Cubo A Mathematical Journal

CUBO, A Mathematical Journal
Vol. 26, no. 01, pp. 153–166, April 2024
DOI: 10.56754/0719-0646.2601.153

Quarter-symmetric metric connection on a p-Kenmotsu manifold

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ABSTRACT

In the present paper we study para-Kenmotsu (p-Kenmotsu) manifold equipped with quarter-symmetric metric connection and discuss certain derivation conditions.

RESUMEN

En el presente artículo estudiamos variedades para-Kenmotsu (p-Kenmotsu) equipadas con conexiones métricas cuarto-simétricas y discutimos ciertas condiciones derivadas.

Keywords and Phrases: Para-Kenmotsu manifold, quarter-symmetric metric connection, curvature tensor, η-Einstein manifold.

2020 AMS Mathematics Subject Classification: 53C15, 53C25.

Published: 10 April, 2024
Accepted: 27 February, 2024
Received: 13 September, 2023

1 Introduction

Kenmotsu in 1971, introduced a class of almost contact Riemannian manifolds satisfying some special conditions, called Kenmotsu manifold [10]. Many researchers including U.C. De and R. N. Singh studied some properties of Kenmotsu manifolds endowed with various conditions [2, 3, 9, 15]. Sato [13] in 1976, introduced the notion of an almost para-contact structure on Riemannian manifolds which is similar to the almost contact structure on Riemannian manifolds. In 1995, B. B. Sinha and K. L. Sai Prasad [16] defined a class of almost para contact metric manifolds analogous to the class of Kenmotsu manifolds, known as para-Kenmotsu (p-Kenmotsu) manifolds. T. Satyanarayana *et al.* [14] studied curvature properties in a p-Kenmotsu manifold.

Friedmann and Schouten in 1924 [6], presented the idea of semi-symmetric connection on a differentiable manifold. Yano introduced semi-symmetric metric connection in 1970 using the idea of metric connection given by Hayden in 1932. M. M. Tripathi [19] and Tang *et al.* [18] studied semi-symmetric metric connection in a Kenmotsu manifold. A linear connection $\bar{\nabla}$ on a Riemannian manifold M is said to be a semi- symmetric connection if the torsion tensor T given by

$$T(X,Y) = \bar{\nabla}_X Y - \bar{\nabla}_Y X - [X,Y]$$

satisfies

$$T(X,Y) = \eta(Y)X - \eta(X)Y,$$

where η is a 1-form and $g(X,\xi) = \eta(X)$, ξ is a vector field and for all vector fields $X, Y \in \chi(M)$, $\chi(M)$ is the set of all differentiable vector fields on M.

Gołąb [7] in 1975 studied quarter-symmetric metric connection in differentiable manifolds with affine connections. Further S. C. Biswas, U. C. De and many others [1, 4, 5, 17] studied quarter-symmetric metric connection in Riemannian manifolds equipped with various structures. A quarter-symmetric connection is considered as a generalisation of semi-symmetric connection since its torsion tensor T satisfies

$$T(X,Y) = \eta(Y)\phi X - \eta(X)\phi Y,$$

where ϕ is a $(1,1)$ tensor field. If quarter-symmetric connection $\bar{\nabla}$ satisfies the condition

$$(\bar{\nabla}_X g)(Y,Z) = 0,$$

where $X, Y, Z \in \chi(M)$, then $\bar{\nabla}$ is said to be a quarter-symmetric metric connection. Let M be an n-dimensional Riemannian manifold and ∇ be its Levi-Civita connection. The Riemannian curvature tensor R, the concircular curvature tensor W, Weyl projective curvature tensor P of M

are defined by [11, 12]

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \tag{1.1}$$

$$W(X,Y)Z = R(X,Y)Z - \frac{r}{n(n-1)}[g(Y,Z)X - g(X,Z)Y], \tag{1.2}$$

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{n-1}[S(Y,Z)X - S(X,Z)Y], \tag{1.3}$$

where $X, Y, Z \in \chi(M)$ and r is the scalar curvature.

The paper is organised as follows: In section 2, a brief introduction of p-Kenmotsu manifolds is given. In section 3, the relation between the curvature tensors of Riemannian connection and the quarter-symmetric metric connection in a p-Kenmotsu manifold is obtained. The study of a p-Kenmotsu manifold with respect to the quarter-symmetric metric connection satisfying the curvature condition $\bar{R} \cdot \bar{S}$ is contained in section 4. In section 5, we study ϕ-concircularly flat p-Kenmotsu manifold with respect to quarter-symmetric metric connection. The curvature condition $\bar{P} \cdot \bar{S} = 0$ and ϕ-Weyl projective flat p-Kenmotsu manifold with respect to quarter-symmetric metric connection are respectively studied in the sections 6 and 7. Finally we give an example of a 5-dimensional p-Kenmotsu manifold.

2 Preliminaries

Let M be a $(2n+1)$-dimensional differentiabe manifold endowed with an almost para-contact structure (ϕ, ξ, η), where ϕ is a $(1,1)$-tensor field, ξ is a vector field, and η is a 1-form on M, then

$$\phi^2 X = X - \eta(X)\xi, \quad \eta(\xi) = 1. \tag{2.1}$$

$$\phi(\xi) = 0, \quad \eta(\phi X) = 0, \quad \text{rank } (\phi) = 2n. \tag{2.2}$$

where X is a vector field on M. The manifold M endowed with (ϕ, ξ, η) is called an almost para-contact manifold [13].

Let g be a Riemannian metric on M compatible to the structure (ϕ, ξ, η), *i.e.*, the following equations are satisfied

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \quad g(X,\xi) = \eta(X), \tag{2.3}$$

for all vector fields X and Y on M. Then the manifold M is said to admit an almost para-contact Riemannian structure (ϕ, ξ, η, g).

If moreover, (ϕ, ξ, η, g) satisfy the following conditions

$$(\nabla_X \eta)Y = g(X,Y) - \eta(X)\eta(Y), \tag{2.4}$$

$$\nabla_X \xi = X - \eta(X)\xi = \phi^2(X), \tag{2.5}$$

$$(\nabla_X \phi)Y = -g(\phi X, Y)\xi - \eta(Y)\phi X, \tag{2.6}$$

then M is called a para-Kenmotsu (p-Kenmotsu) manifold [16].

In a p-Kenmotsu manifold the following relations hold [16]:

$$S(\phi X, \phi Y) = S(X,Y) + (n-1)\eta(X)\eta(Y), \tag{2.7}$$

$$S(X,\xi) = -(n-1)\eta(X), \quad \text{where } g(QX,Y) = S(X,Y), \tag{2.8}$$

$$\eta(R(X,Y)Z) = g(X,Z)\eta(Y) - g(Y,Z)\eta(X), \tag{2.9}$$

$$R(\xi, X)Y = \eta(Y)X - g(X,Y)\xi, \tag{2.10}$$

$$R(X,Y)\xi = \eta(X)Y - \eta(Y)X, \tag{2.11}$$

where S is the Ricci tensor and Q is the symmetric endomorphism of the tangent space at each point corresponding to the Ricci tensor and R is the Riemannian curvature.

If the Ricci curvature tensor S is of the form

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y), \tag{2.12}$$

then M is called η-Einstein manifold and if $b = 0$ then it is said to be Einstein manifold. M is called generalized η-Einstein manifold, if S is of the form

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y) + cg(\phi X, Y), \tag{2.13}$$

where a, b, c are scalar functions on M.

In a p-Kenmotsu manifold M, the connection $\bar{\nabla}$ given by

$$\bar{\nabla}_X Y = \nabla_X Y + \eta(Y)\phi X - g(\phi X, Y)\xi \tag{2.14}$$

is a quarter-symmetric metric connection [8].

3 Curvature tensor of para-Kenmotsu manifold with respect to the quarter-symmetric metric connection

Let M be a p-Kenmotsu manifold. The curvature tensor $\bar{R}$ of a p-Kenmotsu manifold with respect to the quarter-symmetric metric connection $\bar{\nabla}$ is defined by

$$\bar{R}(X,Y)Z = \bar{\nabla}_X\bar{\nabla}_Y Z - \bar{\nabla}_Y\bar{\nabla}_X Z - \bar{\nabla}_{[X,Y]}Z.$$

Using equations (2.1)-(2.6) and (2.13) we get

$$\begin{aligned}\bar{R}(X,Y)Z = R(X,Y)Z &+ g(X,Z)\phi Y - g(Y,Z)\phi X + g(\phi X,Z)Y - g(\phi Y,Z)X\\ &+ g(\phi X,Z)\phi Y - g(\phi Y,Z)\phi X,\end{aligned} \tag{3.1}$$

where R is the Riemannian curvature tensor of the connection ∇ given in (1.1).

Now from (3.1), we have

$$\bar{R}(X,Y)Z + \bar{R}(Y,Z)X + \bar{R}(Z,X)Y = 0, \tag{3.2}$$

or equivalently

$$\bar{R}(X,Y,Z,W) + \bar{R}(Y,Z,X,W) + \bar{R}(Z,X,Y,W) = 0, \tag{3.3}$$

where $\bar{R}(X,Y,Z,W) = g(\bar{R}(X,Y)Z,W)$. Thus the curvature tensor with respect to the quarter-symmetric metric connection satisfies the Bianchi first identity. Taking inner product of (3.1) with respect to W, we get

$$\begin{aligned}\bar{R}(X,Y,Z,W) = R(X,Y,Z,W) &+ g(X,Z)g(\phi Y,W) - g(Y,Z)g(\phi X,W)\\ &+ g(\phi X,Z)g(Y,W) - g(\phi Y,Z)g(X,W)\\ &+ g(\phi X,Z)g(\phi Y,W) - g(\phi Y,Z)g(\phi X,W).\end{aligned} \tag{3.4}$$

Contracting (3.4) over X and W, we get

$$\bar{S}(Y,Z) = S(Y,Z) + (1-2n-\psi)g(\phi Y,Z) + (1-\psi)g(Y,Z) - \eta(Y)\eta(Z), \tag{3.5}$$

where $\psi = trace\ \phi$, S and $\bar{S}$ are the Ricci tensors with respect to the connections ∇ and $\bar{\nabla}$ respectively on M. Now contracting (3.5), we have

$$\bar{r} = r + 2n(1-2\psi) - \psi^2, \tag{3.6}$$

where r and $\bar{r}$ denote the scalar curvatures with respect to the connections ∇ and $\bar{\nabla}$ respectively on M. Now we state the following theorem.

Theorem 3.1. *For a p-Kenmotsu manifold M with respect to the quarter-symmetric metric connection $\bar{\nabla}$*

(1) *The curvature tensor $\bar{R}$ satisfies the Bianchi first identity and is given by*

$$\begin{aligned}\bar{R}(X,Y)Z &= R(X,Y)Z + g(X,Z)\phi Y - g(Y,Z)\phi X \\ &\quad + g(\phi X,Z)Y - g(\phi Y,Z)X + g(\phi X,Z)\phi Y - g(\phi Y,Z)\phi X.\end{aligned}$$

(2) *The Ricci tensor $\bar{S}$ is given by*

$$\bar{S}(Y,Z) = S(Y,Z) + (1-2n-\psi)g(\phi Y,Z) + (1-\psi)g(Y,Z) - \eta(Y)\eta(Z).$$

(3) *The relation between r and $\bar{r}$, respectively the scalar curvatures with respect to ∇ and $\bar{\nabla}$, is given by*

$$\bar{r} = r + 2n(1-2\psi) - \psi^2.$$

Proof. The proof follows from the equations (3.1), (3.2), (3.3), (3.5) and (3.6). □

Some properties of the curvature tensor with respect to the quarter- symmetric metric connection are given in the following lemma.

Lemma 3.2. *In a $(2n+1)$-dimensional p-Kenmotsu manifold with the structure (ϕ,ξ,η,g) with respect to the quarter-symmetric metric connection, the following hold*

$$\bar{R}(X,Y)\xi = \eta(X)Y - \eta(Y)X + \eta(X)\phi Y - \eta(Y)\phi X, \tag{3.7}$$

$$\bar{R}(\xi,Y)Z = \eta(Z)Y + \eta(Z)\phi Y - g(Y,Z)\xi - g(\phi Y,Z)\xi, \tag{3.8}$$

$$\bar{R}(\xi,Y)\xi = Y + \phi Y - \eta(Y)\xi, \tag{3.9}$$

$$\bar{S}(Y,\xi) = (1-n-\psi)\eta(Y), \tag{3.10}$$

$$\bar{S}(\xi,\xi) = (1-n-\psi). \tag{3.11}$$

4 p-Kenmotsu manifold satisfying $\bar{R} \cdot \bar{S} = 0$.

In this section we consider a p-Kenmotsu manifold with respect to the quarter-symmetric metric connection $\bar{\nabla}$ satisfying

$$\bar{R}(X,Y) \cdot \bar{S} = 0.$$

This equation implies

$$\bar{S}(\bar{R}(X,Y)U,V) + \bar{S}(U,\bar{R}(X,Y)V) = 0 \tag{4.1}$$

where $X, Y, U, V \in \chi(M)$. Putting $X = \xi$ in (4.1), we have

$$\bar{S}(\bar{R}(\xi,Y)U,V) + \bar{S}(U,\bar{R}(\xi,Y)V) = 0 \tag{4.2}$$

By the equations (3.5), (3.8) and (3.10), equation (4.2) yields

$$\begin{gathered}\eta(U)\bar{S}(Y,V) + \eta(U)\bar{S}(\phi Y,V) - (1-n-\psi)g(Y,U)\eta(V) - (1-n-\psi)g(\phi Y,U)\eta(V)\\ +\eta(V)\bar{S}(U,Y) + \eta(V)\bar{S}(\phi Y,U) - (1-n-\psi)g(Y,V)\eta(U) - (1-n-\psi)g(\phi Y,V)\eta(U) = 0.\end{gathered}$$

Putting $U = \xi$ and using (2.1) and (2.2), it follows that

$$\bar{S}(Y,V) + \bar{S}(\phi Y,V) = (1-n-\psi)g(Y,V) + (1-n-\psi)g(\phi Y,V). \tag{4.3}$$

Making use of (3.5), (4.3) takes form

$$S(Y,V) + S(\phi Y,V) = (\psi+n-1)g(Y,V) + (2-2n-\psi)\eta(Y)\eta(V) + (\psi+n-1)g(\phi Y,V). \tag{4.4}$$

Therefore we have the following theorem:

Theorem 4.1. *If a p-Kenmotsu manifold with respect to the quarter-symmetric metric connection satisfying the condition $\bar{R} \cdot \bar{S} = 0$, then the Ricci tensor S of the manifold satisfies*

$$S(X,Y) + S(\phi X,Y) = (\psi+n-1)g(X,Y) + (2-2n-\psi)\eta(X)\eta(Y) + (\psi+n-1)g(\phi X,Y).$$

5 ϕ-concircularly flat p-Kenmotsu manifolds with respect to the quarter-symmetric metric connection

Analogous to the definition of the concircular curvature given in (1.2), $\bar{W}$, the concircular curvature with respect to quarter-symmetric metric connection is given by

$$\bar{W}(X,Y)Z = \bar{R}(X,Y)Z - \frac{\bar{r}}{n(n-1)}[g(Y,Z)X - g(X,Z)Y]. \tag{5.1}$$

A p-Kenmotsu manifold is said to be ϕ-concircularly flat with respect to the quarter-symmetric metric connection if

$$\bar{W}(\phi X, \phi Y, \phi Z, \phi W) = 0, \tag{5.2}$$

where $X, Y, Z, W \in \chi(M)$.

Taking inner-product of (5.1) with respect to U and replacing X by ϕX, Y by ϕY, Z by ϕZ and U by ϕU, we get

$$\bar{R}(\phi X, \phi Y, \phi Z, \phi W) = \frac{\bar{r}}{n(n-1)}[g(\phi Y, \phi Z)g(\phi X, \phi W) - g(\phi X, \phi Z)g(\phi Y, \phi W)].$$

In view of (3.1) and (3.6), (5.3) takes the form

$$\begin{aligned} R(\phi X, \phi Y, \phi Z, \phi W) &= g(\phi Y, \phi Z)g(X, \phi W) - g(\phi X, \phi Z)g(Y, \phi W) + g(Y, \phi Z)g(\phi X, \phi W) \\ &- g(X, \phi Z)g(\phi Y, \phi W) - g(X, \phi Z)g(Y, \phi W) + g(Y, \phi Z)g(X, \phi W) \\ &+ \frac{r + 2n(1-2\psi) - \psi^2}{n(n-1)}\Big[g(\phi Y, \phi Z)g(\phi X, \phi W) - g(\phi X, \phi Z)g(\phi Y, \phi W)\Big]. \end{aligned} \tag{5.3}$$

Let $\{e_1, e_2, \ldots, e_{2n}, e_{2n+1} = \xi\}$ be a local orthonormal ϕ-basis of vector fields in M, so that $\{\phi e_1, \phi e_2, \ldots, \phi e_{2n}, \xi\}$ is also a local orthonormal basis in M. Putting $X = W = e_i$ in the last equation and summing over i, we get

$$\begin{aligned} S(Y,Z) &= \frac{(r + 2n(1-2\psi) - \psi^2)(2n-1) + n(n-1)\psi}{n(n-1)} g(Y,Z) \\ &- \frac{(r + 2n(1-2\psi) - \psi^2)(2n-1) + n(n-1)(n-1+\psi)}{n(n-1)} \eta(Y)\eta(Z) \\ &- (2 - 2n - \psi)g(\phi Y, Z). \end{aligned} \tag{5.4}$$

Thus we state the following theorem:

Theorem 5.1. *A ϕ-concircularly flat p-Kenmotsu manifold with respect to the quarter-symmetric metric connection is a generalized η-Einstein manifold with the scalar curvature r given by* (5.4).

6 p-Kenmotsu manifold satisfying $\bar{P} \cdot \bar{S} = 0$ with respect to quarter-symmetric metric connection.

Analogous to (1.3), the Weyl projective curvature $\bar{P}$ with respect to quarter-symmetric metric connection is given by

$$\bar{P}(X,Y)Z = \bar{R}(X,Y)Z - \frac{1}{n-1}[\bar{S}(Y,Z)X - \bar{S}(X,Z)Y].$$

Using (3.1) and (3.5), this equation implies

$$\begin{aligned}\bar{P}(X,Y)Z = {} & R(X,Y)Z + g(X,Z)\phi Y - g(Y,Z)\phi X + g(\phi X,Z)Y \\ & - g(\phi Y,Z)X + g(\phi X,Z)\phi Y - g(\phi Y,Z)\phi X - \frac{1}{n-1}\Big[S(Y,Z)X \\ & + (1-2n-\psi)g(\phi Y,Z)X + (1-\psi)g(Y,Z)X - \eta(Y)\eta(Z)X \\ & - S(X,Z)Y - (1-2n-\psi)g(\phi X,Z)Y - (1-\psi)g(X,Z)Y + \eta(X)\eta(Z)Y\Big].\end{aligned} \tag{6.1}$$

From the equation (6.1), we have the following properties of the Weyl projective curvature $\bar{P}$.

$$\begin{aligned}\bar{P}(\xi,Y)Z = {} & \eta(Y)Z - g(Y,Z)\xi + \eta(Z)\phi Y - g(\phi Y,Z)\xi - \frac{1}{n-1}\Big[S(Y,Z)\xi \\ & + (1-2n-\psi)g(\phi Y,Z)\xi + (1-\psi)g(Y,Z)\xi - \eta(Y)\eta(Z)\xi - S(\xi,Z)Y \\ & - (1-\psi)\eta(Z)Y + \eta(Z)Y\Big].\end{aligned} \tag{6.2}$$

and

$$\bar{P}(\xi,Y)\xi = Y - \eta(Y)\xi + \phi Y - \frac{1}{n-1}\Big[(1-\psi-n)\eta(Y)\xi + (\psi+n-1)Y\Big]. \tag{6.3}$$

Now, we consider a p-Kenmotsu manifold satisfying the curvature condition

$$\bar{P}(X,Y) \cdot \bar{S} = 0,$$

which is equivalent to

$$\bar{S}(\bar{P}(X,Y)U,V) + \bar{S}(U,\bar{P}(X,Y)V) = 0.$$

The last equation implies

$$\bar{S}(\bar{P}(\xi,Y)\xi,V) + \bar{S}(\xi,\bar{P}(\xi,Y)V) = 0. \tag{6.4}$$

Using equation (6.2) and (6.3) in (6.4), we once again get the equation (4.4). Therefore we have the following theorem:

Theorem 6.1. *For a $(2n+1)$-dimensional p-Kenmotsu manifold with respect to the quarter-symmetric metric connection satisfying the condition $\bar{P}\cdot\bar{S}=0$, the Ricci tensor S satisfies*

$$S(X,Y)+S(\phi X,Y)=(\psi+n-1)g(\phi X,Y)+(\psi+n-1)g(X,Y)+(2-2n-\psi)\eta(X)\eta(Y).$$

7 ϕ-Weyl projective flat p-Kenmotsu manifolds with respect to the quarter-symmetric metric connection.

A p-Kenmotsu manifold is said to be ϕ-Weyl projective flat with respect to the quarter-symmetric metric connection if

$$\bar{P}(\phi X,\phi Y,\phi Z,\phi U)=0, \tag{7.1}$$

where $X,Y,Z,U\in\chi(M)$. Taking inner-product of (6.1) with respect to U and replacing X by ϕX, Y by ϕY, Z by ϕZ and U by ϕU, we get

$$\begin{aligned}\bar{P}(\phi X,\phi Y,\phi Z,\phi U)=\bar{R}(\phi X,\phi Y,\phi Z,\phi U)-\frac{1}{n-1}\Big[&S(\phi Y,\phi Z)g(\phi X,\phi U)\\&+(1-2n-\psi)g(Y,\phi Z)g(\phi X,\phi U)+(1-\psi)g(\phi X,\phi Z)\\&-S(\phi X,\phi Z)g(\phi Y,\phi U)-(1-2n-\psi)g(X,\phi Z)g(\phi Y,\phi U)\\&+g(\phi X,\phi U)-(1-\psi)g(\phi X,\phi Z)g(\phi Y,\phi U)\Big].\end{aligned} \tag{7.2}$$

Using (3.1), (7.1) in (7.2), we obtain

$$\begin{aligned}R(\phi X,\phi Y,\phi Z,\phi W)=&-g(\phi X,\phi Z)g(Y,\phi W)+g(\phi Y,\phi Z)g(X,\phi W)-g(X,\phi Z)g(\phi Y,\phi W)\\&+g(Y,\phi Z)g(\phi X,\phi W)-g(X,\phi Z)g(Y,\phi W)+g(Y,\phi Z)g(X,\phi W)\\&+\frac{1}{n-1}\Big[S(\phi Y,\phi Z)g(\phi X,\phi U)+(1-2n-\psi)g(Y,\phi Z)g(\phi X,\phi U)\\&+(1-\psi)g(\phi X,\phi Z)g(\phi X,\phi U)-S(\phi X,\phi Z)g(\phi Y,\phi U)\\&-(1-2n-\psi)g(X,\phi Z)g(\phi Y,\phi U)-(1-\psi)g(\phi X,\phi Z)g(\phi Y,\phi U)\Big].\end{aligned}$$

Let $\{e_1,e_2,\ldots,e_{2n},\xi\}$ be a local orthonormal ϕ-basis of vector fields in M, putting $X=W=e_i$ in the last equation and summing over i, we get

$$S(Y,Z)=\frac{(2n-\psi n-1)}{2(1-n)}g(Y,Z)-\frac{(n^2-3n+n\psi+1)}{2(n-1)}\eta(Y)\eta(Z)+\frac{(2n^2+3n+2\psi-3)}{2(n-1)}g(\phi Y,Z).$$

Thus we state the following theorem:

Theorem 7.1. *If a p-Kenmotsu manifold is ϕ-Weyl projective flat with respect to the quarter-symmetric metric connection, it is a generalized η-Einstein manifold.*

8 Example

Example 8.1. *Consider the 5-dimensional manifold* $M = \{(u, v, x, y, z) \in R^5\}$ *with standard coordinates* (u, v, x, y, z) *in* R^5*. Then the following vector fields*

$$e_1 = z\frac{\partial}{\partial u}, \quad e_2 = z\frac{\partial}{\partial v}, \quad e_3 = z\frac{\partial}{\partial x}, \quad e_4 = z\frac{\partial}{\partial y}, \quad e_5 = -\frac{\partial}{\partial z}$$

are linearly independent at each point of M*. Suppose* g *be the Riemannian metric defined by,*

$$g(e_i, e_j) = \delta_{ij} = \begin{cases} 1 & if\ i = j \\ 0 & if\ i \neq j;\ i, j = 1, 2, 3, 4, 5. \end{cases}$$

Let ϕ *be the tensor field of type* $(1, 1)$ *defined by*

$$\phi(e_1) = e_2, \quad \phi(e_2) = e_1, \quad \phi(e_3) = e_4, \quad \phi(e_4) = e_3, \quad \phi(e_5) = 0,$$

and η *be the* 1*-form defined by* $\eta(X) = g(X, e_5)$*. Using the linearity of* ϕ *and* g*, we have*

$$\eta(e_5) = 1, \quad \phi^2 X = X - \eta(X)e_5, \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),$$

for any vector fields $X, Y \in \chi(M)$*. If we take* $e_5 = \xi$*, the structure* (ϕ, ξ, η, g) *is an almost para-contact Riemannian structure on* M*. Then we have,*

$$\begin{aligned} &[e_1, e_2] = 0, \quad [e_1, e_3] = 0, \quad [e_1, e_4] = 0, \quad [e_1, e_5] = e_1, \quad [e_2, e_3] = 0, \\ &[e_2, e_4] = 0, \quad [e_2, e_5] = e_2, \quad [e_3, e_4] = 0, \quad [e_3, e_5] = e_3, \quad [e_4, e_5] = e_4. \end{aligned}$$

Using Koszul's formula, we obtain the Levi-Civita connection ∇ *of the metric tensor* g *as follows:*

$$\begin{array}{lllll} \nabla_{e_1} e_1 = -e_5, & \nabla_{e_1} e_2 = 0, & \nabla_{e_1} e_3 = 0, & \nabla_{e_1} e_4 = 0, & \nabla_{e_1} e_5 = e_1, \\ \nabla_{e_2} e_1 = 0, & \nabla_{e_2} e_2 = -e_5, & \nabla_{e_2} e_3 = 0, & \nabla_{e_2} e_4 = 0, & \nabla_{e_2} e_5 = e_2, \\ \nabla_{e_3} e_1 = 0, & \nabla_{e_3} e_2 = 0, & \nabla_{e_3} e_3 = -e_5, & \nabla_{e_3} e_4 = 0, & \nabla_{e_3} e_5 = e_3, \\ \nabla_{e_4} e_1 = 0, & \nabla_{e_4} e_2 = 0, & \nabla_{e_4} e_3 = 0, & \nabla_{e_4} e_4 = -e_5, & \nabla_{e_4} e_5 = e_4, \\ \nabla_{e_5} e_1 = 0, & \nabla_{e_5} e_2 = 0, & \nabla_{e_5} e_3 = 0, & \nabla_{e_5} e_4 = 0, & \nabla_{e_5} e_5 = 0. \end{array}$$

Above relations show that equations (2.4)*-*(2.6) *are satisfied. Therefore the manifold is a p-Kenmotsu manifold with the structure* (ϕ, ξ, η, g)*.*

Using (2.14), *we get the quarter symmetric metric connection*

$$
\begin{array}{lllll}
\bar{\nabla}_{e_1}e_1=-e_5, & \bar{\nabla}_{e_1}e_2=-e_5, & \bar{\nabla}_{e_1}e_3=0, & \bar{\nabla}_{e_1}e_4=0, & \bar{\nabla}_{e_1}e_5=e_1+e_2,\\
\bar{\nabla}_{e_2}e_1=-e_5, & \bar{\nabla}_{e_2}e_2=-e_5, & \bar{\nabla}_{e_2}e_3=0, & \bar{\nabla}_{e_2}e_4=0, & \bar{\nabla}_{e_2}e_5=e_1+e_2,\\
\bar{\nabla}_{e_3}e_1=0, & \bar{\nabla}_{e_3}e_2=0, & \bar{\nabla}_{e_3}e_3=-e_5, & \bar{\nabla}_{e_3}e_4=-e_5, & \bar{\nabla}_{e_3}e_5=e_3+e_4,\\
\bar{\nabla}_{e_4}e_1=0, & \bar{\nabla}_{e_4}e_2=0, & \bar{\nabla}_{e_4}e_3=-e_5, & \bar{\nabla}_{e_4}e_4=-e_5, & \bar{\nabla}_{e_4}e_5=e_3+e_4,\\
\bar{\nabla}_{e_5}e_1=0, & \bar{\nabla}_{e_5}e_2=0, & \bar{\nabla}_{e_5}e_3=0, & \bar{\nabla}_{e_5}e_4=0, & \bar{\nabla}_{e_5}e_5=0.
\end{array}
$$

Now we obtain non-zero components of their curvature tensors:

$$
\begin{array}{llll}
R(e_1,e_2)e_1 \;=e_2, & R(e_1,e_3)e_1 \;=e_3, & R(e_1,e_4)e_1 \;=e_4, & R(e_1,e_5)e_1 \;=e_5,\\
R(e_2,e_1)e_2 \;=e_1, & R(e_2,e_3)e_2 \;=e_3, & R(e_2,e_4)e_2 \;=e_4, & R(e_2,e_5)e_2 \;=e_5,\\
R(e_3,e_1)e_3 \;=e_1, & R(e_3,e_2)e_3 \;=e_2, & R(e_3,e_4)e_3 \;=e_4, & R(e_3,e_5)e_3 \;=e_5,\\
R(e_4,e_1)e_4 \;=e_2, & R(e_4,e_2)e_4 \;=e_2, & R(e_4,e_3)e_4 \;=e_3, & R(e_4,e_5)e_4 \;=e_5.
\end{array}
$$

and

$$
\begin{array}{lll}
\bar{R}(e_1,e_3)e_1 \;=e_3+e_4, & \bar{R}(e_1,e_4)e_1 \;=e_3+e_4, & \bar{R}(e_1,e_5)e_1 \;=e_5,\\
\bar{R}(e_2,e_3)e_2 \;=e_3+e_4, & \bar{R}(e_2,e_4)e_2 \;=e_3+e_4, & \bar{R}(e_2,e_5)e_2 \;=e_5,\\
\bar{R}(e_3,e_1)e_3 \;=e_1+e_2, & \bar{R}(e_3,e_2)e_3 \;=e_1+e_2, & \bar{R}(e_3,e_5)e_3 \;=e_5,\\
\bar{R}(e_4,e_1)e_2 \;=e_1+e_2, & \bar{R}(e_4,e_2)e_4 \;=e_1+e_2, & \bar{R}(e_4,e_5)e_4 \;=e_5,\\
\bar{R}(e_5,e_1)e_5 \;=e_1+e_2, & \bar{R}(e_5,e_2)e_5 \;=e_1+e_2, & \bar{R}(e_5,e_3)e_5 \;=e_3+e_4,\\
\bar{R}(e_5,e_4)e_5 \;=e_3+e_4. & &
\end{array}
$$

From the above results, it is easy to find the following non-zero components of Ricci tensors:

$$
S(e_1,e_1)=S(e_2,e_2)=S(e_3,e_3)=S(e_4,e_4)=S(e_5,e_5)=-4,
$$

and

$$
\bar{S}(e_1,e_1)=\bar{S}(e_1,e_2)=\bar{S}(e_2,e_2)=\bar{S}(e_3,e_3)=\bar{S}(e_3,e_4)=-3, \quad \bar{S}(e_4,e_4)=-3, \quad \bar{S}(e_5,e_5)=-4.
$$

Therefore, we get $r=-20$ *and* $\bar{r}=-16$. *Hence the statement of Theorem 3.1 is verified. Also by the relations mentioned above, the results in sections 5 and 6 are easily verified.*

Acknowledgement

The first author is thankful to the Department of Science and Technology (DST), New Delhi for the financial support in the form of JRF (IF200486).

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