nn}, f\left(\sum_{1\leq i\neq j\leq n} x_{ij}, \sum_{1\leq k\neq l\leq n} y_{kl}\right)_{nn}\right)_{pp} = 0$$

and so

$$B\left(a_{nn}, f\left(\sum_{1\leq i\neq j\leq n} x_{ij}, \sum_{1\leq k\neq l\leq n} y_{kl}\right)_{nn}\right) = B\left(a_{nn}, f\left(\sum_{1\leq i\neq j\leq n} x_{ij}, \sum_{1\leq k\neq l\leq n} y_{kl}\right)_{nn}\right)_{nn} = 0,$$

by Proposition 1.2. It follows that $f\left(\sum_{1\leq i\neq j\leq n} x_{ij}, \sum_{1\leq k\neq l\leq n} y_{kl}\right)_{nn} = 0$, by condition (iii), which implies

$$\sum_{1\leq p<n} f\left(\sum_{1\leq i\neq j\leq n} x_{ij}, \sum_{1\leq k\neq l\leq n} y_{kl}\right)_{pp} = 0,$$

by (2.1). Consequently, we have

$$f\left(\sum_{1\leq i\neq j\leq n} x_{ij}, \sum_{1\leq k\neq l\leq n} y_{kl}\right) = 0.$$

Third case. Here, in the third case, we are interested in checking

$$f\left(\sum_{1\le p<n} x_{pp} + \sum_{1\le p\ne q\le n} x_{pq}, \sum_{1\le k<n} u_{kk} + \sum_{1\le k\ne l\le n} u_{kl}\right) = 0.$$

In view of second case, we observe that

$$\begin{aligned}
&B\left(f\left(\sum_{1\le p<n} x_{pp} + \sum_{1\le p\ne q\le n} x_{pq}, \sum_{1\le k<n} u_{kk} + \sum_{1\le k\ne l\le n} u_{kl}\right), a_{rs}\right)\\
&= f\left(B\left(\sum_{1\le p<n} x_{pp} + \sum_{1\le p\ne q\le n} x_{pq}, a_{rs}\right), B\left(\sum_{1\le k<n} u_{kk} + \sum_{1\le k\ne l\le n} u_{kl}, a_{rs}\right)\right)\\
&= f\left(\sum_{1\le p<n} B(x_{pp}, a_{rs}), \sum_{1\le k<n} B(u_{kk}, a_{rs})\right)\\
&= 0.
\end{aligned}$$

It follows that

$$\sum_{1\le t\le n} B\left(f\left(\sum_{1\le p<n} x_{pp} + \sum_{1\le p\ne q\le n} x_{pq}, \sum_{1\le k<n} u_{kk} + \sum_{1\le k\ne l\le n} u_{kl}\right)_{tt}, a_{rs}\right) = 0.$$

Similarly, we have

$$\sum_{1\le t\le n} B\left(a_{rs}, f\left(\sum_{1\le p<n} x_{pp} + \sum_{1\le p\ne q\le n} x_{pq}, \sum_{1\le k<n} u_{kk} + \sum_{1\le k\ne l\le n} u_{kl}\right)_{tt}\right) = 0.$$

It follows that

$$\begin{aligned}
&\sum_{1\le t<n} f\left(\sum_{1\le p<n} x_{pp} + \sum_{1\le p\ne q\le n} x_{pq}, \sum_{1\le k<n} u_{kk} + \sum_{1\le k\ne l\le n} u_{kl}\right)_{tt}\\
&\qquad + \left(-f\left(\sum_{1\le p<n} x_{pp} + \sum_{1\le p\ne q\le n} x_{pq}, \sum_{1\le k<n} u_{kk} + \sum_{1\le k\ne l\le n} u_{kl}\right)_{nn}\right) \in \mathfrak{Z}(\mathfrak{G})
\end{aligned}$$

by condition (iv). But

$$\begin{aligned}
&B\left(f\left(\sum_{1\le p<n} x_{pp} + \sum_{1\le p\ne q\le n} x_{pq}, \sum_{1\le k<n} u_{kk} + \sum_{1\le k\ne l\le n} u_{kl}\right), a_{nn}\right)\\
&= f\left(B\left(\sum_{1\le p<n} x_{pp} + \sum_{1\le p\ne q\le n} x_{pq}, a_{nn}\right), B\left(\sum_{1\le k<n} u_{kk} + \sum_{1\le k\ne l\le n} u_{kl}, a_{nn}\right)\right)
\end{aligned}$$

$$
\begin{aligned}
&= f\left(B\left(\sum_{1\le p\ne q\le n} x_{pq}, a_{nn}\right), B\left(\sum_{1\le k\ne l\le n} u_{kl}, a_{nn}\right)\right)\\
&= f\left(\sum_{1\le p\ne q\le n} B\left(x_{pq}, a_{nn}\right), \sum_{1\le k\ne l\le n} B\left(u_{kl}, a_{nn}\right)\right)\\
&= 0
\end{aligned}
$$

by second case. As a result, we have

$$
\sum_{1\le r\ne s\le n} f\left(\sum_{1\le p<n} x_{pp} + \sum_{1\le p\ne q\le n} x_{pq}, \sum_{1\le k<n} u_{kk} + \sum_{1\le k\ne l\le n} u_{kl}\right)_{rs} = 0 \quad \text{by condition } (ii).
$$

Hence from the second case

$$
\begin{aligned}
&B\left(a_{nn}, f\left(\sum_{1\le p<n} x_{pp} + \sum_{1\le p\ne q\le n} x_{pq}, \sum_{1\le k<n} u_{kk} + \sum_{1\le k\ne l\le n} u_{kl}\right)\right)\\
&= f\left(B\left(a_{nn}, \sum_{1\le p<n} x_{pp} + \sum_{1\le p\ne q\le n} x_{pq}\right), B\left(a_{nn}, \sum_{1\le k<n} u_{kk} + \sum_{1\le k\ne l\le n} u_{kl}\right)\right)\\
&= f\left(B\left(a_{nn}, \sum_{1\le p\ne q\le n} x_{pq}\right), B\left(a_{nn}, \sum_{1\le k\ne l\le n} u_{kl}\right)\right)\\
&= f\left(\sum_{1\le p\ne q\le n} B\left(a_{nn}, x_{pq}\right), \sum_{1\le k\ne l\le n} B\left(a_{nn}, u_{kl}\right)\right)\\
&= 0.
\end{aligned}
$$

This implies

$$
B\left(a_{nn}, f\left(\sum_{1\le p<n} x_{pp} + \sum_{1\le p\ne q\le n} x_{pq}, \sum_{1\le k<n} u_{kk} + \sum_{1\le k\ne l\le n} u_{kl}\right)_{nn}\right) = 0.
$$

Thus

$$
f\left(\sum_{1\le p<n} x_{pp} + \sum_{1\le p\ne q\le n} x_{pq}, \sum_{1\le k<n} u_{kk} + \sum_{1\le k\ne l\le n} u_{kl}\right)_{nn} = 0
$$

implying

$$
\sum_{1\le t<n} f\left(\sum_{1\le p<n} x_{pp} + \sum_{1\le p\ne q\le n} x_{pq}, \sum_{1\le k<n} u_{kk} + \sum_{1\le k\ne l\le n} u_{kl}\right)_{tt} = 0
$$

by Proposition 1.2. Therefore,

$$f\left(\sum_{1\le p<n} x_{pp} + \sum_{1\le p\ne q\le n} x_{pq}, \sum_{1\le k<n} u_{kk} + \sum_{1\le k\ne l\le n} u_{kl}\right) = 0.$$

Now we are interested in checking

$$f\left(\sum_{1\le p<n} x_{pp} + \sum_{1\le p\ne q\le n} x_{pq} + \sum_{1\le q<n} x_{qq}, \sum_{1\le k<n} u_{kk} + \sum_{1\le k\ne l\le n} u_{kl} + \sum_{1\le l<n} u_{ll}\right) = 0.$$

In view of second case, we Observe that

$$\begin{aligned}
&B\left(f\left(\sum_{1\le p<n} x_{pp} + \sum_{1\le p\ne q\le n} x_{pq} + + \sum_{1\le q<n} x_{qq}, \sum_{1\le k<n} u_{kk} + \sum_{1\le k\ne l\le n} u_{kl} + \sum_{1\le l<n} u_{ll}\right), a_{rs}\right)\\
&= f\left(B\left(\sum_{1\le p<n} x_{pp} + \sum_{1\le p\ne q\le n} x_{pq} + \sum_{1\le q<n} x_{qq}, a_{rs}\right), B\left(\sum_{1\le k<n} u_{kk} + \sum_{1\le k\ne l\le n} u_{kl} + \sum_{1\le l<n} u_{ll}, a_{rs}\right)\right)\\
&= 0.
\end{aligned}$$

It follows that

$$\sum_{1\le t\le n} B\left(f\left(\sum_{1\le p<n} x_{pp} + \sum_{1\le p\ne q\le n} x_{pq} + \sum_{1\le q<n} x_{qq}, \sum_{1\le k<n} u_{kk} + \sum_{1\le k\ne l\le n} u_{kl} + \sum_{1\le l<n} u_{ll}\right)_{tt}, a_{rs}\right) = 0.$$

Similarly, we have

$$\sum_{1\le t\le n} B\left(a_{rs}, f\left(\sum_{1\le p<n} x_{pp} + \sum_{1\le p\ne q\le n} x_{pq} + \sum_{1\le q<n} x_{qq}, \sum_{1\le k<n} u_{kk} + \sum_{1\le k\ne l\le n} u_{kl} + \sum_{1\le l<n} u_{ll}\right)_{tt}\right) = 0.$$

It follows that

$$\begin{aligned}
&\sum_{1\le t<n} f\left(\sum_{1\le p<n} x_{pp} + \sum_{1\le p\ne q\le n} x_{pq} + \sum_{1\le q<n} x_{qq}, \sum_{1\le k<n} u_{kk} + \sum_{1\le k\ne l\le n} u_{kl} + \sum_{1\le l<n} u_{ll}\right)_{tt}\\
&+\left(-f\left(\sum_{1\le p<n} x_{pp} + \sum_{1\le p\ne q\le n} x_{pq} + \sum_{1\le q<n} x_{qq}, \sum_{1\le k<n} u_{kk} + \sum_{1\le k\ne l\le n} u_{kl} + \sum_{1\le l<n} u_{ll}\right)_{nn}\right) \in \mathfrak{Z}(\mathfrak{G})
\end{aligned}$$

by condition (iv). But

$$\begin{aligned}
&B\left(f\left(\sum_{1\leq p<n} x_{pp}+\sum_{1\leq p\neq q\leq n} x_{pq}+\sum_{1\leq q<n} x_{qq}, \sum_{1\leq k<n} u_{kk}+\sum_{1\leq k\neq l\leq n} u_{kl}+\sum_{1\leq l<n} u_{ll}\right), a_{nn}\right)\\
&=f\left(B\left(\sum_{1\leq p<n} x_{pp}+\sum_{1\leq p\neq q\leq n} x_{pq}+\sum_{1\leq q<n} x_{qq}, a_{nn}\right), B\left(\sum_{1\leq k<n} u_{kk}+\sum_{1\leq k\neq l\leq n} u_{kl}+\sum_{1\leq l<n} u_{ll}, a_{nn}\right)\right)\\
&=0
\end{aligned}$$

by second case. As a result, we have

$$\sum_{1\leq r\neq s\leq n} f\left(\sum_{1\leq p<n} x_{pp}+\sum_{1\leq p\neq q\leq n} x_{pq}, \sum_{1\leq k<n} u_{kk}+\sum_{1\leq k\neq l\leq n} u_{kl}\right)_{rs}=0 \quad \text{by condition } (ii).$$

Hence from the second case

$$\begin{aligned}
&B\left(a_{nn}, f\left(\sum_{1\leq p<n} x_{pp}+\sum_{1\leq p\neq q\leq n} x_{pq}+\sum_{1\leq q<n} x_{qq}, \sum_{1\leq k<n} u_{kk}+\sum_{1\leq k\neq l\leq n} u_{kl}+\sum_{1\leq l<n} u_{ll}\right)\right)\\
&=f\left(B\left(a_{nn}, \sum_{1\leq p<n} x_{pp}+\sum_{1\leq p\neq q\leq n} x_{pq}+\sum_{1\leq q<n} x_{qq}\right), B\left(a_{nn}, \sum_{1\leq k<n} u_{kk}+\sum_{1\leq k\neq l\leq n} u_{kl}+\sum_{1\leq l<n} u_{ll}\right)\right)\\
&=0.
\end{aligned}$$

This implies

$$B\left(a_{nn}, f\left(\sum_{1\leq p<n} x_{pp}+\sum_{1\leq p\neq q\leq n} x_{pq}+\sum_{1\leq q<n} x_{qq}, \sum_{1\leq k<n} u_{kk}+\sum_{1\leq k\neq l\leq n} u_{kl}+\sum_{1\leq l<n} u_{ll}\right)_{nn}\right)=0.$$

Thus

$$f\left(\sum_{1\leq p<n} x_{pp}+\sum_{1\leq p\neq q\leq n} x_{pq}+\sum_{1\leq q<n} x_{qq}, \sum_{1\leq k<n} u_{kk}+\sum_{1\leq k\neq l\leq n} u_{kl}+\sum_{1\leq l<n} u_{ll}\right)_{nn}=0$$

implying

$$\sum_{1\leq t<n} f\left(\sum_{1\leq p<n} x_{pp}+\sum_{1\leq p\neq q\leq n} x_{pq}+\sum_{1\leq q<n} x_{qq}, \sum_{1\leq k<n} u_{kk}+\sum_{1\leq k\neq l\leq n} u_{kl}+\sum_{1\leq l<n} u_{ll}\right)_{tt}=0$$

by Proposition 1.2. Therefore,

$$f\left(\sum_{1\leq p<n} x_{pp}+\sum_{1\leq p\neq q\leq n} x_{pq}+\sum_{1\leq q<n} x_{qq}, \sum_{1\leq k<n} u_{kk}+\sum_{1\leq k\neq l\leq n} u_{kl}+\sum_{1\leq l<n} u_{ll}\right)=0.$$

Fourth case. Finally in the last case we show that $f = 0$.

Since $B\left(\sum_{1\le p,q\le n} x_{pq}, y_{rs}\right) \subseteq \mathfrak{G}_{rs}$ we have $B(f(x,u), a_{rs}) = f(B(x,a_{rs}), B(u,a_{rs})) = 0$. Then by second case, we obtain

$$B\left(\sum_{1\le p\le n} f(x,u)_{pp}, a_{rs}\right) = 0.$$

Similarly, we have

$$B\left(a_{rs}, \sum_{1\le p\le n} f(x,u)_{pp}\right) = 0.$$

It follows from condition (iv) that $\sum_{1\le p<n} f(x,u)_{pp} + (-f(x,u)_{nn}) \in \mathfrak{Z}(\mathfrak{G})$.

Now as $B\left(\sum_{1\le r<n} y_{rr}, y\right) \subseteq \sum_{1\le r<n} \mathfrak{G}_{rr} + \sum_{1\le r\neq s\le n} \mathfrak{G}_{rs}$ then by third case, we have

$$B\left(\sum_{1\le r<n} a_{rr}, f(x,u)\right) = f\left(B\left(\sum_{1\le r<n} a_{rr}, x\right), B\left(\sum_{1\le r<n} a_{rr}, u\right)\right) = 0.$$

It follows that $B\left(\sum_{1\le r<n} a_{rr}, \sum_{1\le r<n} f(x,u)_{rr} + \sum_{1\le r\neq s\le n} f(x,u)_{rs}\right) = 0$ implying

(1) $B\left(\sum_{1\le r<n} a_{rr}, \sum_{1\le r<n} f(x,u)_{rr}\right) = 0,$

(2) $B\left(\sum_{1\le r<n} a_{rr}, \sum_{1\le r\neq s\le n} f(x,u)_{rs}\right) = 0.$

By identity (1) above we have $\sum_{1\le r<n} B\big(a_{rr}, f(x,u)_{rr}\big) = 0$ resulting $B\big(a_{rr}, f(x,u)_{rr}\big) = 0$ for all $1 \le r < n$. We deduce

$$\begin{aligned} 0 &= B\big(a_{rr}, f(x,u)_{rr}\big)a_{rn}a_{nn} = B\big(f(x,u)_{rr}, a_{rr}\big)a_{rn}a_{nn} \\ &= a_{rr}a_{rn}B\big(-f(x,u)_{nn}, a_{nn}\big) = a_{rr}a_{rn}B\big(a_{nn}, -f(x,u)_{nn}\big) \end{aligned}$$

for all $r < n$, by condition (v). It follows that $B\big(a_{nn}, f(x,u)_{nn}\big) = 0$ which implies $f(x,u)_{nn} = 0$, by condition (iii). Thus, we have $\sum_{1\le p<n} f(x,u)_{pp} = 0$. Now, by identity (2), we have $\sum_{1\le r\neq s\le n} f(x,u)_{rs} = 0$ by condition (ii). Hence, we conclude that $f = 0$. $\square$

As a consequence, we can apply our result to a particular case, *i.e.* the n-generalized matrix ring that satisfy the special conditions and $\mathfrak{G}_{pq}\mathfrak{G}_{qs} = 0$ as follows:

Corollary 2.2. *Let $\mathfrak{G}$ be a n-generalized matrix ring such that*

(i) for $a_{ii} \in \mathfrak{R}_i$, if $a_{ii}\mathfrak{R}_i = 0$, then $a_{ii} = 0$;

(ii) for $b_{jj} \in \mathfrak{R}_j$, if $\mathfrak{R}_j b_{jj} = 0$, then $b_{jj} = 0$.

Let k be a positive integer. If a map $f : \mathfrak{G} \times \mathfrak{G} \longrightarrow \mathfrak{G}$ satisfies

(i) $f(\mathfrak{G}, 0) = f(0, \mathfrak{G}) = 0$;

(ii) $f(x, y)z_1z_2 \cdots z_k = f(xz_1z_2 \cdots z_k, yz_1z_2 \cdots z_k)$;

(iii) $z_1z_2 \cdots z_k f(x, y) = f(z_1z_2 \cdots z_k x, z_1z_2 \cdots z_k y)$,

for all $x, y, z_1, z_2, \ldots, z_k \in \mathfrak{G}$, then $f = 0$.

Proof. We first claim that $f(x, y)z = f(xz, yz)$ and $zf(x, y) = f(zx, zy)$ for all $x, y, z \in \mathfrak{G}$. Indeed, since

$$f(x, y)(zz_1)z_2 \cdots z_k = f(xzz_1z_2 \cdots z_k, yzz_1z_2 \cdots z_k) = f(xz, yz)z_1z_2 \cdots z_k,$$

that is, $(f(x, y)z - f(xz, yz))\mathfrak{G}^k = 0$. Hence $f(x, y)z = f(xz, yz)$ by Proposition 1.3. Analogously, $zf(x, y) = f(zx, zy)$. Define $B : \mathfrak{G} \times \mathfrak{G} \longrightarrow \mathfrak{G}$ by $B(x, y) = xy$. It is easy to check that B and f satisfy the all conditions of Theorem 2.1. Hence $f = 0$. □

3 Applications

In this section we apply our main result to the case of n-generalized matrix ring satisfying the special conditions and $\mathfrak{G}_{pq}\mathfrak{G}_{qs} = 0$.

Theorem 3.1. *Let $\mathfrak{G}$ be a n-generalized matrix ring such that*

(i) For $a_{ii} \in \mathfrak{R}_i$, if $a_{ii}\mathfrak{R}_i = 0$, then $a_{ii} = 0$;

(ii) For $b_{jj} \in \mathfrak{R}_j$, if $\mathfrak{R}_j b_{jj} = 0$, then $b_{jj} = 0$.

Then every m-multiplicative isomorphism from $\mathfrak{G}$ onto a ring $\mathfrak{R}$ is additive.

Proof. Suppose that φ is a m-multiplicative isomorphism from $\mathfrak{G}$ onto a ring $\mathfrak{R}$. Since φ is onto, $\varphi(x) = 0$ for some $x \in \mathfrak{G}$. Then $\varphi(0) = \varphi(0 \cdots 0x) = \varphi(0) \cdots \varphi(0)\varphi(x) = \varphi(0) \cdots \varphi(0)0 = 0$ and so $\varphi^{-1}(0) = 0$. Let us check that the conditions of the Corollary 2.2 are satisfied. For every $x, y \in \mathfrak{G}$ we define $f(x, y) = \varphi^{-1}(\varphi(x + y) - \varphi(x) - \varphi(y))$, we see that $f(x, 0) = f(0, x) = 0$ for all $x \in \mathfrak{G}$. It is easy to check that φ^{-1} is also a m-multiplicative isomorphism. Thus, for any $u_1, \ldots, u_{m-1} \in \mathfrak{G}$, we have

$$\begin{aligned} f(x,y)u_1 \cdots u_{m-1} &= \varphi^{-1}(\varphi(x+y) - \varphi(x) - \varphi(y))\varphi^{-1}(\varphi(u_1)) \cdots \varphi^{-1}(\varphi(u_{m-1})) \\ &= \varphi^{-1}((\varphi(x+y) - \varphi(x) - \varphi(y))\varphi(u_1) \cdots \varphi(u_{m-1})) \\ &= f(xu_1 \cdots u_{m-1}, yu_1 \cdots u_{m-1}). \end{aligned}$$

Similarly we have $u_1 \cdots u_{m-1} f(x,y) = f(u_1 \cdots u_{m-1}x, u_1 \cdots u_{m-1}y)$. Therefore by Corollary 2.2, $f = 0$. That is, $\varphi(x+y) = \varphi(x) + \varphi(y)$ for all $x, y \in \mathfrak{G}$. $\square$

Theorem 3.2. *Let $\mathfrak{G}$ be a n-generalized matrix ring such that*

(i) For $a_{ii} \in \mathfrak{R}_i$, if $a_{ii}\mathfrak{R}_i = 0$, then $a_{ii} = 0$;

(ii) For $b_{jj} \in \mathfrak{R}_j$, if $\mathfrak{R}_j b_{jj} = 0$, then $b_{jj} = 0$.

Then any m-multiplicative derivation d of $\mathfrak{G}$ is additive.

Proof. We define $f(x,y) = d(x+y) - d(x) - d(y)$, for any $x, y \in \mathfrak{G}$. Hence f defined in this way satisfy the conditions of Corollary 2.2. Therefore $f = 0$ and so $d(x+y) = d(x) + d(y)$. $\square$

It is worth noting that the technique used to prove the main result of this article is still not enough to answer the result obtained in Corollary 2.2, without the $\mathfrak{G}_{pq}\mathfrak{G}_{qs} = 0$ condition.

We therefore end our work with two open questions:

(a) When are m-multiplicative isomorphism additive?

(b) When are m-multiplicative derivation additive?

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