Curvature properties of $\alpha$-cosymplectic manifolds with $\ast\eta$-Ricci-Yamabe solitons

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ABSTRACT

In this research article, we study $\ast\eta$-Ricci-Yamabe solitons on an $\alpha$-cosymplectic manifold by giving an example in the support and also prove that it is an $\eta$-Einstein manifold. In addition, we investigate an $\alpha$-cosymplectic manifold admitting $\ast\eta$-Ricci-Yamabe solitons under some conditions. Lastly, we discuss the concircular, conformal, conharmonic, and $W_2$-curvatures on the said manifold admitting $\ast\eta$-Ricci-Yamabe solitons.

RESUMEN

En el presente artículo, estudiamos solitones $\ast\eta$-Ricci-Yamabe en una variedad $\alpha$-cosimpléctica dando un ejemplo que lo soporta y también probamos que es una variedad $\eta$-Einstein. Adicionalmente, investigamos una variedad $\alpha$-cosimpléctica que admite solitones $\ast\eta$-Ricci-Yamabe bajo ciertas condiciones. Finalmente, discutimos las curvaturas concircular, conforme, con-armónica y $W_2$ en dicha variedad admitiendo solitones $\ast\eta$-Ricci-Yamabe.

Keywords and Phrases: $\ast\eta$-Ricci-Yamabe soliton, $\alpha$-cosymplectic manifold, curvature, $\eta$-Einstein manifold.

2020 AMS Mathematics Subject Classification: 53B20, 53C21, 53C44, 53C25, 53C50, 53D35
1 Introduction

In the year 1982, R. S. Hamilton [9] investigated the concept of Ricci flow on a smooth Riemannian manifold (shortly, RM). A self-similar solution to the Ricci flow is nothing but a Ricci soliton if it moves only by a one parameter family of diffeomorphism and scaling. After introducing the idea of Ricci flow, the theory of Yamabe flow was also initiated by Hamilton in [10] to construct Yamabe metrics on a compact RM. A Yamabe soliton is again corresponded to a self-similar solution of the Yamabe flow.

S. Guler and M. Crasmareanu gave a new class of geometric flow of type \((\rho, q)\), known as Ricci-Yamabe flow in [7]. They proposed the idea of Ricci-Yamabe soliton (shortly, RYS) if it moves only by one parameter group of diffeomorphism and scaling. The metric of the RM \((M^n, h), n > 2\), is said to be RYS \((h, V, \Lambda, \rho, q)\) if it satisfies the following [20]:

\[
\mathcal{L}_V h + 2\rho \text{Ric} = [2\Lambda - qr] h, \tag{1.1}
\]

where Lie derivative operator of the metric \(h\) along the vector field \(V\) represented by \(\mathcal{L}_V h\), the Ricci curvature tensor by \(\text{Ric}\) (the Ricci operator \(Q\) defined by \(\text{Ric}(A, B) = h(QA, B)\) for \(A, B \in \chi(M)\), \(\chi(M)\) being the Lie algebra of vector fields of \(M\)), the scalar curvature by \(r\) and the real scalars by \(\Lambda, \rho, q\). According to \(\Lambda\), RYS will be expanding, steady or shrinking if \(\Lambda\) is negative, zero or positive, respectively.

The concept of \(\eta\)-Ricci-Yamabe solitons (\(\eta\)-RYS) was defined by M. D. Siddiqi, et al. [20] in 2020 as a new generalization of RYS and it is defined as

\[
\mathcal{L}_V h + 2\text{Ric} + [2\Lambda - qr] h + 2\mu \eta \otimes \eta = 0, \tag{1.2}
\]

where \(\mu\) is a constant and \(\eta\) is a 1-form on \(M\).

On the other hand, S. Dey and S. Roy [5] inaugurated a new generalization of \(\eta\)-Ricci soliton (\(\eta\)-RS) [3], namely \(\ast\)-\(\eta\)-Ricci soliton (\(\ast\)-\(\eta\)-RS), defined below:

\[
\mathcal{L}_V h + 2\text{Ric}^\ast + 2\Lambda h + 2\mu \eta \otimes \eta = 0, \tag{1.3}
\]

where \(\ast\)-Ricci tensor (shortly, \(\ast\)-RT) is denoted by \(\text{Ric}^\ast\).

Tachibana [22] brought up the concept of \(\ast\)-RT on almost Hermitian manifolds and afterwards Hamada [8] studied \(\ast\)-RT on real hypersurfaces of non-flat complex space forms. Such geometrical works inspired S. Roy, et al. to come up with new idea \(\ast\)-\(\eta\)-Ricci-Yamabe soliton (shortly, \(\ast\)-\(\eta\)-RYS) of type \((\rho, q)\), which is RM and fulfilling [18]
\[ \mathcal{L}_\psi h + 2\rho \text{Ric}^* + [2\Lambda - qr^*]h + 2\mu \eta \otimes \eta = 0, \quad (1.4) \]

where \( r^* (= \text{trace}(\text{Ric}^*)) \) is the \(*\)-scalar curvature and \( \Lambda, \rho, q, \mu \) are real scalars. The \(*\eta\)-RYS is shrinking, steady or expanding if \( \Lambda \) is negative, zero or positive respectively. And they discussed \(*\eta\)-RYS on \( \alpha\)-cosymplectic manifolds with a quarter-symmetric metric (shortly, QSM) connection.

Further, A. Haseeb, R. Prasad and F. Mofarreh [12] obtained some interesting results on an \( \alpha\)-Sasakian manifold admitting \(*\eta\)-RYS with the potential vector field \( \zeta \) satisfying conditions \( \text{Rim}(\zeta, X).\text{Ric} = 0, \ Q(h, \text{Ric}) = 0 \) and pseudo-Ricci symmetric and also showed that \( \alpha\)-Sasakian admitting \(*\eta\)-RYS is an \( \eta\)-EM.

In last few years, numerous authors have worked on the characterizations of Ricci, Ricci-Yamabe, \( \eta\)-Ricci-Yamabe and \(*\eta\)-Ricci-Yamabe solitons (respectively, RS, RYS, \( \eta\)-RYS and \(*\eta\)-RYS) in contact geometry. First, the study of RS in contact geometry was proposed by Sharma in [19]. After the initial work on Ricci solitons, some notable classes of contact geometry explored by H. I. Yoldas in [25,26] where Ricci solitons have been investigated. Later on, D. Dey [2] provided the idea of an almost Kenmotsu metric as RYS. Also, P. Zhang et al. [27] have studied conformal RYS on perfect fluid space-time. New type of soliton namely \(*\)-RYS on contact geometry introduced by M. D. Siddiqi and Akyol in [20] and they have discussed the notion of \( \eta\)-RYS for geometrical structure on Riemannian submersions admitting \( \eta\)-RYS with the potential field. In recent years, a Kenmotsu metric in terms of \( \eta\)-RYS was measured by Yoldas in [23]. Next, the notion of \(*\eta\)-RYS was studied by many authors on different odd dimensional Riemannian manifolds. It should be noted that the geometry of \(*\eta\)-RYS and gradient \(*\eta\)-RYS on Kenmotsu manifolds were given by S. Dey and S. Roy in [4].

We organize this paper as follows: In section 2, we review some basic definitions and tools of an \( \alpha\)-cosymplectic manifold \( M \). The main results are stated in section 3. In fact, we prove that an \( n\)-dimensional \( M \) admitting a \(*\eta\)-RYS is an \( \eta\)-Einstein manifold. Then some curvature tensor conditions are studied on \( M \) with \(*\eta\)-RYS. Finally, in section 4, we discuss some results on \( M \) when it is \( \zeta\)-concircularly flat, \( \zeta\)-conharmonically flat, \( \zeta\)-\( W_2\) flat and \( \zeta\)-conformal.

### 2 Preliminaries

On an \( n(= 2m + 1)\)-dimensional RM \( M \), if an almost contact metric structure \((\Phi, \zeta, \eta, h)\) satisfies the following relations, then \( M \) is called an almost contact metric manifold:

\[ \Phi^2 A = A - \eta(A)\zeta \quad (2.1) \]

\[ \eta(\zeta) = 1, \quad \Phi(\zeta) = 0, \quad \eta(\Phi\zeta) = 0 \quad (2.2) \]
\[ h(A, \Phi B) = -h(\Phi A, B), \quad (2.3) \]
\[ h(A, \zeta) = \eta(A), \quad h(\Phi A, \Phi B) = h(A, B) - \eta(A)\eta(B), \quad (2.4) \]

for all \( A, B \in \chi(M) \), where \( \Phi \) denotes a \((1,1)\) tensor field, \( \zeta \) is a vector field, \( \eta \) is a 1-form and \( h \) is the compatible Riemannian metric.

The fundamental form \( \phi \) on \( M \) is defined as \[1\]
\[ \phi(A, B) = h(\Phi A, B), \quad (2.5) \]

for all \( A, B \in \chi(M) \).

If the Nijenhuis tensor field of \( \Phi \) on \( M \) satisfies \[ N_\Phi(A, B) + 2d\eta(A, B)\zeta = 0, \]
then \( M \) is called a normal almost contact metric manifold. Here

\[ N_\Phi(A, B) = \Phi^2[A, B] + [\Phi A, \Phi B] - \Phi[A, \Phi B] - \Phi[\Phi A, B], \]

for any \( A, B \in \chi(M) \).

Under the following conditions, a normal almost contact metric manifold \( M \) is known as an \( \alpha \)-cosymplectic manifold (shortly, \( \alpha \)-CM):

1. \( d\eta = 0 \),
2. \( d\phi = 2\alpha \eta \wedge \phi \),

for \( \alpha \in \mathbb{R} \).

We note that an \( \alpha \)-CM can be

1. a cosymplectic manifold provided that \( \alpha = 0 \),
2. an \( \alpha \)-Kenmotsu manifold provided that \( \alpha \neq 0 \).

For an \( \alpha \)-CM \( M \), we have

\[ (\nabla_A \Phi)B = \alpha(h(\Phi A, B)\zeta - \eta(B)\Phi A) \quad (2.6) \]

and

\[ \nabla_A \zeta = -\alpha \Phi^2 A = \alpha[A - \eta(A)\zeta], \quad (2.7) \]

where \( \nabla \) is the Levi-Civita connection associated with \( h \).

The main examples and curvature characteristics of \( \alpha \)-CM were firstly obtained in \([11,14,15]\). Also, we have the following relations for the Riemannian curvature tensor \( R_{\text{Riem}} \) and the Ricci curvature
In [11], the Ricci tensor $\nabla$ of $M$:

$$
\nabla(A, B) = \alpha^2 [\eta(A)B - \eta(B)A],
$$
\n(2.8)

$$
\nabla(\zeta, A)B = \alpha^2 [\eta(B)A - h(A, B)\zeta],
$$
\n(2.9)

$$
\nabla(\zeta, A)\zeta = \alpha^2 [A - \eta(A)\zeta],
$$
\n(2.10)

$$
\eta(\nabla(A, B)C) = \alpha^2 [\eta(B)h(A, C) - \eta(A)h(B, C)],
$$
\n(2.11)

$$
\nabla(A, C) = -\alpha^2(n-1)\eta(A),
$$
\n(2.12)

for all $A, B, C \in \chi(M)$.

In [11], the $\ast$-RT $\nabla^\ast$ of type $(0, 2)$ on an $n$-dimensional $\alpha$-CM $M$ is obtained as

$$
\nabla^\ast(B, C) = \nabla(B, C) + \alpha^2(n-2)h(B, C) + \alpha^2\eta(B)\eta(C),
$$
\n(2.13)

for any $B, C \in \chi(M)$.

Let $\{E_i|i = 1, 2, \ldots, n\}$ be an orthonormal basis of $T_p(M)$, $p \in M$. We set $B = C = E_i$ and it is easy to derive the $\ast$-scalar curvature $r^\ast = \text{trace}(\nabla^\ast)$ as

$$
r^\ast = r + \alpha^2(n-1)^2.
$$
\n(2.14)

On the other hand, $\alpha$-CM $M$ is said to be an $\eta$-EM if the Ricci curvature tensor has the following form [24]:

$$
\nabla(A, B) = uh(A, B) + v\eta(A)\eta(B),
$$
\n(2.15)

for $A, B \in \chi(M)$, where $u$ and $v$ being constants.

For this paper, we need some curvature tensors on a RM $(M^n, h)$, which are given below [17]:

$$
\nabla(A, B)C = \nabla(A, B)C - \frac{r}{n(n-1)}[h(B, C)A - h(A, C)B],
$$
\n(2.16)

$$
H(A, B)C = \nabla(A, B)C - \frac{1}{n-2}[h(B, C)QA - h(A, C)QB + Ric(B, C)A - Ric(A, C)B],
$$
\n(2.17)

$$
W_2(A, B)C = \nabla(A, B)C + \frac{1}{n-1}[h(A, C)QB - h(B, C)QA],
$$
\n(2.18)

$$
\nabla^\ast(A, B)C = \nabla(A, B)C - \frac{1}{n-2}[Ric(B, C)A - Ric(A, C)B + h(B, C)QA - h(A, C)QB] + \frac{r}{(n-1)(n-2)}[h(B, C)A - h(A, C)B],
$$
\n(2.19)

where $\nabla$, $H$, $W_2$ and $\nabla^\ast$ represent the concircular curvature tensor [16], the conharmonic curvature tensor [13], the $W_2$-curvature tensor [16] and the conformal curvature tensor [6].
3 On $\alpha$-CM admitting $\ast$-$\eta$-RYS

Let us take a $\ast$-$\eta$-RYS $\left( h, \zeta, \Lambda, \mu, \rho, q \right)$ on an $n$-dimensional $\alpha$-CM $M$, which is given by

$$ (\mathcal{L}_\zeta h)(A, B) + 2\rho \text{Ric}^*(A, B) + [2\Lambda - qr^*] h(A, B) + 2\mu \eta(A)\eta(B) = 0, \quad (3.1) $$

for any $A, B \in \chi(M)$.

**Theorem 3.1.** An $n$-dimensional $\alpha$-CM $M$ admitting $\ast$-$\eta$-RYS $\left( h, \zeta, \Lambda, \mu, \rho, q \right)$ is an $\eta$-EM of the constant scalar curvature $r$. Moreover, the scalars $\Lambda$ and $\mu$ are related by

$$ \Lambda + \mu = \frac{qr}{2} + \frac{q\alpha^2(n-1)^2}{2}. \quad (3.2) $$

**Proof.** From (2.4) and (2.7), we arrive at

$$ (\mathcal{L}_\zeta h)(A, B) = h(\nabla_A \zeta, B) + h(A, \nabla_B \zeta) = 2\alpha \left( h(A, B) - \eta(A)\eta(B) \right). \quad (3.3) $$

Substitute (3.3) into (3.1) to get

$$ \text{Ric}^*(A, B) = -\frac{1}{\rho} \left( \Lambda - \frac{qr^*}{2} + \alpha \right) h(A, B) - \frac{(\mu - \alpha)}{\rho} \eta(A)\eta(B). \quad (3.4) $$

By using (2.13) and (2.14) in (3.4), we obtain

$$ \text{Ric}(A, B) = \left[ -\frac{1}{\rho} \left( \Lambda - \frac{qr}{2} - \frac{q\alpha^2(n-1)^2}{2} + \alpha \right) - \alpha^2(n-2) \right] h(A, B) - \left( \frac{(\mu - \alpha)}{\rho} + \alpha^2 \right) \eta(A)\eta(B), \quad (3.5) $$

that is,

$$ \text{Ric}(A, B) = \sigma_1 h(A, B) + \sigma_2 \eta(A)\eta(B), \quad (3.6) $$

where

$$ \sigma_1 = -\frac{1}{\rho} \left( \Lambda - \frac{qr}{2} - \frac{q\alpha^2(n-1)^2}{2} + \alpha \right) - \alpha^2(n-2), \quad \sigma_2 = -\left( \frac{(\mu - \alpha)}{\rho} + \alpha^2 \right). $$

Now, if we fix $B = \zeta$ in (3.6), then we can easily get the following relation:

$$ \text{Ric}(A, \zeta) = \left[ -\frac{1}{\rho} \left( \Lambda - \frac{qr}{2} - \frac{q\alpha^2(n-1)^2}{2} + \mu \right) - \alpha^2(n-1) \right] \eta(A). \quad (3.7) $$

Using (2.12) and values of $\sigma_1$ and $\sigma_2$ in (3.7), we can have (3.2). Also, on contracting (3.6) and using the values of $\sigma_1$ and $\sigma_2$, we find

$$ r = (n-1) \left( \frac{\mu}{\rho} - \frac{\alpha}{\rho} - \alpha^2(n-1) \right), \quad (3.8) $$
where $\mu$ and $\rho(\neq 0)$ are constant.

Thus, (3.6) together with (3.2) and (3.8) give the relation of $\Lambda$ and $\mu$, which shows that $\ast\eta$-RYS on $\alpha$-CM is an $\eta$-EM.

\[\tag{3.9}\]

**Remark 3.2.** For the particular value of $\rho = 0$ in (3.1), an $n$-dimensional $\alpha$-CM $M$ endowed with $\ast\eta$-RYS $(h, \zeta, \Lambda, \mu, \rho, q)$ furnishes the scalar quantities as $\Lambda = -\alpha + \frac{qr^*}{2}$ and $\mu = \alpha$.

First we give the more general construction of $\alpha$-cosymplectic manifold:

**Example 3.3.** Let $(N, J, \tilde{h})$ be a Kähler manifold. Denote by $\mathbb{R} \times_{\sigma} N$ the manifold $(\mathbb{R} \times_{\sigma} N, \Phi, \zeta, \eta, h)$, where $\Phi$ is the tensor field such that

\[\Phi \left( \frac{d}{dt} \right) = 0, \quad \Phi(A) = J(A), \quad A \in TN,\]

\[\zeta = \frac{d}{dt}, \quad \eta = dt, \quad h = dt \otimes dt + \exp(2\alpha t)\tilde{h}, \quad \alpha \in \mathbb{R}.\]

Putting $\sigma = \exp(\alpha t)$, $h$ is the warped product metric of the Euclidean metric and $\tilde{h}$ by means of the function $\sigma$. Then $\mathbb{R} \times_{\sigma} N$ is $\alpha$-cosymplectic and $(N, \tilde{h})$ is a totally umbilical submanifold with mean curvature vector $-\alpha \zeta$. Assume that $\alpha \neq 0$. Applying well-known curvature formulas, one relates the Ricci tensors of $N$ and $\mathbb{R} \times_{\sigma} N$. But here we consider the flat Kähler manifold $\mathbb{R}^4$ endowed with the canonical complex structure and then the $\alpha$-cosymplectic manifold $\mathbb{R} \times_{\sigma} \mathbb{R}^4$. If $\alpha = 0$, one has $\sigma = 1$, $\mathbb{R} \times_{\sigma} N = \mathbb{R} \times N$ is cosymplectic and $N$ is totally geodetic. In this case the Ricci tensors are related by:

\[Ric(A, B) = Ric(A - \eta(A)\zeta, B - \eta(B)\zeta).\]

It follows that if $N$ is an Einstein manifold, then $\mathbb{R} \times N$ is $\eta$-Einstein.

Next, by giving the following example we can show the existence of this soliton in $\alpha$-cosymplectic manifold:

**Example 3.4.** Recall an example of 5-dimensional $\alpha$-CM in [11], that is,

\[M = \{(x_1, x_2, y_1, y_2, z) \in \mathbb{R}^5, \Phi, \zeta, \eta, h\},\]

where $(x_1, x_2, y_1, y_2, z)$ are the standard coordinates in $\mathbb{R}^5$.

The linearly independent vector fields on $M$ are denoted by $E_1 = \exp^{az} \partial x_1$, $E_2 = \exp^{az} \partial x_2$, $E_3 = \exp^{az} \partial y_1$, $E_4 = \exp^{az} \partial y_2$ and $E_5 = \zeta = -\partial z$ for $i = \{1, 2\}$. Thus, $h$ and $\Phi$ are respectively defined as

\[h(E_i, E_i) = 1, \quad h(E_i, E_j) = 0, \quad i \neq j = \{1, 2, 3, 4, 5\}\]
and
\[ \Phi E_1 = -E_2, \quad \Phi E_2 = E_1, \quad \Phi E_3 = -E_4, \quad \Phi E_4 = E_3, \quad \Phi E_5 = \Phi \zeta = 0. \]

By the linearity of these tensors, it is quite easy to compute (2.1)-(2.4). Also, (2.6) and (2.7) are verified in [11].

By applying Koszul’s formula, Rim of \( M \) (see [11]) can be obtained easily and hence the components Ric of Ricci tensor of \( M \) are:
\[ \text{Ric}(E_i, E_i) = -4\alpha^2 \] for \( i = \{1, 2, 3, 4, 5\} \). Since \( r = \sum_{i=1}^{5} \text{Ric}(E_i, E_i) \), so we have \( r = -20\alpha^2 \).

Now, we use (3.7) and find
\[ \text{Ric}(E_5, E_5) = \text{Ric}(\zeta, \zeta) = -\frac{1}{\rho} (\Lambda + 2q\alpha^2 + \mu) - 4\alpha^2. \]

By equating the values of \( \text{Ric}(\zeta, \zeta) \), we arrive at a relation: \( \Lambda + \mu = -2q\alpha^2 \). We also verify this relation for \( n = 5 \) by using (3.2). Thus, \( h \) gives an \(*\zeta\)-RYS \((h, \zeta, \Lambda, \mu, \rho, q)\) on an \( \alpha \)-cosymplectic manifold \( M \) of dimension 5.

On the other hand, suppose that an \( n \)-dimensional \( \alpha \)-CM \( M \) admitting \(*\zeta\)-RYS \((h, \zeta, \Lambda, \mu, \rho, q)\) satisfies
\[ Q(h, \text{Ric})(A, B, C, D) = 0, \quad (3.10) \]

where \( Q(h, \text{Ric})(A, B, C, D) = (h(A, B).\text{Ric})(C, D) \), for all vector fields \( A, B, C, D \) on \( M \). This can be expressed as
\[ Q(h, \text{Ric})(A, B, C, D) = h(B, C)\text{Ric}(A, D) - h(A, C)\text{Ric}(B, D) \]
\[ + h(B, D)\text{Ric}(A, C) - h(A, D)\text{Ric}(B, C). \quad (3.11) \]

Theorem 3.5. If \(*\zeta\)-RYS on an \( \alpha \)-CM \( M \) satisfies \( Q(h, \text{Ric}) = 0 \), then
\[ \Lambda = \frac{q}{2} (r + \alpha^2(n - 1)^2) - \alpha (1 - \alpha\rho), \quad (3.12) \]
\[ \mu = \alpha(1 - \alpha\rho). \quad (3.13) \]

Proof. From the expressions (3.6), (3.10) and (3.11), we have
\[ \sigma_2[h(B, C)\eta(A)\eta(D) - h(A, C)\eta(B)\eta(D) + h(B, D)\eta(A)\eta(C) - h(A, D)\eta(B)\eta(C)] = 0. \quad (3.14) \]

Above equation follows that \( \sigma_2 = 0 \), which implies that
\[ \mu = \alpha(1 - \alpha\rho). \]
We obtain the following from (3.2)

\[ \Lambda = \frac{q}{2} (r + \alpha^2(n - 1)^2) - \alpha(1 - \alpha \rho). \]  

(3.15)

Now, by using these values of \( \sigma_1, \sigma_2 \) and \( \Lambda \) as well as \( \mu \) in (3.6), we calculate

\[ \text{Ric}(A, B) = -\alpha^2(n - 1)h(A, B). \]  

(3.16)

Thus, from above we can state the following result:

**Corollary 3.6.** If \( \ast \eta \)-RYS on an \( \alpha \)-CM \( M \) satisfies \( Q(h, \text{Ric}) = 0 \), then \( M \) is an EM.

Next, we have

\[ \text{Rim}(\zeta, A) \text{Ric} = 0, \]  

(3.17)

then we have

\[ \text{Ric}(\text{Rim}(\zeta, A)B, C) + \text{Ric}(B, \text{Rim}(\zeta, A)C) = 0, \]  

(3.18)

for all vector fields \( A, B, C \) on \( M \).

**Theorem 3.7.** If \( \ast \eta \)-RYS on an \( \alpha \)-CM \( M \) satisfies \( \text{Rim}(\zeta, A) \text{Ric} = 0 \), then either \( M \) becomes CM or we have

\[ \Lambda = \frac{q}{2} (r + \alpha^2(n - 1)^2) - \alpha(1 - \alpha \rho) \]  

(3.19)

\[ \mu = \alpha(1 - \alpha \rho). \]  

(3.20)

**Proof.** In view of (3.6) and (3.18), we compute

\[ \alpha^2 \sigma_2(2\eta(A)\eta(B)\eta(C) - \eta(C)h(A, B) - \eta(B)h(A, C)) = 0. \]  

(3.21)

Putting \( C = \zeta \) into (3.21) and using (2.4), it is quite easy to see

\[ \alpha^2 \sigma_2 h(\Phi A, \Phi B) = 0, \]  

(3.22)

which implies either \( \alpha = 0 \) or \( \sigma_2 = 0 \). Further, from later case we find \( \mu = \alpha(1 - \alpha \rho) \) and hence from (3.2), we calculate the value of \( \Lambda \). From the first case we can also say that \( M \) is CM. This is the desired result.

Next, by using the values of \( \Lambda \) as well \( \mu \) in (3.6), we have

\[ \text{Ric}(A, B) = -\alpha^2(n - 1)h(A, B). \]  

(3.23)
Thus, we can state the following:

**Corollary 3.8.** If $\ast - \eta$-RYS on an $\alpha$-CM $M$ satisfies $\text{Rim}(\zeta, A)Ric = 0$ then $M$ is either an EM or CM.

The non-flat manifold $M$ of $n$-dimension is named pseudo Ricci symmetric, if $\text{Ric}(\neq 0)$ of $M$ satisfies the condition:

$$\left(\nabla_C Ric\right)(A, B) = 2\kappa(C)Ric(A, B) + \kappa(A)Ric(C, B) + \kappa(B)Ric(C, A),$$

(3.24)

where $\kappa$ is a non-zero 1-form. In particular, $M$ is said to be Ricci symmetric if $\kappa = 0$.

**Theorem 3.9.** If an $\alpha$-CM $M$ admitting $\ast$-$\eta$-RYS is pseudo-Ricci-symmetric, then $M$ is either Ricci symmetric or CM.

**Proof.** The covariant derivative of (3.6) leads

$$\left(\nabla_C Ric\right)(A, B) = \nabla_C [\sigma_1 h(A, B) + \sigma_2 \eta(A)\eta(B)] = \alpha \sigma_2 [h(\Phi A, \Phi C)\eta(B) + \eta(A)h(\Phi B, \Phi C)].$$

(3.25)

Further, we use the relations (3.6), (3.24), (3.25) and obtain

$$2\kappa(C) [\sigma_1 h(A, B) + \sigma_2 \eta(A)\eta(B)] + \kappa(A) [\sigma_1 h(C, B) + \sigma_2 \eta(C)\eta(B)]$$

$$+ \kappa(B) [\sigma_1 h(C, A) + \sigma_2 \eta(C)\eta(A)] = \alpha \sigma_2 [h(\Phi A, \Phi C)\eta(B) + \eta(A)h(\Phi B, \Phi C)].$$

(3.26)

Taking $C = B = \zeta$ in (3.26), we get

$$(\sigma_1 + \sigma_2)(\kappa(A) + 3\eta(A)\kappa(\zeta)) = 0,$$

which gives either

$$\kappa(A) = -3\eta(A)\kappa(\zeta)$$

(3.27)

or

$$\sigma_1 + \sigma_2 = 0.$$  

(3.28)

Putting $A = \zeta$ in (3.27), we have $\kappa(\zeta) = 0$, which further implies that $\kappa(A) = 0$. Also, from (3.28) and (3.2), we can have $\alpha^2(n - 1) = 0$. This implies that $\alpha = 0$ because $n \neq 1$. Thus, we arrive at our desired result.
4 Some curvature properties on $\alpha$-CM admitting $\ast$-$\eta$-RYS

This section deals with the curvature properties on $M$ admitting $\ast$-$\eta$-RYS. We mainly discuss the conditions that $M$ is $\zeta$-concircularly flat, $\zeta$-conharmonically flat, $\zeta$-$W_2$ flat and $\zeta$-conformal flat.

**Theorem 4.1.** Let $M$ be an $n$-dimensional $\alpha$-CM admitting $\ast$-$\eta$-RYS $(h, \zeta, \Lambda, \mu, \rho, q)$, where $\zeta$ being the Reeb vector field on $M$. Then $M$ is

(1) $\zeta$-concircularly flat if and only if $\mu = \alpha - \rho \alpha^2$.

(2) $\zeta$-conformal curvature flat.

(3) $\zeta$-conharmonically flat if and only if $\mu = \alpha + (n - 1)\alpha^2 \rho$.

(4) $\zeta$-$W_2$-curvature flat if and only if $\mu = \alpha - \rho \alpha^2$.

**Proof.** By using the property $h(QA, B) = \text{Ric}(A, B)$ in (3.6), we arrive at

$$QB = \sigma_1 B + \sigma_2 \eta(B)\zeta,$$

where $\sigma_1 = -\frac{1}{\rho} \left( \Lambda - \frac{q^2}{2} - \frac{q \alpha^2 (n-1)^2}{2} + \alpha \right) - \alpha^2 (n-2)$ and $\sigma_2 = -\left( \frac{\mu - \alpha}{\rho} + \alpha^2 \right)$.

Firstly, we put $C = \zeta$ into (2.16) and use the relations (2.4), (2.8) and (3.8), we have

$$\mathcal{C}(A, B)\zeta = \frac{1}{n} \left( \frac{\mu - \alpha}{\rho} + \alpha^2 \right) (\eta(A)B - \eta(B)A),$$

which gives $\mathcal{C}(A, B)\zeta = 0$ if and only if $\mu = \alpha - \rho \alpha^2$.

Secondly, if we put $C = \zeta$ and use (2.8), (2.12), (4.1) in (2.19), then we have

$$\mathcal{C}^*(A, B)\zeta = \left( \frac{\alpha^2 - \sigma_1}{n-2} + \frac{r}{(n-1)(n-2)} \right) (\eta(A)B - \eta(B)A).$$

Again, using the value of $\sigma_1$, (3.2) and (3.8), we have

$$\mathcal{C}^*(A, B)\zeta = 0.$$

Thirdly, we take $C = \zeta$ in (2.17) and make use of (2.8), (4.1) and (2.12), we get

$$H(A, B)\zeta = \left( \frac{\sigma_1 - \alpha^2}{n-2} \right) (\eta(A)B - \eta(B)A).$$

This implies

$$H(A, B)\zeta = 0.$$
if and only if
\[ \sigma_1 = \alpha^2. \]
Thus,
\[ H(A, B)\zeta = 0 \]
if and only if
\[ \mu = \alpha + (n - 1)\alpha^2 \rho. \]
Lastly, by taking \( C = \zeta \) and using (2.8) and (4.1) in (2.18), we conclude
\[
W_2(A, B)\zeta = \left( \alpha^2 + \frac{\sigma_1}{n - 1} \right) (\eta(A)B - \eta(B)A). \quad (4.6)
\]
From (4.6),
\[ W_2(A, B)\zeta = 0 \]
if and only if
\[ \alpha^2 + \frac{\sigma_1}{n - 1} = 0. \]
This further implies that
\[ W_2(A, B)\zeta = 0 \]
if and only if
\[ \mu = \alpha - \rho \alpha^2. \]

**Remark 4.2.** We observe that above results are true only for \( \alpha \)-Kenmotsu manifolds because \( \Lambda \) and \( \mu \) are depending on \( \alpha \). But for \( \alpha = 0 \), one puts in (3.1) \( B = \xi \) obtains
\[ \Lambda + \mu = \frac{1}{2} qr. \]
Then (3.1) implies
\[ \text{Ric} = \frac{\mu}{\rho} (h - \eta \otimes \eta). \]
So, according to the cases \( \mu \) is zero or non-zero, \( M \) is Ricci-flat or \( \eta \)-Einstein for \( \alpha = 0 \). This is consistent with the formula (3.9), when \( N \) is Einstein.

**Remark 4.3.** If \( M \) is a cosymplectic manifold, then we have
\[
\mathcal{C}(A, B)\zeta = - \left( \frac{\mu}{n - \rho} \right) (\eta(B)A - \eta(A)B). \]
and similar relations can be obtained for \( H(A, B)\zeta \) and \( W_2(A, B)\zeta \), while
\[ \mathcal{C}^*(A, B)\zeta = 0. \]
By the above formulas, one has $\mu = 0$ if and only if $\overline{C}(A, B)\zeta = 0$ if and only if $H(A, B)\zeta = 0$ if and only if $W_2(A, B)\zeta = 0$.

Acknowledgments

The authors would like to thank the reviewer for the valuable comments and constructive suggestions.

Conflict of Interest: The authors declare no competing interests.

Funding: Not Applicable.

Data Availability: Not Applicable.

Ethical Conduct: The manuscript is not currently being submitted for publication elsewhere and has not been previously publish.
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