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On a class of fractional $p(x,y)-$Kirchhoff type problems with indefinite weight

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ABSTRACT

This paper is concerned with a class of fractional $p(x,y)-$Kirchhoff type problems with Dirichlet boundary data along with indefinite weight of the following form

$$\begin{cases} M\left(\int_Q \frac{1}{p(x,y)} \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\,dx\,dy\right) \\ \left(-\Delta_{p(x)}\right)^s u(x) + |u(x)|^{q(x)-2}u(x) \\ = \lambda V(x)|u(x)|^{r(x)-2}u(x) & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N \backslash \Omega. \end{cases}$$

By means of direct variational approach and Ekeland's variational principle, we investigate the existence of nontrivial weak solutions for the above problem in case of the competition between the growth rates of functions p and r involved in above problem, this fact is essential in describing the set of eigenvalues of this problem.

RESUMEN

Este artículo estudia una clase de problemas de tipo $p(x,y)-$Kirchhoff fraccionarios con data Dirichlet en el borde junto con un peso indefinido de la siguiente forma

$$\begin{cases} M\left(\int_Q \frac{1}{p(x,y)} \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\,dx\,dy\right) \\ \left(-\Delta_{p(x)}\right)^s u(x) + |u(x)|^{q(x)-2}u(x) \\ = \lambda V(x)|u(x)|^{r(x)-2}u(x) & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N \backslash \Omega. \end{cases}$$

A través del enfoque variacional directo y el principio variacional de Ekeland, investigamos la existencia de soluciones débiles no triviales para el problema anterior en el caso de competencia entre las tasas de crecimiento de las funciones p y r involucradas en el problema. Este hecho es esencial para describir el conjunto de valores propios de este problema.

Keywords and Phrases: Kirchhoff type problems, indefinite weight, Ekeland's variational principle, variable exponent, fractional $p(x,y)-$Laplacian problems.

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1 Introduction

Fractional differential equations have been an area of great interest recently. This is because of both the intensive development of the theory of fractional calculus itself and the applications of such constructions in various scientific fields such as physics, mechanics, chemistry, engineering, etc. In [5], a non-Kirchhoff equation was investigated, which had an indefinite weight function.

In [3], a Kirchhoff-type equation was surveyed, which lacked an indefinite weight function. We combine these equations and, using the methods applied in [3] and [5], open a corridor to an equation that is both Kirchhoff-type and equipped with an indefinite weight function. In this paper, we aim to discuss the existence of a nontrivial solution for a fractional $p(x,y)-$Kirchhoff type eigenvalue problem

$$\begin{cases} M\left(\int_Q \frac{1}{p(x,y)} \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\,dx\,dy\right)\left(-\Delta_{p(x)}\right)^s u(x) + |u(x)|^{q(x)-2}u(x) = \lambda V(x)|u(x)|^{r(x)-2}u(x), & \text{in } \Omega, \\ u = 0, \quad \text{in } \mathbb{R}^N \backslash \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a Lipschitz bounded open domain and $Q := \mathbb{R}^{2N} \backslash (C\Omega \times C\Omega)$ with $C\Omega = \mathbb{R}^N \backslash \Omega$, $N \geq 3$, $p : \overline{Q} \to (1, +\infty)$ is continuous, $q,\ r \in C_+\left(\overline{\Omega}\right)$, $V : \Omega \to \mathbb{R}$ is an indefinite weight function in the sense that it is allowed to change sign in Ω, λ is a positive constant and $s \in (0,1)$ and $M : \mathbb{R}^+ \to \mathbb{R}$ is a continuous function which satisfies the (polynomial growth condition)

($\mathbf{M}_1$): There exist $m_2 \geq m_1 > 0$ and $\alpha > 1$ such that

$$m_1 t^{\alpha-1} \leq M(t) \leq m_2 t^{\alpha-1} \quad \text{for all} \quad t \in \mathbb{R}^+.$$

Here the operator $\left(-\Delta_{p(x)}\right)^s$ is the fractional $p(x)-$Laplacian operator defined as follows

$$\left(-\Delta_{p(x)}\right)^s u(x) = p \cdot v \cdot \int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))}{|x-y|^{N+sp(x,y)}}\,dy, \quad \text{for all } x \in \mathbb{R}^N,$$

where $p \cdot v \cdot$ is a commonly used abbreviation in the principal value sense.

Throughout this paper, we assume that

$$\alpha p(x,x) < q(x) < p_s^*(x) := \frac{Np(x,x)}{N - sp(x,x)}, \quad p(x,y) < \frac{N}{s}, \quad \forall x, y \in \overline{\Omega}, \tag{1.2}$$

where $p_s^*(x)$ is the so-called critical exponent in fractional Sobolev space with variable exponent.

If $s = 1$ problem (1.1) becomes the $p(\cdot)-$Kirchhoff Laplacian problem.

Problem (1.1) is related to the stationary version of the Kirchhoff equation

$$\rho\frac{\partial^2 u}{\partial t^2}-\left(\frac{\rho_0}{h}+\frac{E}{2L}\int_0^L\left|\frac{\partial u}{\partial x}\right|^2dx\right)\frac{\partial^2 u}{\partial x^2}=0, \tag{1.3}$$

which extends the classical D'Alembert's wave equation, by considering the effect of the changing in the length of the string during the vibration. A distinguishing feature of Eq. (1.3) is that the equation contains a nonlocal coefficient $\frac{\rho_0}{h}+\frac{E}{2L}\int_0^L\left|\frac{\partial u}{\partial x}\right|^2dx$ which depends on the average $\frac{1}{2L}\int_0^L\left|\frac{\partial u}{\partial x}\right|^2dx$, and hence the equation is no longer a pointwise identity. The parameters in (1.3) have the following meanings: L is the length of the string, h is the area of the crosssection, E is the Young modulus of the material, ρ is the mass density and ρ_0 is the initial tension.

This paper is organised as follows. In Section 2, we give some definitions and fundamental properties of generalized Lebesgue spaces $L^{q(x)}(\Omega)$ and fractional Sobolev spaces with variable exponent $W^{s,q(x),p(x,y)}(\Omega)$, moreover, we compare the space $W^{s,q(x),p(x,y)}(\Omega)$ with the fractional Sobolev space X and we study the completeness, reflexivity and separability of these spaces. Furthermore, we establish a continuous and compact embedding theorem of these spaces into variable exponent Lebesgue spaces. In Section 3, we discuss the existence of nontrivial weak solutions of problem in sublinear case, when $1<r(x)<p^-$ for all $x\in\overline{\Omega}$. We apply Ekeland's variational principle.

2 Preliminaries

Consider the set

$$C_+\left(\overline{\Omega}\right)=\{h\in C\left(\overline{\Omega}\right):h(x)>1\quad\text{for all } x\in\overline{\Omega}\}.$$

For all $h\in C_+\left(\overline{\Omega}\right)$, we define

$$h^+=\sup_{x\in\overline{\Omega}}h(x)\quad\text{and}\quad h^-=\inf_{x\in\overline{\Omega}}h(x)\quad\text{such that,}\quad 1<h^-\leq h(x)\leq h^+<+\infty. \tag{2.1}$$

For any $h\in C_+\left(\overline{\Omega}\right)$, we define the variable exponent Lebesgue space as

$$L^{h(x)}(\Omega)=\left\{u:\ u\text{ is a measurable real-valued function, }\int_\Omega|u(x)|^{h(x)}dx<+\infty\right\}.$$

This vector space endowed with the Luxemburg norm, which is defined by

$$\|u\|_{L^{h(x)}(\Omega)}=\inf\left\{\lambda>0:\int_\Omega\left|\frac{u(x)}{\lambda}\right|^{h(x)}dx\leq 1\right\}$$

is a separable reflexive Banach space.

Let $\hat{h} \in C_+\left(\overline{\Omega}\right)$ be the conjugate exponent of h, that is, $1/h(x) + 1/\hat{h}(x) = 1$.

Then we have the following Hölder type inequality

$$\left|\int_\Omega uv\,dx\right| \le \left(\frac{1}{h^-} + \frac{1}{\hat{h}^-}\right)\|u\|_{L^{h(x)}(\Omega)}\|v\|_{L^{\hat{h}(x)}(\Omega)} \le 2\|u\|_{L^{h(x)}(\Omega)}\|v\|_{L^{\hat{h}(x)}(\Omega)}$$

Moreover, if $h_1, h_2, h_3 \in C_+\left(\overline{\Omega}\right)$ and $1/h_1 + 1/h_2 + 1/h_3 = 1$, then for any $u \in L^{h_1(x)}(\Omega)$, $v \in L^{h_2(x)}(\Omega)$ and $w \in L^{h_3(x)}(\Omega)$ we have

$$\left|\int_\Omega uvw\,dx\right| \le \left(\frac{1}{{h_1}^-} + \frac{1}{{h_2}^-} + \frac{1}{{h_3}^-}\right)\|u\|_{L^{h_1(x)}(\Omega)}\|v\|_{L^{h_2(x)}(\Omega)}\|w\|_{L^{h_3(x)}(\Omega)}. \tag{2.2}$$

Note that $L^{h_1(x)}(\Omega) \hookrightarrow L^{h_2(x)}(\Omega)$ for all functions h_1 and h_2 in $C_+\left(\overline{\Omega}\right)$ satisfying $h_1(x) \le h_2(x)$ for all $x \in \left(\overline{\Omega}\right)$. In addition this embedding is continuous.

The modular of the $L^{h(x)}(\Omega)$ space is the mapping $\rho_{h(\cdot)} : L^{h(x)}(\Omega) \to \mathbb{R}$ defined by

$$u \mapsto \rho_{h(\cdot)}(u) = \int_\Omega |u(x)|^{h(x)} dx.$$

Proposition 2.1. *Let $u \in L^{h(x)}(\Omega)$, then we have*

(i) $\|u\|_{L^{h(x)}(\Omega)} < 1$ *(resp.* $= 1, > 1$*)* $\iff \rho_{h(\cdot)}(u) < 1$ *(resp.* $= 1, > 1$*),*

(ii) $\|u\|_{L^{h(x)}(\Omega)} < 1 \implies \|u\|^{h^+}_{L^{h(x)}(\Omega)} \le \rho_{h(\cdot)}(u) \le \|u\|^{h^-}_{L^{h(x)}(\Omega)}$,

(iii) $\|u\|_{L^{h(x)}(\Omega)} > 1 \implies \|u\|^{h^-}_{L^{h(x)}(\Omega)} \le \rho_{h(\cdot)}(u) \le \|u\|^{h^+}_{L^{h(x)}(\Omega)}$.

Proposition 2.2. *If $u, u_k \in L^{h(x)}(\Omega)$ and $k \in \mathbb{N}$, then the following assertions are equivalent*

(i) $\lim_{k\to+\infty} \|u_k - u\|_{L^{h(x)}}(\Omega) = 0$,

(ii) $\lim_{k\to+\infty}\rho_{h(\cdot)}(u_k - u) = 0$,

(iii) $u_k \to u$ *in measure in* Ω *and* $\lim_{k\to+\infty}\rho_{h(\cdot)}(u_k) = \rho_{h(\cdot)}(u)$.

From [8, Theorems 1.6 and 1.10], we obtain the following proposition:

Proposition 2.3. *Suppose that* (2.1) *is satisfied. If Ω is a bounded open domain, then $(L^{h(x)}(\Omega)$, $\|u\|_{L^{h(x)}(\Omega)})$ is a reflexive uniformly convex and separable Banach space.*

Proposition 2.4 (see [7])**.** *Let h_1 and h_2 be measurable functions such that $h_1 \in L^\infty(\mathbb{R}^N)$ and $1 \le h_1(x)h_2(x) \le +\infty$ for a.e. $x \in \mathbb{R}^N$. Let $u \in L^{h_2(x)}(\mathbb{R}^N)$, $u \ne 0$. Then we have the following assertions*

$$\|u\|_{L^{h_1(x)h_2(x))}(\mathbb{R}^N)} \le 1 \implies \|u\|^{h_1^+}_{L^{h_1(x)h_2(x))}(\mathbb{R}^N)} \le \||u|^{h_1(x)}\|_{L^{h_2(x))}(\mathbb{R}^N)} \le \|u\|^{h_1^-}_{L^{h_1(x)h_2(x))}(\mathbb{R}^N)},$$

$$\|u\|_{L^{h_1(x)h_2(x))}(\mathbb{R}^N)} \geq 1 \implies \|u\|^{h_1^-}_{L^{h_1(x)h_2(x))}(\mathbb{R}^N)} \leq \||u|^{h_1(x)}\|_{L^{h_2(x))}(\mathbb{R}^N)} \leq \|u\|^{h_1^+}_{L^{h_1(x)h_2(x))}(\mathbb{R}^N)}.$$

In particular, if $h_1(x) = h_1$ is a constant, then it holds that

$$\||u|^{h_1}\|_{L^{h_2(x)}(\mathbb{R}^N)} = \|u\|^{h_1}_{L^{h_1(x)h_2(x)}(\mathbb{R}^N)}.$$

Let Ω be a Lipschitz bounded open set in $\mathbb{R}^N$ and let $p : \overline{\Omega}\times\overline{\Omega} \to (1,+\infty)$ be a continuous bounded function. We assume that

$$1 < p^- := \min_{(x,y)\in\overline{\Omega}\times\overline{\Omega}} p(x,y) \leq p(x,y) \leq p^+ := \max_{(x,y)\in\overline{\Omega}\times\overline{\Omega}} p(x,y) < +\infty, \tag{2.3}$$

and

$$p \text{ is symmetric, that is, } p(x,y) = p(y,x) \quad \text{for all } (x,y) \in \overline{\Omega}\times\overline{\Omega}. \tag{2.4}$$

Set

$$\overline{p}(x) = p(x,x) \quad \text{for any } x \in \overline{\Omega}.$$

Throughout this paper s is a fixed real number such that $0 < s < 1$.

We define the fractional Sobolev space with variable exponent via Gagliardo approach as follows

$$W = W^{s,q(x),p(x,y)}(\Omega) = \left\{ u \in L^{q(x)}(\Omega), \int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^{p(x,y)}}{\lambda^{p(x,y)}|x-y|^{N+sp(x,y)}}\,dx\,dy < +\infty \text{ for some } \lambda > 0 \right\}.$$

The space $W^{s,q(x),p(x,y)}(\Omega)$ is a Banach space if it is equipped with the norm

$$\|u\|_W = \|u\|_{L^{q(x)}(\Omega)} + [u]_{s,p(x,y)},$$

where $[\cdot]_{s,p(x,y)}$ is a Gagliardo seminorm with variable exponent, which is defined by

$$[u]_{s,p(x,y)} = [u]_{s,p(x,y)}(\Omega) := \inf\left\{ \lambda > 0 : \int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^{p(x,y)}}{\lambda^{p(x,y)}|x-y|^{N+sp(x,y)}}\,dx\,dy \leq 1 \right\}.$$

Due to [9, Lemma 3.1], $(W, \|.\|_W)$ is a separable and reflexive Banach space.

Proposition 2.5 (see [9])**.** *Let $\Omega \subset \mathbb{R}^N$ be a Lipschitz bounded domain and $s \in (0,1)$. Let $q(x), p(x,y)$ be continuous variable exponents with $sp(x,y) < N$ for all $(x,y) \in \overline{\Omega}\times\overline{\Omega}$ and $q(x) > p(x,x)$ for all $x \in \overline{\Omega}$. Assume that $r : \overline{\Omega} \to (1,+\infty)$ is a continuous function such that*

$$p_s^*(x) := \frac{Np(x,x)}{N - sp(x,x)} > r(x) \geq r^- > 1$$

for all $x \in \overline{\Omega}$. Then, there exists a constant $c = c(N,s,p,q,r,\Omega)$ such that for every $u \in W =$

$W^{s,q(x),p(x,y)}(\Omega)$, *it holds that*

$$\|u\|_{L^{r(x)}(\Omega)} \leq c\|u\|_W.$$

That is, if $1 < r(x) < p_s^*(x)$ *for all* $x \in \overline{\Omega}$ *then the space* W *is continuously embedded in* $L^{r(x)}(\Omega)$. *Moreover, this embedding is compact.*

It is important to encode the boundary condition $u = 0$ in $\mathbb{R}^N \backslash \Omega$ in the weak formulation. For this purpose, we introduce the new fractional Sobolev space as follows

$$\left\{ \begin{array}{c} u : \mathbb{R}^N \to \mathbb{R} \text{ measurable, such that } u_{|\Omega} \in L^{q(x)}(\Omega) \text{ with} \\ \int_Q \frac{|u(x)-u(y)|^{p(x,y)}}{\lambda^{p(x,y)}|x-y|^{N+sp(x,y)}}\, dx\, dy < +\infty \text{ for some } \lambda > 0 \end{array} \right\},$$

where $p : \overline{Q} \to (1, +\infty)$ satisfies (2.3) and (2.4) on $\overline{Q}$. The space X is endowed with the following norm

$$\|u\|_X = \|u\|_{L^{q(x)}(\Omega)} + [u]_X,$$

where $[u]_X$ is a Gagliardo seminorm with variable exponent, defined by

$$[u]_X = [u]_{s,p(x,y)}(Q) := \inf \left\{ \lambda > 0 : \int_Q \frac{|u(x) - u(y)|^{p(x,y)}}{\lambda^{p(x,y)}|x - y|^{N+sp(x,y)}}\, dx\, dy \leq 1 \right\}.$$

Similar to the space $(W, \|.\|_W)$ we have that $(X, \|.\|_X)$ is a separable reflexive Banach space.

Remark 2.6. *Note that the norms* $\|.\|_X$ *and* $\|.\|_W$ *are not the same, because* $\Omega \times \Omega$ *is strictly contained in* Q. *This makes the fractional Sobolev space with variable exponent* $W = W^{s,q(x),p(x,y)}(\Omega)$ *not sufficient for studying the nonlocal problems.*

Now let X_0 denote the following linear subspace of X

$$X_0 = \left\{ u \in X : u = 0 \text{ a.e. in } \mathbb{R}^N \backslash \Omega \right\},$$

with the norm

$$\|u\|_{X_0} = \|u\|_X = \inf \left\{ \lambda > 0 : \int_Q \frac{|u(x) - u(y)|^{p(x,y)}}{\lambda^{p(x,y)}|x - y|^{N+sp(x,y)}}\, dx\, dy \leq 1 \right\}.$$

It is easy to check that $\|.\|_{X_0}$ is a norm on X_0.

Similar to [3, Theorem 2.2] we have

Theorem 2.7. *Let* Ω *be a Lipschitz bounded domain in* $\mathbb{R}^N$ *and let* $s \in (0, 1)$. *Let* $p : \overline{Q} \to (1, +\infty)$ *be a continuous function satisfying* (2.3) *and* (2.4) *on* $\overline{Q}$ *with* $sp^+ < N$. *Then the following assertions hold:*

(i) *If* $u \in X$, *then* $u \in W$. *Moreover,*

$$\|u\|_W \leq \|u\|_X,$$

(ii) *If* $u \in X_0$, *then* $u \in W^{s,q(x),p(x,y)}(\mathbb{R}^N)$. *Moreover,*

$$\|u\|_W \leq \|u\|_{W^{s,q(x),p(x,y)}(\mathbb{R}^N)} = \|u\|_X,$$

(iii) *If* $r : \overline{\Omega} \to (1,+\infty)$ *be a continuous variable exponent such that*

$$1 < r^- \leq r(x) < p_s^*(x) = \frac{N\overline{p}(x)}{N - s\overline{p}(x)} \quad \text{for all } x \in \overline{\Omega},$$

then, there exists a constant $C = C(N,s,p,q,r,\Omega) > 0$ *such that, for any* $u \in W$,

$$\|u\|_{L^{r(x)}(\Omega)} \leq C\|u\|_X.$$

That is, the space X *is continuously embedded in* $L^{r(x)}(\Omega)$. *Moreover, this embedding is compact.*

Remark 2.8. *(i) The assertion (iii) in Theorem 2.7 remains true if we replace* X *by* X_0.

(ii) Since by (1.2) *we have* $1 < q^- \leq q(x) < p_s^*(x)$ *for all* $x \in \overline{\Omega}$. *then by Theorem 2.7 (iii) we have that* $\|\cdot\|_{X_0} = [\cdot]_X$ *and* $\|\cdot\|_X$ *are equivalent on* X_0.

Definition 2.9. *Let* $p : \overline{Q} \to (1,+\infty)$ *be a continuous variable exponent and let* $s \in (0,1)$, *we define the modular* $\rho_{p(.,.)} : X_0 \to \mathbb{R}$, *by*

$$\rho_{p(.,.)}(u) = \int_Q \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\,dx\,dy.$$

Then $\|u\|_{\rho_{p(.,.)}} = \inf\left\{\lambda > 0 : \rho_{p(.,.)}\left(\frac{u}{\lambda}\right) \leq 1\right\} = [u]_X$.

The modular $\rho_{p(.,.)}$ checks the following result, which is similar to [2, Proposition 2.1 and Lemma 2.2].

Lemma 2.10. *Let* $p : \overline{Q} \to (1,+\infty)$ *be a continuous variable exponent and let* $s \in (0,1)$, *for any* $u \in X_0$, *we have*

(i) $1 \leq \|u\|_{X_0} \implies \|u\|_{X_0}^{p^-} \leq \rho_{p(.,.)}(u) \leq \|u\|_{X_0}^{p^+}$,

(ii) $\|u\|_{X_0} \leq 1 \implies \|u\|_{X_0}^{p^+} \leq \rho_{p(.,.)}(u) \leq \|u\|_{X_0}^{p^-}$.

Remark 2.11. *Note that* $\rho_{p(.,.)}$ *satisfies the results of Proposition 2.2.*

Similar to [3, Lemma 2.3] we have

Lemma 2.12. *$(X_0, \|.\|_{X_0})$ is a separable, reflexive, and uniformly convex Banach space.*

Let denote by $\mathcal{L}$ the operator associated to the $\left(-\Delta_{p(x)}\right)^s$ defined as

$$\mathcal{L} : X_0 \to X_0^*, \quad u \mapsto \mathcal{L}(u) : X_0 \to \mathbb{R}, \quad \varphi \mapsto \langle \mathcal{L}(u), \varphi \rangle$$

such that

$$\langle \mathcal{L}(u), \varphi \rangle = \int_Q \frac{|u(x) - u(y)|^{p(x,y)-2}(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x - y|^{N+sp(x,y)}}\, dx\, dy,$$

where X_0^* is the dual space of X_0.

Lemma 2.13 (see [4]). *Under the conditions of Proposition 2.5, the following assertions hold true:*

(i) *$\mathcal{L}$ is a bounded and strictly monotone operator.*

(ii) *$\mathcal{L}$ is a mapping of type (s_+), that is, if $u_k \rightharpoonup u$ in X_0 and $\limsup_{k\to+\infty} \langle \mathcal{L}(u_k) - \mathcal{L}(u), u_k - u \rangle \leq 0$, then $u_k \to u$ in X_0.*

(iii) *$\mathcal{L}$ is a homeomorphism.*

Throughout this paper, for simplicity, we use c_i to denote the general nonnegative or positive constant (the exact value may change from line to line).

3 The main result and proof of the theorem

Definition 3.1. *We say that $u \in X_0$ is a weak solution of problem (1.1) if*

$$M(\sigma_{p(x,y)}(u)) \int_Q \frac{|u(x) - u(y)|^{p(x,y)-2}((u(x) - u(y))(\varphi(x) - \varphi(y))}{|x - y|^{N+sp(x,y)}}\, dx\, dy \tag{3.1}$$

$$+ \int_\Omega |u(x)|^{q(x)-2} u(x)\varphi(x) dx - \lambda \int_\Omega V(x)|u(x)|^{r(x)-2} u(x)\varphi(x) dx = 0$$

for all $\varphi \in X_0$, where

$$\sigma_{p(x,y)}(u) = \int_Q \frac{1}{p(x,y)} \frac{|u(x) - u(y)|^{p(x,y)}}{|x - y|^{N+sp(x,y)}}\, dx\, dy.$$

Let us consider the Euler-Lagrange functional associated to (1.1), defined by

$$\begin{aligned}\mathcal{J}_\lambda : X_0 \to \mathbb{R}, \quad \mathcal{J}_\lambda(u) &= \widehat{M}\left(\int_Q \frac{1}{p(x,y)} \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\, dx\, dy\right) \\ &\quad + \int_\Omega \frac{1}{q(x)}|u(x)|^{q(x)}dx - \lambda \int_\Omega \frac{V(x)}{r(x)}|u(x)|^{r(x)}dx \\ &= \widehat{M}\left(\sigma_{p(x,y)}(u)\right) + \int_\Omega \frac{1}{q(x)}|u(x)|^{q(x)}dx - \lambda \int_\Omega \frac{V(x)}{r(x)}|u(x)|^{r(x)}dx,\end{aligned}$$

where $\widehat{M}(t) = \int_0^t M(\tau)d\tau$.

Theorem 3.2. *Under the same assumptions of Theorem 2.7, if we assume that* (M_1) *holds and* $\sigma, r \in C_+(\overline{\Omega})$ *satisfy the following conditions:*

(H_1) $1 < r^- \le r(x) \le r^+ < p^- \le p^+ < \frac{N}{s} < \sigma(x)$ *for all* $x \in \overline{\Omega}$,

(H_2) $V \in L^{\sigma(x)}(\Omega)$ *and there exists a measurable set* $\Omega_0 \subset\subset \Omega$ *of positive measure such that* $V(x) > 0$ *for all* $x \in \Omega_0$.

Then there exists $\bar{\lambda} > 0$ *such that any* $\lambda \in (0, \bar{\lambda})$ *is an eigenvalue of problem* (1.1).

Proof. For each $\lambda > 0$, let us consider the functional $\mathcal{J}_\lambda : X_0 \to \mathbb{R}$ associated with problem (1.1) by the formula

$$\mathcal{J}_\lambda(u) = \Phi(u) - \lambda\Psi(u),$$

where

$$\Phi(u) = \widehat{M}(\sigma_{p(x,y)}(u)) + \int_\Omega \frac{1}{q(x)}|u(x)|^{q(x)}dx, \quad \Psi(u) = \frac{V(x)}{r(x)}|u(x)|^{r(x)}dx.$$

From conditions $(H_1) - (H_2)$ and Proposition 2.4, for all $u \in X_0$, we get

$$|\Phi(u)| \le \frac{2}{r^-}\|V\|_{L^{\sigma(x)}(\Omega)}\||u|^{r(x)}\|_{L^{\sigma(x)/(\sigma(x)-1)}(\Omega)}$$

$$\le \begin{cases} \frac{2}{r^-}\|V\|_{L^{\sigma(x)}(\Omega)}\|u\|^{r^-}_{L^{\sigma(x)r(x)/(\sigma(x)-1)}(\Omega)} & \text{if } \|u\|_{L^{\sigma(x)r(x)/(\sigma(x)-1)}(\Omega)} \le 1, \\ \frac{2}{r^-}\|V\|_{L^{\sigma(x)}(\Omega)}\|u\|^{r^+}_{L^{\sigma(x)r(x)/(\sigma(x)-1)}(\Omega)} & \text{if } \|u\|_{L^{\sigma(x)r(x)/(\sigma(x)-1)}(\Omega)} \ge 1. \end{cases} \tag{3.2}$$

We also deduce from (H_1) that $\beta(x) = \sigma(x)r(x)/(\sigma(x)-r(x)) < p_s^*(x)$ and $\gamma(x) = \sigma(x)r(x)/(\sigma(x)-1) < p_s^*(x)$ for all $x \in \overline{\Omega}$. In view of (Theorem 2.7 (iii) and Remark 2.8 (i)) the embeddings $X_0 \hookrightarrow L^{\beta(x)}(\Omega)$ and $X_0 \hookrightarrow L^{\gamma(x)}(\Omega)$ are continuous and compact. Thus, the functional $\mathcal{J}_\lambda$ is well-defined on X_0. The proof of Theorem 3.2 is divided into following four steps.

Step 1. We show that $\mathcal{J}_\lambda \in C^1(X_0, \mathbb{R})$ and its derivative is

$$\langle \mathcal{J}'_\lambda(u), \varphi \rangle = M(\sigma_{p(x,y)}(u)) \int_Q \frac{|u(x)-u(y)|^{p(x,y)-2}((u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+sp(x,y)}}\, dx\, dy$$
$$+ \int_\Omega |u(x)|^{q(x)-2} u(x)\varphi(x) dx - \lambda \int_\Omega V(x)|u(x)|^{r(x)-2} u(x)\varphi(x) dx$$

for all $u, \varphi \in X_0$. This means that weak solutions for problem (1.1) can be found as the critical points of the functional $\mathcal{J}_\lambda$ in the space X_0.

Using the same method as in the proof of [1, Lemma 4.1] and [6, Lemma 3.1] and the continuity of M, we can show that $\Phi \in C^1(X_0, \mathbb{R})$ and

$$\langle \Phi'(u), \varphi \rangle = M(\sigma_{p(x,y)}(u)) \int_Q \frac{|u(x)-u(y)|^{p(x,y)-2}((u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+sp(x,y)}}\, dx\, dy$$
$$+ \int_\Omega |u(x)|^{q(x)-2} u(x)\varphi(x) dx$$

for all $u, \varphi \in X_0$.

Also it has been proved by Chung in [5] that $\Psi \in C^1(X_0, \mathbb{R})$ and

$$\langle \Psi'(u), \varphi \rangle = \int_\Omega V(x)|u(x)|^{r(x)-2} u(x)\varphi(x) dx, \quad \forall u, \varphi \in X_0$$

and thus Step 1 is completed.

Step 2. We prove that there exists $\bar{\lambda} > 0$ such that for any $\lambda \in (0, \bar{\lambda})$, there exist constants $R, \rho > 0$ such that $\mathcal{J}_\lambda(u) \geq R$ for all $u \in X_0$ with $\|u\|_{X_0} = \rho$.

Indeed, since $\gamma(x) = \sigma(x)r(x)/(\sigma(x)-1) < p_s^*(x)$ for all $x \in \overline{\Omega}$, the embedding $X_0 \hookrightarrow L^{\gamma(x)}(\Omega)$ is continuous and there exists $c_2 > 0$ such that

$$\|u\|_{L^{\gamma(x)}(\Omega)} \leq c_2 \|u\|_{x_0}, \quad \forall u \in X_0.$$

Hence, by relation (3.2), for any $u \in X_0$ with $\|u\| = \rho$ small enough,

$$\begin{aligned}
\mathcal{J}_\lambda(u) &= \widehat{M}(\sigma_{p(x,y)}(u)) + \int_\Omega \frac{1}{q(x)} |u(x)|^{q(x)} dx - \lambda \int_\Omega \frac{V(x)}{r(x)} |u(x)|^{r(x)} dx \\
&\geq \frac{m_1}{\alpha(p^+)^\alpha} \|u\|_{X_0}^{\alpha p^+} - \lambda \frac{2c_2^{r^-}}{r^-} \|V\|_{L^{\sigma(x)}(\Omega)} \|u\|_{X_0}^{r^-} = \frac{m_1}{\alpha(p^+)^\alpha} \rho^{\alpha p^+} - \lambda \frac{2c_2^{r^-}}{r^-} \|V\|_{L^{\sigma(x)}(\Omega)} \rho^{r^-} \\
&= \rho^{r^-} \left(\frac{m_1}{\alpha(p^+)^\alpha} \rho^{\alpha p^+ - r^-} - \lambda \frac{2c_2^{r^-}}{r^-} \|V\|_{L^{\sigma(x)}(\Omega)} \right).
\end{aligned}$$

Putting

$$\bar{\lambda} = \frac{m_1}{2\alpha(p^+)^\alpha} \rho^{\alpha p^+ - r^-} \cdot \frac{r^-}{2c_2^{r^-} \|V\|_{L^{\sigma(x)}(\Omega)}} > 0,$$

for any $\lambda \in (0,\bar{\lambda})$ and $u \in X_0$ with $\|u\| = \rho$, there exists $R = \frac{m_1 \rho^{\alpha p^+}}{2\alpha(p^+)^\alpha}$ such that $\mathcal{J}_\lambda(u) \geq R > 0$.

Step 3. We prove that there exists $\varphi_0 \in X_0$ such that $\varphi_0 \geq 0$, $\varphi_0 \neq 0$ and $\mathcal{J}_\lambda(t\varphi_0) < 0$ for all $t > 0$ small enough.

Indeed, condition (H_1) implies that $r(x) < \min\{p^-, q^-\} = p^-$ for all $x \in \overline{\Omega_0}$. In the sequel, we use the notation $r_0^- = \inf_{x\in\Omega_0} r(x)$. Let $\varepsilon_0 > 0$ be such that $r_0^- + \varepsilon_0 < p^-$. We also have since $r \in C(\overline{\Omega_0})$ that there exists an open subset $\Omega_1 \subset \Omega_0$ such that

$$|r(x) - r_0^-| < \varepsilon_0, \quad \forall x \in \Omega_1$$

and thus

$$r(x) \leq r_0^- + \varepsilon_0 < p^- < \alpha p^-, \quad \forall x \in \Omega_1.$$

Let $\varphi_0 \in C_0^\infty(\Omega_0)$ such that $\overline{\Omega_1} \subset \mathrm{supp}(\varphi_0)$, $\varphi_0(x) = 1$ for all $x \in \overline{\Omega_1}$ and $0 \leq \varphi_0 \leq 1$ in Ω_0. Then, using the above information and assumption(M_1), for any $t \in (0,1)$ we have

$$\begin{aligned}
\mathcal{J}_\lambda(t\varphi_0) &= \widehat{M}\left(\sigma_{p(x,y)}(t\varphi_0)\right) + \int_\Omega \frac{1}{q(x)}|t\varphi_0|^{q(x)}dx - \lambda\int_\Omega \frac{V(x)}{r(x)}|t\varphi_0|^{r(x)}dx \\
&\leq \frac{m_2}{\alpha}\left(\sigma_{p(x,y)}(t\varphi_0)\right)^\alpha + \frac{t^{q^-}}{q^-}\int_{\Omega_0}|\varphi_0|^{q(x)}dx - \lambda\int_{\Omega_0}\frac{V(x)}{r(x)}t^{r(x)}|\varphi_0|^{r(x)}dx \\
&\leq \frac{m_2}{\alpha(p^-)^\alpha}t^{\alpha p^-}\left(\rho_{p(.,.)}(\varphi_0)\right)^\alpha + \frac{t^{q^-}}{q^-}\int_{\Omega_0}|\varphi_0|^{q(x)}dx - \frac{\lambda t^{r_0^-+\varepsilon_0}}{r_0^+}\int_{\Omega_1}V(x)|\varphi_0|^{r(x)}dx \\
&\leq kt^{\alpha p^-}\left(\left(\rho_{p(.,.)}(\varphi_0)\right)^\alpha + \int_{\Omega_0}|\varphi_0|^{q(x)}dx\right) - \frac{\lambda t^{r_0^-+\varepsilon_0}}{r_0^+}\int_{\Omega_1}V(x)|\varphi_0|^{r(x)}dx
\end{aligned}$$

where

$$k = \max\left\{\frac{m_2}{\alpha(p^-)^\alpha}, \frac{1}{q^-}\right\}.$$

Therefore

$$\mathcal{J}_\lambda(t\varphi_0) < 0 \quad \text{for } 0 < t < \delta^{1/(\alpha p^- - r_0^- - \varepsilon_0)}$$

with

$$0 < \delta < \min\left\{1, \frac{\lambda}{kr_0^+}\cdot\frac{\int_{\Omega_1}V(x)|\varphi_0|^{r(x)}dx}{\left(\rho_{p(.,.)}(\varphi_0)\right)^\alpha + \int_\Omega|\varphi_0|^{q(x)}dx}\right\}.$$

The above fraction is meaningful if we can show that

$$\left(\rho_{p(.,.)}(\varphi_0)\right)^\alpha + \int_\Omega|\varphi_0|^{q(x)}dx > 0.$$

Since $\varphi_0(x) = 1$ for all $x \in \overline{\Omega_1}$, we have

$$\int_\Omega|\varphi_0|^{q(x)}dx > 0.$$

Thus, the above fraction is meaningful.

Indeed, it is clear that

$$\int_{\Omega_1} |\varphi_0|^{r(x)} dx \leq \int_{\Omega} |\varphi_0|^{r(x)} dx \leq \int_{\Omega} |\varphi_0|^{r^-} dx.$$

On the other hand, the space X_0 is continuously embedded in $L^{r^-}(\Omega)$ and thus, there exists $c_3 > 0$ such that $\|\varphi_0\|_{L^{r^-}(\Omega)} \leq c_3 \|\varphi_0\|_{X_0}$, which implies that $\|\varphi_0\|_{X_0} > 0$. Thus, Step 3 is completed.

By Step 2 we have

$$\inf_{u \in \partial B_\rho(0)} \mathcal{J}_\lambda(u) > 0.$$

We also deduce from Step 2 that, the functional $\mathcal{J}_\lambda$ is bounded from below on $B_\rho(0)$. Moreover, by Step 3, there exists $\varphi \in X$ such that $\mathcal{J}_\lambda(t\varphi) < 0$ for all $t > 0$ small enough.

It follows from Step 2 that

$$\mathcal{J}_\lambda(u) \geq \frac{m_1}{\alpha(p^+)^\alpha} \|u\|_{X_0}^{\alpha p^+} - \lambda \frac{2c_2^{r^-}}{r^-} \|V\|_{L^{\sigma(x)}(\Omega)} \|u\|_{X_0}^{r^-},$$

which yields

$$-\infty < \underline{c}_\lambda = \inf_{u \in \overline{B_\rho(0)}} \mathcal{J}_\lambda(u) < 0.$$

Let us choose $\varepsilon > 0$ such that

$$0 < \varepsilon < \inf_{u \in \partial B_\rho(0)} \mathcal{J}_\lambda(u) - \inf_{u \in \overline{B_\rho(0)}} \mathcal{J}_\lambda(u).$$

Applying the Ekeland variational principle [7] to the functional $\mathcal{J}_\lambda : \overline{B_\rho(0)} \to \mathbb{R}$, it follows that there exists $u_\varepsilon \in \overline{B_\rho(0)}$

$$\mathcal{J}_\lambda(u_\varepsilon) < \inf_{u \in \overline{B_\rho(0)}} \mathcal{J}_\lambda(u) + \varepsilon, \;\; \mathcal{J}_\lambda(u_\varepsilon) < \mathcal{J}_\lambda(u) + \varepsilon \|u - u_\varepsilon\|_{X_0}, \quad u \neq u_\varepsilon,$$

then we infer that

$$\mathcal{J}_\lambda(u_\varepsilon) < \inf_{u \in \partial B_\rho(0)} \mathcal{J}_\lambda(u)$$

and thus

$$u_\varepsilon \in B_\rho(0).$$

Let us consider the functional

$$I_\lambda : \overline{B_\rho(0)} \to \mathbb{R} \quad \text{by} \quad I_\lambda(u) = \mathcal{J}_\lambda(u) + \varepsilon \|u - u_\varepsilon\|_{X_0}.$$

Then u_ε is a minimum point of I_λ and thus

$$\frac{I_\lambda(u_\varepsilon+\tau\varphi)-I_\lambda(u_\varepsilon)}{\tau}\geq 0$$

for all $\tau>0$ small enough and $\varphi\in B_\rho(0)$. The above information shows that

$$\frac{\mathcal{J}_\lambda(u_\varepsilon+\tau\varphi)-\mathcal{J}_\lambda(u_\varepsilon)}{\tau}+\varepsilon\|\varphi\|_{X_0}\geq 0.$$

Letting $\tau\to 0^+$, we deduce that

$$\langle \mathcal{J}'_\lambda(u_\varepsilon),\varphi\rangle+\varepsilon\|\varphi\|_{X_0}\geq 0$$

and we infer that

$$\|\mathcal{J}'_\lambda(u_\varepsilon)\|_{X_0^*}\leq\varepsilon.$$

Therefore, there exists a sequence $\{u_n\}\subset B_\rho(0)$ such that

$$\mathcal{J}_\lambda(u_n)\to\underline{c}_\lambda=\inf_{u\in\overline{B_\rho(0)}}\mathcal{J}_\lambda(u)<0 \quad\text{and}\quad \mathcal{J}'_\lambda(u_n)\to 0 \quad\text{in}\quad X_0^* \quad\text{as}\quad n\to\infty. \tag{3.3}$$

It is clear that the sequence $\{u_n\}$ is bounded in X_0. Now, since X_0 is a reflexive Banach space, there exists $u\in X_0$ such that passing to a subsequence, still denoted by $\{u_n\}$, it converges weakly to u in X_0.

Step 4. We prove that $\{u_n\}$ which is given by (3.3) converges strongly to u in X_0, *i.e.*, $\lim\limits_{n\to+\infty}\|u_n-u\|_{X_0}=0$.

By conditions $(H_1)-(H_2)$, using Hölder's inequality (2.2) and Propositions 2.4 and 2.5 we deduce that

$$\begin{aligned}\left|\int_\Omega |u_n|^{q(x)-2}u_n(u_n-u)dx\right| &\leq 2\||u_n|^{q(x)-2}u_n\|_{L^{q(x)/(q(x)-1)}(\Omega)}\|u_n-u\|_{L^{q(x)}(\Omega)}\\ &\leq 2\|u_n\|^{q^+-1}_{L^{q(x)}(\Omega)}\|u_n-u\|_{L^{q(x)}(\Omega)}\to 0 \quad\text{as}\quad n\to\infty,\end{aligned}$$

and

$$\begin{aligned}\left|\int_\Omega V(x)|u_n|^{r(x)-2}u_n(u_n-u)dx\right| &\leq 3\|V\|_{L^{\sigma(x)}(\Omega)}\||u_n|^{r(x)-2}u_n\|_{L^{r(x)/(r(x)-1)}(\Omega)}\|u_n-u\|_{L^{\beta(x)}(\Omega)}\\ &\leq 3\|V\|_{L^{\sigma(x)}(\Omega)}\left(1+\|u_n\|^{r^+-1}_{L^{r(x)}(\Omega)}\right)\|u_n-u\|_{L^{\beta(x)}(\Omega)}\to 0 \quad\text{as}\quad n\to\infty,\end{aligned}$$

where $\beta(x)=\sigma(x)r(x)/(\sigma(x)-r(x))$. Moreover, by (3.3) we have $\lim_{n\to\infty}\langle\mathcal{J}'_\lambda(u_n),u_n-u\rangle=0$, *i.e.*,

$$M(\sigma_{p(x,y)}(u_n))\mathcal{I}_Q(u_n) + \int_\Omega |u_n|^{q(x)-2}u_n(u_n-u)dx \\ - \lambda \int_\Omega V(x)|u_n|^{r(x)-2}u_n(u_n-u)dx \to 0 \quad \text{as} \quad n \to \infty,$$

which yields

$$M(\sigma_{p(x,y)}(u_n))\mathcal{I}_Q(u_n) \to 0 \tag{3.4}$$

where

$$\mathcal{I}_Q(u_n) = \int_Q \frac{|u_n(x)-u_n(y)|^{p(x,y)-2}((u_n(x)-u_n(y))((u_n(x)-u(x))-(u_n(y)-u(y)))}{|x-y|^{N+sp(x,y)}}\,dx\,dy$$

Since $\{u_n\}$ is bounded in X_0, passing to subsequence, if necessary, we may assume that

$$\sigma_{p(x,y)}(u_n) \overset{n\to+\infty}{\longrightarrow} t_1 \geq 0.$$

If $t_1 = 0$, then $\{u_n\}$ converge strongly to $u = 0$ in X_0, then by (3.3), we obtain

$$\lim_{n\to+\infty} \mathcal{J}_\lambda(u_n) = \mathcal{J}_\lambda(u) = \mathcal{J}_\lambda(0) = 0 = \underline{c}_\lambda < 0.$$

That is a contradiction, thus $t_1 > 0$.

Since the function M is continuous, we have

$$M\left(\sigma_{p(x,y)}(u_n)\right) \overset{n\to+\infty}{\longrightarrow} M\left(t_1\right) > 0.$$

Hence, by (M_1), for n large enough, we get

$$0 < c_4 < M\left(\sigma_{p(x,y)}(u_n)\right) < c_5. \tag{3.5}$$

Combining (3.4) and (3.5), we deduce

$$\lim_{n\to+\infty} \mathcal{I}_Q(u_n) = 0.$$

Using the above information, Lemma 2.13 (ii) and the fact that $u_n \rightharpoonup u$ in X_0, we get

$$\begin{cases} \limsup_{n\to+\infty} \langle \mathcal{L}(u_n), u_n - u\rangle \leq 0, \\ u_n \rightharpoonup u \quad \text{in} \quad X_0, & \Longrightarrow u_n \to u \quad \text{in} \quad X_0. \\ \mathcal{L} \quad \text{is a mapping of type } (S_+). \end{cases}$$

Thus, in view of (3.3), we obtain

$$\mathcal{J}_\lambda(u) = \underline{c}_\lambda < 0 \quad \text{and} \quad \mathcal{J}'_\lambda(u) = 0.$$

This means that u is a non-trivial weak solution of (1.1), *i.e.*, any $\lambda \in (0, +\infty)$ is an eigenvalue of problem (1.1). Theorem 3.2 is completely proved. □

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