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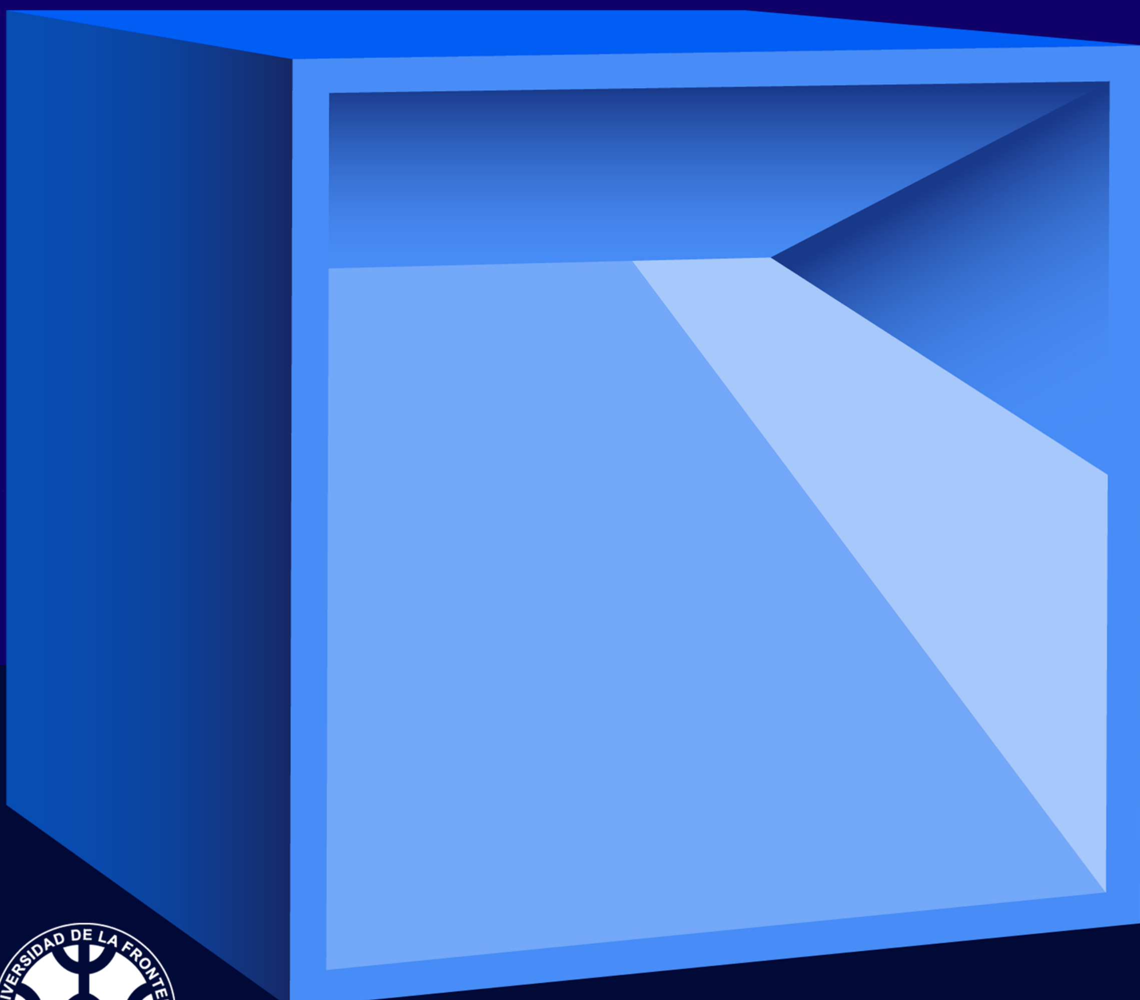

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SUMMARY

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An approach to F. Riesz representation Theorem

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ABSTRACT

In this note we give a direct proof of the F. Riesz representation theorem which characterizes the linear functionals acting on the vector space of continuous functions defined on a set K. Our start point is the original formulation of Riesz where K is a closed interval. Using elementary measure theory, we give a proof for the case K is an arbitrary compact set of real numbers. Our proof avoids complicated arguments commonly used in the description of such functionals.

RESUMEN

En esta nota, damos una demostración directa del teorema de representación de F. Riesz que caracteriza los funcionales lineales actuando en el espacio vectorial de funciones continuas definidas en un conjunto K. Nuestro punto de partida es la formulación original de Riesz, donde K es un intervalo cerrado. Usando teoría elemental de la medida, damos una demostración para el caso en que K es un conjunto arbitrario compacto de números reales. Nuestra demostración evita argumentos complicados comúnmente usados en la descripción de dichos funcionales.

Keywords and Phrases: Riesz representation theorem, positive linear functionals, Riemann-Stieltjes integral.

2010 AMS Mathematics Subject Classification: 46E11,46E15,28A25.

1 Introduction

The Riesz representation theorem is a remarkable result which describes the continuous linear functionals acting on the space of continuous functions defined on a set K. It is very surprising that all these functionals are just integrals and vice versa. In case K is a closed interval of real numbers, any such functional is represented by Riemann-Stieltjes integral, which is a generalization of the usual Riemann integral. This was first announced by F. Riesz in 1909 [14]. In case K is compact set (not necessarily a closed interval), then a more general concept of integral is needed, because the Riemann-Stieltjes integral used by Riesz is defined only for functions on intervals. In this work, we prove that there is a short path between the two cases.

Besides its aesthetic appeal, the above mentioned theorem has far-reaching applications. It allows a short proof of the Kolmogoroff consistency theorem, see [3] thm 10.6.2., and can be used to give an elegant proof of the spectral theorem for selfadjoint bounded operators, see section VII.2 of [13]. Both these theorems are main results in probability and functional analysis respectively. Moreover, the entire theory of integration for general spaces can be recovered using the theorem of Riesz. See for example [19], where the Lebesgue measure on $\mathbb{R}^n$ is constructed. More generally it can also be used to show the existence of the Haar measure on a group, see [3] chap. 9.

In this note we give a short proof of the Riesz representation theorem for the case K is an arbitrary compact set of real numbers, see Theorem 3.1 below. This is interesting because in many situations we have a compact set which is not a closed interval. To prove the spectral theorem, for example, one considers the set of continuous functions defined on the spectrum of selfadjoint bounded operator, which is a compact set of $\mathbb{R}$, but not necessarily a closed interval. We get our result starting from the nondecreasing function that appears in the Riemann-Stieltjes integral representation of Riesz original formulation. To this function we associate a measure which is used to integrate over general compact sets. Then we show how this Lebesgue integral representation can be seen as a Riemann-Stieltjes integral. Our proof is new, avoids technical arguments which appear frequently in proofs of Riesz theorem, it is elementary, direct and quite simple.

2 Preliminaries

Let us introduce first some definitions and notations we shall use.

2.1 Definitions and notation.

Let $\mathcal{C}(K) := \{f : K \to \mathbb{R} : f \text{ continuous}\}$ where K is a compact subset of $\mathbb{R}$, the real numbers. A *functional* is an assignment $L : \mathcal{C}(K) \to \mathbb{R}$. The functional is linear if $L(c_1 f + c_2 g) = c_1 L(f) + c_2 L(g)$ for all $f, g \in \mathcal{C}(K), c_1, c_2 \in \mathbb{R}$. It is continuous if there exists a fixed $M > 0$ such that $|Lf| \leq M\|f\|_\infty$

for all $f \in \mathcal{C}(K)$, where $\|\cdot\|_\infty$ denotes the uniform norm, that is, $\|f\|_\infty = \sup\{|f(x)| : x \in K\}$. We define the norm of such functional as

$$\|L\|_{\mathcal{C}(K)} = \|L\| = \sup\{|L(f)| : f \in \mathcal{C}(K) \text{ and } \|f\|_\infty \leq 1\},$$

We denote the set of the *linear continuous functionals* on $\mathcal{C}(K)$ by $\mathcal{C}(K)^*$. It is called the *dual space*. In general, the dual of normed linear space X is denoted by X^*. A functional L on $\mathcal{C}(K)$ is said to be a *positive* if $L(f) \geq 0$ whenever $f(x) \geq 0$ for every $x \in \mathbb{R}$. We use the notation $\mathcal{C}(K)^*_+$ for the set of *positive linear functionals* on $\mathcal{C}(K)$.

The function $\alpha : [a, b] \longrightarrow \mathbb{R}$ is said to be *normalized*, if $\alpha(a) = 0$ and $\alpha(t) = \alpha(t+)$, $a < t < b$, that is, α is continuous from the right inside the interval (not at a! If it were right continuous at a, theorem (2.1) would not hold for the functional $L(f) = f(a)$). The total variation of a monotone increasing function α is defined as $V(\alpha) = \alpha(b) - \alpha(a)$. We denote the characteristic function of a set $A \subset [a, b]$ by $\mathbf{1}_A$ where $\mathbf{1}_A(x) = 1$ if $x \in A$ and 0 if $x \in [a, b] \setminus A$.

2.2 Representation theorem for functionals on $\mathcal{C}[a, b]$.

We formulate the above-mentioned result by F. Riesz as follows:

Theorem 2.1. *Let $L : \mathcal{C}[a, b] \longrightarrow \mathbb{R}$ be a positive linear functional. There exists a unique normalized monotone function $\alpha : [a, b] \longrightarrow \mathbb{R}$ such that*

$$Lf = \int_a^b f(x) d\alpha(x). \tag{2.1}$$

The integral is understood in the sense of Riemann-Stieltjes. Moreover $\|L\| = V(\alpha)$.

The Riemann-Stieltjes integral is a generalization of the Riemann integral, where instead of taking the length of the intervals, a α-weighted length is taken. For an interval I the α-length is given by $\alpha(I) = \alpha(y) - \alpha(x)$, where x, y are the end points of I and α is a function of finite variation. The integral of a continuous function f on $[a, b]$ is defined as the limit, when it exists, of the sum $\sum_i f(c_i)\alpha(I_i)$ where $\{I_i\}$ is a finite collection of subintervals whose endpoints form a partition of $[a, b]$ and $c_i \in I_i$. See [17] p.122.

There are different proofs of the above theorem, see for example [22]. Here we will give a sketch of the proof which uses the following result about extensions of functionals known as the Hahn-Banach theorem:

Let X a normed linear space, Y a subspace of X, and λ an element of Y^*. Then there exists a $\Lambda \in X^*$ extending λ with the same norm. See [13] for a proof.

Proof of theorem 2.1. . We may assume that $[a, b] = [0, 1]$. Since $L \in \mathcal{C}[0, 1]^*$ we use Hahn-Banach theorem to conclude the existence of $\Lambda \in B[0, 1]^*$ such that $\|\Lambda\| = \|L\|$ and $L = \Lambda$ on $\mathcal{C}[0, 1]$ and where $B[0,1]$ is the set of bounded functions on $[0,1]$.

Let us define the functions $\mathbf{1}_x := \mathbf{1}_{[0,x]}$, that is $\mathbf{1}_x(t) = 1$ when $t \in [0,x]$ and zero otherwise. Set $\alpha(x) = \Lambda(\mathbf{1}_x)$ for all $x \in [0,1]$.

Now for $f \in \mathcal{C}[0,1]$, define

$$f_n = \sum_{j=1}^{n} f(j/n)(\mathbf{1}_{j/n} - \mathbf{1}_{(j-1)/n}).$$

Since f is continuous, it is uniformly continuous on $[0,1]$ and so $\|f_n - f\|_\infty \to 0$. Thus

$$\lim_n \Lambda(f_n) = \Lambda(f) = L(f).$$

Using the definition of α we get

$$\Lambda(f_n) = \sum_{j=1}^{n} f(j/n)(\alpha(j/n) - \alpha((j-1)/n)).$$

This in turn implies

$$\Lambda(f) = \lim_n \Lambda(f_n) = \int_0^1 f\, d\alpha.$$

Now to see that $\|L\| = V(\alpha)$:

Let $\varepsilon > 0$ and choose $f \in \mathcal{C}[0,1]$ such that $\|f\|_\infty \leq 1$ and $\|L\| \leq |L(f)| + \varepsilon$, we apply (2.1) and we get

$$\|L\| \ \leq |L(f)| + \varepsilon = \left| \int_0^1 f(x) d\alpha(x) \right| \ + \varepsilon \leq \alpha(1) - \alpha(0) + \varepsilon = V(\alpha) + \varepsilon.$$

It is possible to normalize α and in this case we easily have the other inequality, that is,

$$V(\alpha) = \alpha(1) - \alpha(0) = \alpha(1) = \Lambda(\mathbf{1}_1) \leq \|\Lambda\| = \|L\|.$$

$\square$

Remarks.

(1) The standard textbook's proof uses Hahn-Banach's theorem ([10],[22]), but the original proof of F. Riesz does not use it. See [17] section 50 and [15],[16].

(2) E. Helly [8] should have similar results. J. Radon extended theorem 2.1 to compact subsets $K \subset \mathbb{R}^n$ [12]. S. Banach and S. Saks extended the result to compact metric spaces, see appendix of [21] and [20]. The proof by S. Saks is particularly elegant and clean. For compact Hausdorff spaces the theorem was proven by S. Kakutani [9] and for normal spaces by A. Markoff [11]. Nowadays this theorem is also known as Riesz-Markoff or Riesz-Markoff-Kakutani theorem. There is a great variety of proofs of F. Riesz theorem using different methods and even category theory see [7]. Our proof only uses basic knowledge of measure theory. More information on the history of this theorem can be found in [5] p. 231, the references therein, [23] p.238 and [6].

(3) Positivity of a linear functional L implies continuity of L. To see it, we take the function $\mathbf{1}(x) = 1$ for all $x \in K$, then $\mathbf{1} \in \mathcal{C}(K)$ and $|f(x)| \le \|f\|_\infty \mathbf{1}(x)$, therefore

$$\|f\|_\infty \mathbf{1}(x) \pm f(x) \ge 0 \qquad \text{implies} \qquad \|f\|_\infty L(\mathbf{1}) \pm L(f) \ge 0$$

so $|L(f)| \le L(\mathbf{1})\|f\|_\infty$. See [5] Prop. 7.1.

3 Main Result

Next theorem is our main result. It is a generalization of Theorem 2.1 to continuous functions defined on arbitrary compact sets $K \subset \mathbb{R}$. Since an ordinary Riemann-Stieltjes integral is not defined for functions on general compact K, we shall introduce the Lebesgue integral which makes sense for such functions. In the Appendix, we collect the basic facts and definitions of measure theory we need.

Theorem 3.1. *Let K a compact subset of $\mathbb{R}$ and let $\ell : \mathcal{C}(K) \to \mathbb{R}$ be a positive linear functional. Then, there is a unique finite Borel measure μ such that $\mu(K) = \|\ell\|_{\mathcal{C}(K)^*}$ and*

$$\ell f = \int_K f d\mu. \tag{3.1}$$

Proof. The proof proceeds in stages.

i) *Integral representation.* Let $[a, b]$ be a closed and bounded interval containing K. Note that the technique used in what follows is independent of this interval. Let $r : \mathcal{C}[a, b] \longrightarrow \mathcal{C}(K)$ be the restriction operator, that is, for every $f \in \mathcal{C}[a, b]$, $r(f)(x) = f(x)$ for $x \in K$. It is clear that r is a bounded linear operator, so we can define its transpose operator, see [23] p.11, also known as adjoint, see [22]. Recall r^t is defined as follows $r^t : \mathcal{C}(K)^* \to \mathcal{C}[a, b]^*$, $r^t(\ell)(f) = \ell(r(f))$ for $f \in \mathcal{C}[a, b]$; the expression $\ell(r(f))$ assigns a scalar to each function $f \in \mathcal{C}[a, b]$.

Let ℓ be a positive linear functional in $\mathcal{C}(K)$ and we define $Lf = r^t(\ell)(f) = \ell(rf)$. Since ℓ and r are positive linear functionals, so is L and we can apply theorem 2.1 and (c) in the Appendix to find a monotone increasing function α and an associated Borel measure μ such that

$$Lf = r^t(\ell)(f) = \int_a^b f d\alpha = \int_a^b f d\mu \tag{3.2}$$

for every $f \in \mathcal{C}[a,b]$.

Denote $K^c := [a,b] \setminus K$. We will show that $\mu(K^c) = 0$. Let $\varepsilon > 0$ and choose F_ε as a closed subset of K^c such that

$$\mu(K^c \setminus F_\varepsilon) < \varepsilon, \tag{3.3}$$

see (a) in the Appendix.

Let $\tilde{f} \in \mathcal{C}[a,b]$ be a continuous function such that $\tilde{f}(x) = 1$ if $x \in K$, $\tilde{f}(x) = 0$ if $x \in F_\varepsilon$ and $\|\tilde{f}\|_\infty \leq 1$. One can take for instance

$$\tilde{f}(x) = \frac{d(x, F_\varepsilon)}{d(x, F_\varepsilon) + d(x, K)},$$

where $d(x, A) = \inf_{y \in A} |x - y|$. Note that since $|d(x, A) - d(y, A)| \leq |x - y|$ the function $d(x, A)$ is even uniformly continuous, (cf. Urysohn's Lemma. [5], 4.15.). Therefore

$$L(\tilde{f}) = \int_a^b \tilde{f} d\mu = \int_K d\mu + \int_{K^c \setminus F_\varepsilon} \tilde{f} d\mu + \int_{F_\varepsilon} \tilde{f} d\mu$$

The third integral on the right is equal zero, by definition of $\tilde{f}$. We can estimate the second integral as follows,

$$0 \leq \int_{K^c \setminus F_\varepsilon} \tilde{f} d\mu \leq \int_{K^c \setminus F_\varepsilon} d\mu = \mu(K^c \setminus F_\varepsilon) < \varepsilon,$$

since $\tilde{f} \leq 1$ and using (3.3). Then

$$L(\tilde{f}) < \int_K d\mu + \varepsilon = \mu(K) + \varepsilon.$$

We have that

$$\mu(K) + \mu(K^c) = \int_a^b d\mu = L(\mathbf{1}_{[a,b]}) = L(\tilde{f}) < \mu(K) + \varepsilon,$$

The third equality follows from $r(\tilde{f}) = r(\mathbf{1}_{[a,b]})$. Thus $0 \leq \mu(K^c) < \varepsilon$, since $\mu(K) < \infty$.

To conclude, let $f \in \mathcal{C}(K)$ and f^* a continuous extension of f to the closed interval $[a,b]$. We can do this extension taking, for example, straight lines as follows: since K^c is an open subset of $[a,b]$, it is at most a countable union of pairwise disjoint open intervals (α_i, β_i)

intersected with the interval $[a, b]$, (see Lindeloef's thm., [18] Prop.9. p.40). For $x \in (\alpha_i, \beta_i)$ we define

$$f^*(x) = (1-t)f(\alpha_i) + tf(\beta_i)$$

if $x = \alpha_i(1-t)+t\beta_i$ for $t \in (0, 1)$. The function f^* is continuous on the interval $[a, b]$ since on K coincides with the continuous function f and on K^c consists of straight lines, (cf. Tietze's Theorem [5], 4.16).

Then we have

$$\ell(f) = \ell(r(f^*)) = Lf^* = \int_a^b f^* d\alpha = \int_a^b f^* d\mu = \int_K f^* d\mu = \int_K f d\mu. \tag{3.4}$$

as was to be shown.

ii) *Conservation of norm.* Take $f \in \mathcal{C}(K)$ such that $\|f\|_\infty \leq 1$. Since (3.1) holds we have,

$$|\ell(f)| = \left|\int_K f d\mu\right| \leq \|f\|_\infty \mu(K) \leq \mu(K).$$

For the reverse inequality, let $\mathbf{1}(x) = 1$ for all x, as defined in remark (3), so

$$\|\ell\| \geq |\ell(\mathbf{1})| = \left|\int_K \mathbf{1}\, d\mu\right| = \mu(K),$$

we can conclude that $\mu(K) = \|\ell\|$.

iii) *Uniqueness.* Suppose μ and ν are finite measures that satisfy (3.1). Since μ and ν are regular measures, from (a) in the Appendix, it is enough to show that $\mu(C) = \nu(C)$ for any closed set C of K. Let C a nonempty closed set of K and set $f_k(x) := \max\{0, 1 - kd(x, C)\}$ for all k and $x \in K$, where $d(x, C) = \inf_{y \in C} |x - y|$. These functions are bounded, by 0 and 1, and continuous. Thus f_k belongs to $\mathcal{C}(K)$ for all k. Notice that they form a sequence that decreases to the indicator of C, i.e., $f_k \downarrow \mathbf{1}_C$, where $\mathbf{1}_C(x) = 1$ if $x \in C$ and $\mathbf{1}_C(x) = 0$ if $x \notin C$. Thus, for all k we must have that $\int_K f_k d\mu = \int_K f_k d\nu$, and so we can use the dominated convergence theorem, see (b) in the Appendix, to conclude that

$$\mu(C) = \lim_k \int_K f_k d\mu = \lim_k \int_K f_k d\nu = \nu(C).$$

$\square$

Remarks

(a) It is possible to represent the linear positive functionals acting on $\mathcal{C}(K)$ as Riemann-Stieltjes integrals, similar to the original work of F. Riesz. This follows immediately from the chain of equalities (3.4). The caveat is that we cannot use f directly in order to define the Riemann-Stieltjes integral, but any continuous extension of f works, cf. theorem 3.2 below. This integral is independent of the extension of f.

(b) As just seen, the use of compact set K above allows us to extend the continuous functions to the entire interval $[a, b]$, using an elementary version of the Tietze's theorem. This construction is in general not possible if K is an arbitrary subset of the real line.

3.1 Isomorphic spaces

As a consequence of the previous results, we shall see that two spaces of functionals are practically the same. One of the spaces consists of Lebesgue integrals on compact subsets of $[a, b]$ and the other of Riemann-Stieltjes integrals over the whole interval $[a, b]$. In this way we show how the Lebesgue integral representation, that was introduced to represent functionals in the case of general compact sets, can be seen as a Riemann-Stieltjes integral. To state this precisely we need to introduce the terms *isomorphic* and *constant in* K^c.

A transformation T which preserves the norm, that is $\|Tx\| = \|x\|$, is called an *isometry.* Two normed vector spaces X and Y are said to be *isomorphic* if there is a linear, bijective, isometry $T : X \to Y$. Such functions are called *isomorphisms.* Since an isomorphism preserves the linear as well as the metric structure of the spaces, two isomorphic spaces can be considered identical, the isomorphism corresponding just to a labeling of the elements. We say that the monotone function α *is constant in* K^c if it is constant in each interval of K^c.

Recall that $\mathcal{C}(X)^*_+$ denotes the set of positive linear functionals on $\mathcal{C}(X)$. Let L_α denote the functional with corresponding monotone function α as introduced in (2.1).

The result mentioned above can be then stated as follows:

Theorem 3.2. *The normed spaces* $\left\{L_\alpha \in \mathcal{C}[a, b]^*_+ : \alpha \text{ is constant in } K^c\right\}$ *and* $\mathcal{C}(K)^*_+$ *are isomorphic.*

Before we prove this theorem we need two preparatory results.

Proposition 3.3. $r^t : \mathcal{C}(K)^*_+ \to \mathcal{C}[a, b]^*_+$ *is an isometry.*

Proof. $\|r^t\ell\|_{\mathcal{C}[a,b]^*_+} = V(\alpha) = \alpha(b) - \alpha(a) = \mu([a, b]) = \mu(K) + \mu([a, b] \setminus K) = \mu(K) = \|\ell\|_{\mathcal{C}(K)^*_+}$.

The first equality follows from Theorem 2.1. The function α depends on ℓ. The second is the definition of the total variation of α and the third is the definition of μ. The last two equalities follow from the construction of Theorem 3.1.

$\square$

We denote the range of r^t by Rang $r^t = \left\{L \in \mathcal{C}[a,b]^*_+ : \exists l \in \mathcal{C}(K)^*_+ \text{ s.t. } L = r^t l\right\}$

Proposition 3.4.

$$\textit{Rang } r^t = \left\{L_\alpha \in \mathcal{C}[a,b]^*_+ : \alpha \textit{ is constant in } K^c\right\}$$

Proof. "$\subset$"

Let $L \in$ Rang $r^t \subset \mathcal{C}[a,b]^*_+$. Then there exists $\ell \in \mathcal{C}(K)^*_+$ such that as in (3.2)

$$r^t(\ell)(f) = Lf = L_\alpha f = \int_a^b f d\alpha = \int_a^b f d\mu$$

As was shown in the proof of Theorem 3.1 i), $\mu(K^c) = 0$. Since K^c is a countable union of intervals, these have μ measure zero. By the relation which is given in (4.1) below, between the measure μ and the monotone function α we conclude that α is constant in each one of the intervals of K^c.

" $\supset$ "

Let $L_\alpha \in \mathcal{C}[a,b]^*_+$ with α constant in each interval of K^c and μ be the measure associated with this α, as in Appendix (c). Define $\ell \in \mathcal{C}(K)^*_+$ as $\ell h = \int_K h d\mu$. We shall show that $r^t(\ell)(f) = L_\alpha f$ for every $f \in \mathcal{C}[a,b]$. Since α constant in each interval of K^c this implies, using again (4.1), that $\mu(K^c) = 0$. Then we have

$$L_\alpha f = \int_a^b f d\alpha = \int_a^b f d\mu = \int_K f d\mu = \int_K r(f) d\mu = \ell(r(f)) = r^t(\ell)(f)$$

where $r(f)$ denotes, as in Theorem (3.1) i) above, the restriction of f to K. □

Proof of Theorem 3.2. .

From Proposition 3.3 and Proposition 3.4 it follows that r^t is a bijective isometry. Since r^t is linear as follows from its definition, then it is an isomorphism. □

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4 Appendix

A collection of subsets $\mathcal{A}$ of X is called an σ-*algebra* if it is closed under finite (countable) union, complements and $X \in \mathcal{A}$. If our space is $\mathbb{R}$, the *Borel* σ-*algebra*, $\mathcal{B}_\mathbb{R}$, is the smallest σ-algebra containing all the open intervals. A function $\mu : \mathcal{A} \to [0,\infty]$, where $\mathcal{A}$ is a σ-algebra, it is called a *measure* if it is countable additive, that is $\mu(\bigcup A_n) = \sum \mu(A_n)$ whenever $\{A_n\}$ is a disjoint sequence of elements in $\mathcal{A}$, and $\mu(\varnothing) = 0$. A *Borel measure* is a measure defined on $\mathcal{B}_\mathbb{R}$. We say that a measure is *regular* if every measurable set can be approximated from above by open measurable sets and from below by compact measurable sets. A function f from $(X, \mathcal{A}, \mu)$ to $(\mathbb{R}, B_{\mathbb{R}})$ is $\mathcal{A}$-measurable if $\{x : f(x) \leq t\} \in \mathcal{A}$ for all $t \in \mathbb{R}$.

The following results are used in the proof of Theorem 3.1.

(a) Every Borel measure in a metric space is regular. We will only use inner regularity, that is, for every Borel set A and every $\varepsilon > 0$ there exist a compact set F_ε such that $F_\varepsilon \subset A$ and $\mu(A \setminus F_\epsilon) < \epsilon$. [2] Thm 7.1.7. or [3] Lemma 1.5.7.

(b) (Dominated convergence theorem) Let $(X, \mathcal{A}, \mu)$ a measure spaces. Let g be a $[0, \infty]$-valued integrable function on X, that is, $\int g d\mu < \infty$, and let $f, f_1, f_2, \ldots$ real-valued $\mathcal{A}$-measurable functions on X such that $f(x) = \lim_n f_n(x)$ and $|f_n(x)| \leq g(x)$. Then f and $\{f_n\}$ are integrable and $\int f d\mu = \lim_n \int f_n d\mu$.

(c) Given a normalized monotone function α in the closed interval $[a, b]$, there is a unique Borel measure μ associated with it. This can be seen as follows (see for example [4]): for $a \leq s \leq t \leq b$ let define $\langle s, t]$ where

Let

$$\mathcal{F}_0 = \left\{ \bigcup_{\text{finite}} \langle s_k, t_k] : \langle s_k, t_k] \subset [a, b] \text{ pairwise disjoint} \right\}$$

Then $\mathcal{F}_0$ is an algebra of subsets of $[a, b]$ and therefore we can define a set function as

$$\mu_0 \left(\bigcup_{\text{finite}} \langle s_k, t_k] \right) = \sum_{\text{finite}} \alpha(t_k) - \alpha(s_k). \tag{4.1}$$

Moreover, μ_0 has a unique extension to a measure in the smallest σ-algebra containing $\mathcal{F}_0$ (Caratheodory's Theorem). See [1] . Moreover, for any continuous function f it happens that

$$\int_a^b f d\alpha = \int_a^b f d\mu \tag{4.2}$$

where the integral on the left is a Riemann-Stieltjes integral, whereas the integral on the right is an integral in the sense of Lebesgue.

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Odd Vertex Equitable Even Labeling of Cycle Related Graphs

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ABSTRACT

Let G be a graph with p vertices and q edges and $A = \{1, 3, ..., q\}$ if q is odd or $A = \{1, 3, ..., q + 1\}$ if q is even. A graph G is said to admit an odd vertex equitable even labeling if there exists a vertex labeling $f : V(G) \to A$ that induces an edge labeling f^* defined by $f^*(uv) = f(u) + f(v)$ for all edges uv such that for all a and b in A, $|v_f(a) - v_f(b)| \leq 1$ and the induced edge labels are $2, 4, ..., 2q$ where $v_f(a)$ be the number of vertices v with $f(v) = a$ for $a \in A$. A graph that admits an odd vertex equitable even labeling is called an odd vertex equitable even graph. Here, we prove that the subdivision of double triangular snake $(S(D(T_n)))$, subdivision of double quadrilateral snake $(S(D(Q_n)))$, $DA(Q_m) \odot nK_1$ and $DA(T_m) \odot nK_1$ are odd vertex equitable even graphs.

RESUMEN

Sea G un grafo con p vértices y q aristas, y $A = \{1, 3, ..., q\}$ si q es impar o $A = \{1, 3, ..., q + 1\}$ si q es par. Se dice que un grafo G admite un etiquetado par equitativo de vértices impares si existe un etiquetado de vértices $f : V(G) \to A$ que induce un etiquetado de ejes f^* definido por $f^*(uv) = f(u) + f(v)$ para todos los ejes uv tales que para todo a y b en A, $|v_f(a) - v_f(b)| \leq 1$ y las etiquetas de ejes inducidas son $2, 4, ..., 2q$ donde $v_f(a)$ es el número de vértices v con $f(v) = a$ para $a \in A$. Un grafo que admite un etiquetado par equitativo de vértices impares se dice grafo par equitativo de vértices impares. Aquí demostramos que la subdivisión de serpientes triangulares dobles $(S(D(T_n)))$, la subdivisión de serpientes cuadriláteras dobles $(S(D(Q_n)))$, $DA(Q_m) \odot nK_1$ y $DA(T_m) \odot nK_1$ son grafos pares equitativos de vértices impares.

Keywords and Phrases: Odd vertex equitable even labeling, odd vertex equitable even graph, double triangular snake, subdivision of double quadrilateral snake, double alternate triangular snake, double alternate quadrilateral snake, subdivision graph.

2010 AMS Mathematics Subject Classification: 05C78.

1 Introduction:

All graphs considered here are simple, finite, connected and undirected. Let $G(V, E)$ be a graph with p vertices and q edges. We follow the basic notations and terminology of graph theory as in [2]. The vertex set and the edge set of a graph are denoted by $V(G)$ and $E(G)$ respectively. A graph labeling is an assignment of integers to the vertices or edges or both, subject to certain conditions and a detailed survey of graph labeling can be found in [1]. The concept of vertex equitable labeling was due to Lourdusamy and Seenivasan [6]. Let G be a graph with p vertices and q edges and $A = \{0, 1, 2, ..., \lceil \frac{q}{2} \rceil\}$. A graph G is said to be vertex equitable if there exists a vertex labeling $f : V(G) \to A$ that induces an edge labeling f^* defined by $f^*(uv) = f(u) + f(v)$ for all edges uv such that for all a and b in A, $|v_f(a) - v_f(b)| \leq 1$ and the induced edge labels are $1, 2, 3, ..., q$, where $v_f(a)$ be the number of vertices v with $f(v) = a$ for $a \in A$. The vertex labeling f is known as vertex equitable labeling. A graph G is said to be a vertex equitable if it admits vertex equitable labeling. Motivated by the concept of vertex equitable labeling [6], Jeyanthi, Maheswari and Vijayalakshmi extend this concept and introduced a new labeling namely odd vertex equitable even (OVEE) labeling in [3]. A graph G with p vertices and q edges and $A = \{1, 3, ..., q\}$ if q is odd or $A = \{1, 3, ..., q + 1\}$ if q is even. A graph G is said to admit an odd vertex equitable even labeling if there exists a vertex labeling $f : V(G) \to A$ that induces an edge labeling f^* defined by $f^*(uv) = f(u) + f(v)$ for all edges uv such that for all a and b in A, $v_f(a) - v_f(b) \leq 1$ and the induced edge labels are $2, 4, ..., 2q$ where $v_f(a)$ be the number of vertices v with $f(v) = a$ for $a \in A$. A graph that admits an odd vertex equitable even (OVEE) labeling then G is called an odd vertex equitable even (OVEE) graph. In [3], [4] and [5] the same authors proved that nC_4-snake, $CS(n_1, n_2, ..., n_k$, $n_i \equiv 0(mod4)$, $n_i \geq 4$, be a generalized kC_n -snake, $T\widehat{O}QS_n$ and $T\widetilde{O}QS_n$ are odd vertex equitable even graphs. They also proved that the graphs path, $P_n \odot P_m (n, m \geq 1)$, $K_{1,n} \cup K_{1,n-2} (n \geq 3)$, $K_{2,n}$, T_p-tree, cycle C_n ($n \equiv$ 0 or 1 (mod4)), quadrilateral snake Q_n, ladder L_n, $L_n \odot K_1$, arbitrary super subdivision of any path P_n, $S(L_n)$, $L_m \widehat{O} P_n$, $L_n \odot \overline{K_m}$ and $\left\langle L_n \widehat{O} K_{1,m} \right\rangle$ are odd vertex equitable even graphs. Also they proved that the graphs $K_{1,n}$ is an odd vertex equitable even graph iff $n \leq 2$ and the graph $G = K_{1,n+k} \cup K_{1,n}$ is an odd vertex equitable even graph if and only if $k = 1, 2$ and cycle C_n is an odd vertex equitable even graph if and only if $n \equiv 0$ or 1(mod4). Let G be a graph with p vertices and q edges and $p \leq \lceil \frac{q}{2} \rceil + 1$, then G is not an odd vertex equitable even graph. In addition they proved that if every edge of a graph G is an edge of a triangle, then G is not an odd vertex equitable even graph.
We use the following definitions in the subsequent section.

Definition 1.1. *The double triangular snake* $D(T_n)$ *is a graph obtained from a path* P_n *with vertices* $v_1, v_2, ..., v_n$ *by joining* v_i *and* v_{i+1} *to the new vertices* w_i *and* u_i *for* $i = 1, 2, ..., n - 1$.

Definition 1.2. *The double quadrilateral snake* $D(Q_n)$ *is a graph obtained from a path* P_n *with vertices* $u_1, u_2, ..., u_n$ *by joining* u_i *and* u_{i+1} *to the new vertices* v_i, x_i *and* w_i, y_i *respectively and then joining* v_i, w_i *and* x_i, y_i *for* $i = 1, 2, ..., n - 1$.

Definition 1.3. *A double alternate triangular snake* $DA(T_n)$ *consists of two alternate triangular snakes that have a common path. That is, a double alternate triangular snake is obtained from* *a path* $u_1, u_2, ..., u_n$ *by joining* u_i *and* u_{i+1} *(alternatively) to the two new vertices* v_i *and* w_i *for* $i = 1, 2, ..., n-1$.

Definition 1.4. *A double alternate quadrilateral snake* $DA(Q_n)$ *consists of two alternate quadrilateral snakes that have a common path. That is, a double alternate quadrilateral snake is obtained from a path* $u_1, u_2, ..., u_n$ *by joining* u_i *and* u_{i+1} *(alternatively) to the two new vertices* v_i, x_i *and* w_i, y_i *respectively and adding the edges* v_iw_i *and* x_iy_i *for* $i = 1, 2, ..., n-1$.

Definition 1.5. *Let* G *be a graph. The subdivision graph* $S(G)$ *is obtained from* G *by subdividing each edge of* G *with a vertex.*

Definition 1.6. *The corona* $G_1 \odot G_2$ *of the graphs* G_1 *and* G_2 *is defined as the graph obtained by taking one copy of* G_1 *(with* p *vertices) and* p *copies of* G_2 *and then joining the* i^{th} *vertex of* G_1 *to every vertex of the* i^{th} *copy of* G_2.

2 Main Results

In this section, we prove that $S(D(T_n))$, $S(D(Q_n))$, $DA(Q_m) \odot nK_1$ and $DA(T_m) \odot nK_1$ are odd vertex equitable even graphs.

Theorem 2.1. *Let* $G_1(p_1, q_1)$, $G_2(p_2, q_2)$,...,$G_m(p_m, q_m)$ *be an odd vertex equitable even graphs with each* q_i *is even for* $i = 1, 2, ..., m-1$, q_m *is even or odd and let* u_i, v_i *be the vertices of* $G_i(1 \leq i \leq m)$ *labeled by 1,* q_i *if* q_i *is odd or* $q_i + 1$ *if* q_i *is even. Then the graph* G *obtained by identifying* v_1 *with* u_2 *and* v_2 *with* u_3 *and* v_3 *with* u_4 *and so on until we identify* v_{m-1} *with* u_m *is also an odd vertex equitable even graph.*

Proof. The graph G has $p_1 + p_2 + ... + p_m - (m-1)$ vertices and $\sum_{i=1}^{m} q_i$ edges and f_i be an odd vertex equitable even labeling of $G_i(1 \leq i \leq m)$.

Let $A = \left\{ \begin{array}{ll} 1, 3, 5, ..., \sum_{i=1}^{m} q_i, & \text{if} \quad \sum_{i=1}^{m} q_i \text{ is odd} \\ 1, 3, 5, ..., \sum_{i=1}^{m} q_i + 1, & \text{if} \quad \sum_{i=1}^{m} q_i \text{ is even} \end{array} \right\}$.

Define a vertex labeling $f : V(G) \to A$ as follows: $f(x) = f_1(x)$ if $x \in V(G_1)$, $f(x) = f_i(x) + \sum_{k=1}^{i-1} q_k$ if $x \in V(G_i)$ for $2 \leq i \leq m$. The edge labels of the graph G_1 will remain fixed, the edge labels of the graph $G_i(2 \leq i \leq m)$ are $2q_1 + 2, 2q_1 + 4, ..., 2(q_1 + q_2)$;$2(q_1 + q_2) + 2, 2(q_1 + q_2) + 4, ..., 2(q_1 + q_2 + q_3)$;...,$2\sum_{i=1}^{m-1} q_i + 2, 2\sum_{i=1}^{m-1} q_i + 4, ..., 2\sum_{i=1}^{m} q_i$. Hence the edge labels of G are distinct and is $\left\{2, 4, 6, ..., 2\sum_{i=1}^{m} q_i\right\}$. Also $|v_f(a) - v_f(b)| \leq 1$ for all $a, b \in A$. Hence G is an odd vertex equitable even graph. $\square$

Theorem 2.2. *The graph* $S(D(T_n))$ *is an odd vertex equitable even graph.*

Proof. Let $G_i = S(D(T_2))$ $1 \leq i \leq n-1$ and u_i, v_i be the vertices with labels 1 and $q + 1$ respectively. By Theorem 2.1, $S(D(T_2))$ admits an odd vertex equitable even labeling. An odd vertex equitable even labeling of $G_i = S(D(T_2))$ is given in Figure 1.

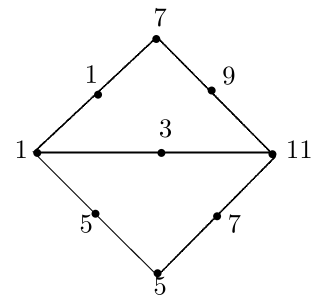


Figure 1. □

Theorem 2.3. *The graph* $S(D(Q_n))$ *is an odd vertex equitable even graph.*

Proof. Let $G_i = S(D(Q_2))$ $1 \leq i \leq n-1$ and u_i, v_i be the vertices with labels 1 and $q+1$ respectively. By Theorem 2.1, $S(D(Q_2))$ admits an odd vertex equitable even labeling. An odd vertex equitable even labeling of $G_i = S(D(Q_2))$ is given in Figure 2.

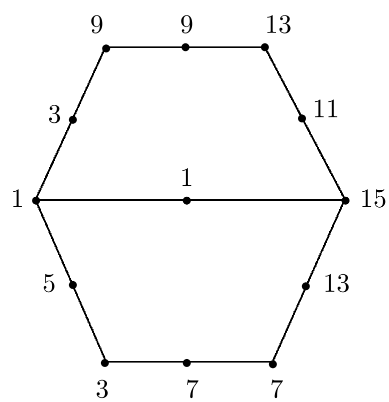


Figure 2. □

Theorem 2.4. *The double quadrilateral graph* $D(Q_{2n})$ *is an odd vertex equitable even graph.*

Proof. Let $G_i = D(Q_4)$ $1 \leq i \leq n-1$ and u_i, v_i be the vertices with labels 1 and $q+1$ respectively. By Theorem 2.1, $D(Q_4)$ admits an odd vertex equitable even labeling. An odd vertex equitable even labeling of $G_i = D(Q_4)$ is given in Figure 3.

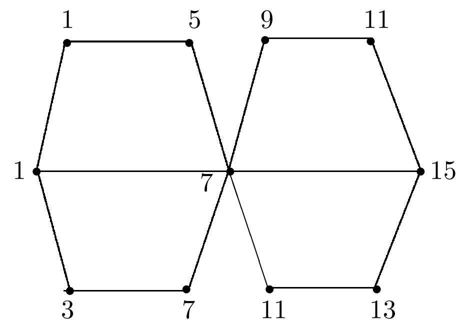


Figure 3. □

Theorem 2.5. *Let $G_1(p_1, q), G_2(p_2, q), ..., G_m(p_m, q)$ be an odd vertex equitable even graphs with q odd and u_i,v_i be vertices of $G_i(1 \le i \le m)$ labeled by 1 and q. Then the graph G obtained by joining v_1 with u_2 and v_2 with u_3 and v_3 with u_4 and so on until joining v_{m-1} with u_m by an edge is also an odd vertex equitable even graph.*

Proof. The graph G has $p_1 + p_2 + ... + p_m$ vertices and $mq + (m - 1)$ edges.
Let f_i be the odd vertex equitable even labeling of $G_i(1 \le i \le m)$ and
let $A = \{1, 3, ..., mq + (m - 1)\}$.
Define a vertex labeling $f : V(G) \to A$ as
$f(x) = f_i(x) + (i - 1)(q + 1)$ if $x \in G_i$ for $1 \le i \le m$.
The edge labels of G_i are incresed by $2(i - 1)(q + 1)$ for $i = 1, 2, ..., m$ under the new labeling f.
The bridge between the two graphs G_i, G_{i+1} will get the label $2i(q + 1)$, $1 \le i \le m - 1$.
Hence the edge labels of G are distinct and is $\{2, 4, ..., 2(mq + m - 1)\}$.
Also $|v_f(a) - v_f(b)| \le 1$ for all $a, b \in A$.
Then the graph G is an odd vertex equitable even graph. □

Theorem 2.6. *The graph $DA(T_2) \odot nK_1$ is an odd vertex equitable even graph for $n \ge 1$.*

Proof. Let $G = DA(T_2) \odot nK_1$. Let $V(G) = \{u_1, u_2, u, w\} \cup \{u_{ij} : 1 \le i \le 2, 1 \le j \le n\} \cup \{v_i, w_i : 1 \le i \le n\}$ and
$E(G) = \{u_1u_2, u_1v, vu_2, u_1w, wu_2\} \cup \{u_iu_{ij} : 1 \le i \le 2, 1 \le j \le n\} \cup \{vv_i, ww_i : 1 \le i \le n\}$.
Here $|V(G)| = 4(n + 1)$ and $|E(G)| = 4n + 5$.
Let $A = \{1, 3, ..., 4n + 5\}$.
Define a vertex labeling $f : V(G) \to A$ as follows:
For $1 \le i \le n$ $f(u_1) = 1$, $f(u_2) = 4n + 5$, $f(v) = 2n + 1$, $f(w) = 2n + 5$, $f(u_{1i}) = 2i - 1$, $f(u_{2i}) = 4n + 5 - 2(i - 1)$,

$$f(v_i) = \begin{cases} 3 & \text{if } i=1 \\ 2i + 3 & \text{if } 2 \le i \le n, \end{cases}$$

$$f(w_i) = \begin{cases} 2(n + i) + 1 & \text{if } 1 \le i \le n - 1 \\ 4n + 3 & \text{if } i=n. \end{cases}$$

It can be verified that the induced edge labels of $DA(T_2) \odot nK_1$ are $2, 4, ..., 8n+10$ and $|v_f(a) - v_f(b)| \leq 1$ for all $a, b \in A$.
Hence f is an odd vertex equitable even labeling $DA(T_2) \odot nK_1$.
An odd vertex equitable even labeling of $DA(T_2) \odot 3K_1$ is shown in Figure 4.

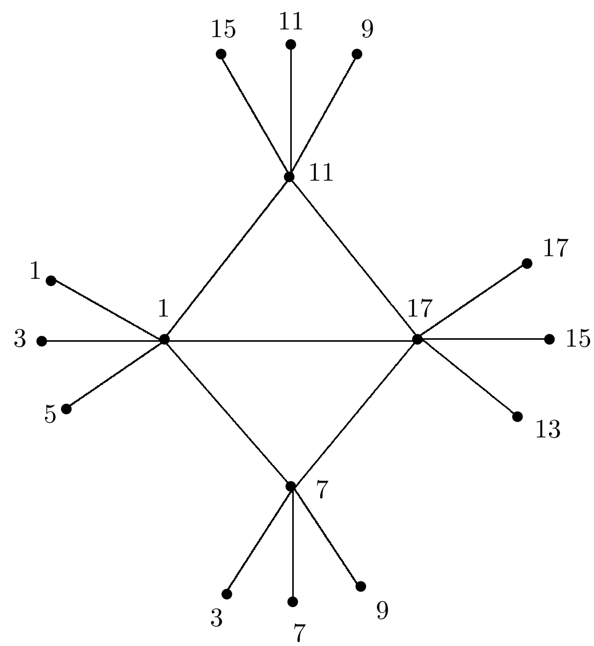


Figure 4.

□

Theorem 2.7. *The graph* $DA(Q_2) \odot nK_1$ *is an odd vertex equitable even graph for* $n \geq 1$.

Proof. Let $G = DA(Q_2) \odot nK_1$. Let $V(G) = \{u_1, u_2, v, w, x, y\} \cup \{v_i, w_i, x_i, y_i : 1 \leq i \leq n\} \cup \{u_{ij} : 1 \leq i \leq 2, 1 \leq j \leq n\}$ and
$E(G) = \{u_1u_2, u_1v, vw, wu_2, u_1x, xy, yu_2\} \cup \{vv_i, ww_i, xx_i, yy_i : 1 \leq i \leq n\} \cup \{u_iu_{ij} : 1 \leq i \leq 2, 1 \leq j \leq n\}$.
Here $|V(G)| = 6(n+1)$ and $|E(G)| = 6n+7$.
Let $A = \{1, 3, ..., 6n+7\}$.
Define a vertex labeling $f : V(G) \to A$ as follows:
For $1 \leq i \leq n$ $f(u_1) = 1$, $f(u_2) = 6n+7$, $f(u_{1i}) = 2i-1$, $f(u_{2i}) = 6n-2i+9$, $f(v) = 2n+1$, $f(w) = 2n+3$, $f(x) = 4n+5$, $f(y) = 4n+7$, $f(v_i) = 2i+1$, $f(w_i) = f(x_i) = 2n+2i+3$, $f(y_i) = 4n+2i+5$.
It can be verified that the induced edge labels of $DA(Q_2) \odot nK_1$ are $2, 4, ..., 12n+14$ and $|v_f(a) - v_f(b)| \leq 1$ for all $a, b \in A$.
Hence f is an odd vertex equitable even labeling of $DA(Q_2) \odot nK_1$.

An odd vertex equitable even labeling of $DA(Q_2) \odot 4K_1$ is shown in Figure 5.

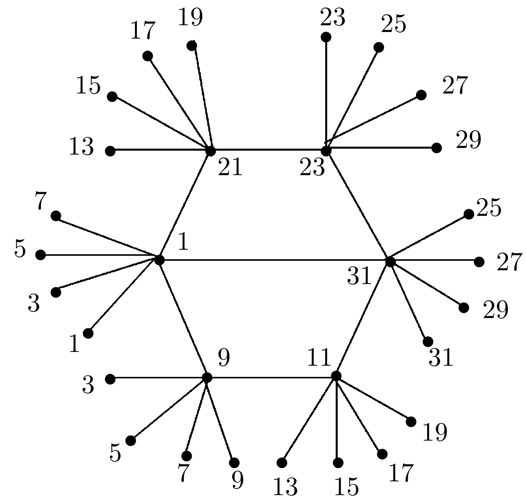
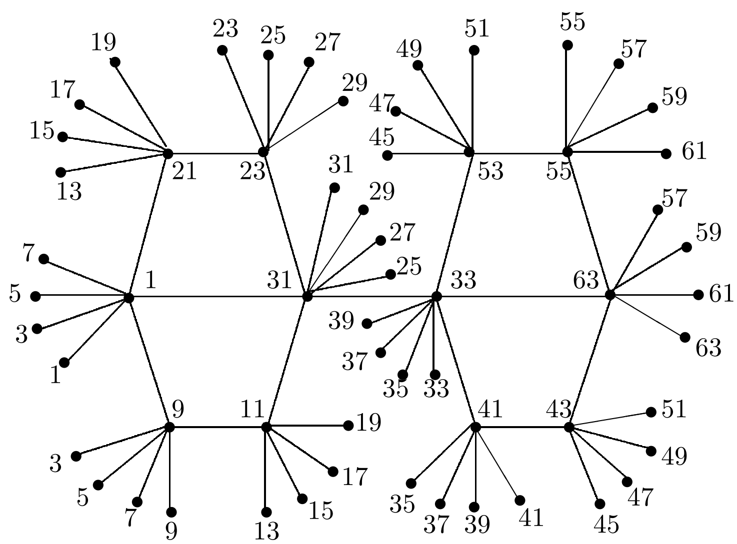


Figure 5. $\square$

Theorem 2.8. *The graph* $DA(Q_m) \odot nK_1$ *is an odd vertex equitable even graph for* $m, n \geq 1$.

Proof. By Theorem 2.7, $DA(Q_2) \odot nK_1$ is an odd vertex equitable even graph. Let $G_i = DA(Q_2) \odot nK_1$ for $1 \leq i \leq m-1$. Since each G_i has 6n+7 edges, by Theorem 2.5, $DA(Q_m) \odot nK_1$ admits odd vertex equitable even labeling.
An odd vertex equitable even labeling of $DA(Q_4) \odot 4K_1$ is shown in Figure 6.



$\square$

Figure 6.

Theorem 2.9. *The graph* $DA(T_m) \odot nK_1$ *is an odd vertex equitable even graph for* $m, n \geq 1$.

Proof. By Theorem 2.6, $DA(T_2) \odot nK_1$ is an odd vertex equitable even graph. Let $G_i = DA(T_2) \odot nK_1$ for $1 \leq i \leq m-1$. Since each G_i has 4n+5 edges, by Theorem 2.5, $DA(T_m) \odot nK_1$ admits odd vertex equitable even labeling.
An odd vertex equitable even labeling of $DA(T_4) \odot 3K_1$ is shown in Figure 7.

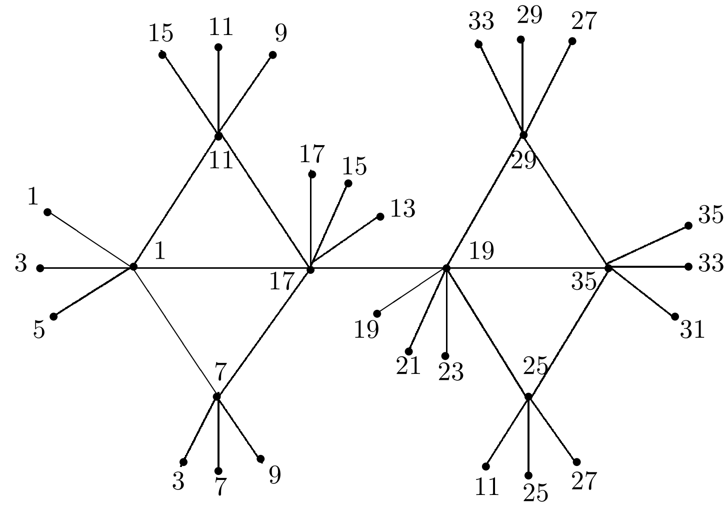


Figure 7.

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New approach to prove the existence of classical solutions for a class of nonlinear parabolic equations

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ABSTRACT

In this article, we consider a class of nonlinear parabolic equations. We use an integral representation combined with a sort of fixed point theorem to prove the existence of classical solutions for the initial value problem (1.1), (1.2). We also obtain a result on continuous dependence on the initial data. We propose a new approach for investigation for existence of classical solutions of some classes nonlinear parabolic equations.

RESUMEN

En este artículo, consideramos una clase de ecuaciones parabólicas nolineales. Usamos una representación integral combinada con una especie de teorema de punto fijo para probar la existencia de soluciones clásicas para el problema de valor inicial (1.1), (1.2). También obtenemos un resultado sobre la dependencia continua de la data inicial. Proponemos una estrategia nueva para la investigación de la existencia de soluciones clásicas de algunas clases de ecuaciones parabólicas nolineales.

Keywords and Phrases: parabolic equation, existence, differentiability with respect to the initial data

2010 AMS Mathematics Subject Classification: 35K55, 35K45.

1 Introduction

Here, we consider the Cauchy problem

$$u_t - u_{xx} = f(t,x,u,u_x) \quad \text{in} \quad (0,\infty)\times\mathbb{R}, \tag{1.1}$$

$$u(0,x) = \phi(x) \quad \text{in} \quad \mathbb{R}, \tag{1.2}$$

where $\phi \in \mathcal{C}^2(\mathbb{R})$, $f : [0,\infty)\times\mathbb{R}\times\mathbb{R}\times\mathbb{R} \longmapsto \mathbb{C}$ is a given continuous function, $u : [0,\infty)\times\mathbb{R} \longmapsto \mathbb{C}$ is the main unknown.

Our main results are as follows.

Theorem 1.1. *Let* $f \in \mathcal{C}([0,\infty)\times\mathbb{R}\times\mathbb{R}\times\mathbb{R})$, $\phi \in \mathcal{C}^2(\mathbb{R})$. *Then there exists* $m \in (0,1)$ *such that the problem (1.1), (1.2) has a solution* $u \in \mathcal{C}^1([0,m],\mathcal{C}^2([0,1]))$.

Theorem 1.2. *Let* $f \in \mathcal{C}([0,\infty)\times\mathbb{R}\times\mathbb{R}\times\mathbb{R})$, $\phi \in \mathcal{C}^2(\mathbb{R})$. *Then there exists* $m \in (0,1)$ *such that the problem (1.1), (1.2) has a solution* $u \in \mathcal{C}^1([0,m],\mathcal{C}^2(\mathbb{R}))$.

For $O_1, O_2 \subset \mathbb{R}$ with $\mathcal{C}^1(O_1,\mathcal{C}^2(O_2))$ we denote the space of all continuous functions u on $O_1\times O_2$ such that u_t, u_x and u_{xx} exist and are continuous on $O_1\times O_2$.

Example 1.3. *Let* $p > 1$ *and* $a \in \mathbb{C}$ *be chosen so that* $a^{p-1} = -\dfrac{1}{p-1}$. *Consider the Cauchy problem*

$$u_t - u_{xx} = u^p \quad \textit{in} \quad (0,\infty)\times\mathbb{R}$$

$$u(0,x) = a \quad \textit{in} \quad \mathbb{R}.$$

Then $u(t,x) = a(t+1)^{-\frac{1}{p-1}}$ *is its solution. Actually,*

$$u_t(t,x) = -\frac{a}{p-1}(t+1)^{-\frac{p}{p-1}},$$

and

$$u_{xx}(t,x) = 0,$$

and

$$(u(t,x))^p = -\frac{a}{p-1}(t+1)^{-\frac{p}{p-1}}.$$

Therefore

$$u_t(t,x) - u_{xx}(t,x) = (u(t,x))^p \quad \textit{in} \quad (0,\infty)\times\mathbb{R}$$

and

$$u(0,x) = a \quad \textit{in} \quad \mathbb{R}.$$

To prove our main result we propose new integral representation of the solutions of the initial value problem (1.1), (1.2). Many works have been devoted to the investigation of initial value problems for parabolic equations and systems (see, for example, [13]-[16] and the references therein). We note that in the references the IVP (1.1), (1.2) is connected with the dimension n, Fujita exponent, Sobolev critical exponents, bounded and unbounded domain. In this article we propose new idea which tell us that the local existence of classical solutions of the IVP is connected with the integral representation of the solutions, it is not connected with the dimension n and if the domain is bounded or not.

As an application of our new integral representation we deduce some results connected with the continuous dependence on the initial data and parameters of the problem (1.1), (1.2).

Theorem 1.4. *Let* $f \in \mathcal{C}([0,\infty)\times\mathbb{R}\times\mathbb{R}\times\mathbb{R})$, $\frac{\partial f}{\partial u}$, $\frac{\partial f}{\partial u_x}$ *exist and are continuous in* $[0,\infty)\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}$, $\phi \in \mathcal{C}^2(\mathbb{R})$. *Let also,* $u(t,x,\phi) \in \mathcal{C}^1([0,m],\mathcal{C}^2([c,d]))$ *be a solution to the problem (1.1), (1.2) for some* $m \in (0,1)$ *and for some* $[c,d] \subset \mathbb{R}$. *Then* $u(t,x,\phi)$ *is differentiable with respect to* ϕ *and* $v(t,x) = \frac{\partial u}{\partial \phi}(t,x,\phi)$ *satisfies the following initial value problem*

$$\begin{aligned} v_t - v_{xx} &= \tfrac{\partial f}{\partial u}(t,x,u(t,x,\phi),u_x(t,x,\phi))v \\ &\quad + \tfrac{\partial f}{\partial u_x}(t,x,u(t,x,\phi),u_x(t,x,\phi))v_x \quad \text{in} \quad [0,m]\times[c,d], \end{aligned} \tag{1.3}$$

$$v(0,x) = 1 \quad \text{in} \quad [c,d]. \tag{1.4}$$

2 Auxiliary results

We will start with the following useful lemma.

Lemma 2.1. *Let* $f \in \mathcal{C}([a,b]\times[c,d]\times\mathbb{R}\times\mathbb{R})$, $g \in \mathcal{C}^2([c,d])$. *Then the function* $u \in \mathcal{C}^1([a,b],\mathcal{C}^2([c,d]))$ *is a solution to the problem*

$$u_t - u_{xx} = f(t,x,u,u_x) \quad \text{in} \quad (a,b]\times[c,d], \tag{2.1}$$

$$u(a,x) = g(x) \quad \text{in} \quad [c,d], \tag{2.2}$$

if and only if it is a solution to the integral equation

$$\begin{aligned} &\int_c^x\int_c^y (u(t,z)-g(z))\,dzdy - \int_a^t (u(\tau,x)-u(\tau,c)-(x-c)u_x(\tau,c))\,d\tau \\ &= \int_a^t\int_c^x\int_c^y f(\tau,z,u(\tau,z),u_x(\tau,z))dzdyd\tau, \qquad x\in[c,d], \quad t\in[a,b]. \end{aligned} \tag{2.3}$$

Proof. (1) Let $u \in \mathcal{C}^1([a,b],\mathcal{C}^2([c,d]))$ is a solution to the problem (2.1), (2.2).

We integrate the equation (2.1) with respect to x and we get

$$\begin{aligned} &\int_c^x u_t(t,z)dz - \int_c^x u_{xx}(t,z)dz \\ &= \int_c^x f(t,z,u(t,z),u_x(t,z))dz, \qquad x\in[c,d], \quad t\in[a,b], \end{aligned}$$

or

$$\int_c^x u_t(t,z)dz - u_x(t,x) + u_x(t,c)$$

$$= \int_c^x f(t,z,u(t,z),u_x(t,z))dz, \qquad x \in [c,d], \quad t \in [a,b].$$

Now we integrate the last equation with respect to x and we find

$$\int_c^x \int_c^y u_t(t,z)dzdy - \int_c^x \left(u_x(t,z) - u_x(t,c)\right) dz$$

$$= \int_c^x \int_c^y f(t,z,u(t,z),u_x(t,z))dzdy, \qquad x \in [c,d], \quad t \in [a,b],$$

or

$$\int_c^x \int_c^y u_t(t,z)dzdy - u(t,x) + u(t,c) + (x-c)u_x(t,c)$$

$$= \int_c^x \int_c^y f(t,z,u(t,z),u_x(t,z))dzdy, \qquad x \in [c,d], \quad t \in [a,b].$$

We integrate the last equality with respect to t and we obtain

$$\int_a^t \int_c^x \int_c^y u_t(s,z)dzdyds - \int_a^t \left(u(s,x) - u(s,c) - (x-c)u_x(s,c)\right) ds$$

$$= \int_a^t \int_c^x \int_c^y f(s,z,u(s,z),u_x(s,z))dzdyds, \qquad x \in [c,d], \quad t \in [a,b],$$

or

$$\int_c^x \int_c^y \left(u(t,z) - g(z)\right) dzdy - \int_a^t \left(u(s,x) - u(s,c) - (x-c)u_x(s,c)\right) ds$$

$$= \int_a^t \int_c^x \int_c^y f(s,z,u(s,z),u_x(s,z))dzdyds, \qquad x \in [c,d], \quad t \in [a,b],$$

i.e., u satisfies the equation (2.3).

(2) Let $u \in \mathcal{C}^1([a,b],\mathcal{C}^2([c,d]))$ be a solution to the integral equation (2.3).
We differentiate the equation (2.3) with respect to x and we get

$$\int_c^x \left(u(t,z) - g(z)\right) dz - \int_a^t \left(u_x(s,x) - u_x(s,c)\right) ds$$

$$= \int_a^t \int_c^x f(s,z,u(s,z),u_x(s,z))dzds, \qquad x \in [c,d], \quad t \in [a,b].$$

Again we differentiate with respect to x and we find

$$\begin{aligned} & u(t,x) - g(x) - \int_a^t u_{xx}(s,x)ds \\ & = \int_a^t f(s,x,u(s,x),u_x(s,x))ds, \qquad x \in [c,d], \quad t \in [a,b]. \end{aligned} \tag{2.4}$$

Now we put $t = a$ in the last equation and we find

$$u(a,x) = g(x), \qquad x \in [c,d],$$

i.e., the function u satisfies (2.2).

Now we differentiate the equation (2.4) with respect to t and we find

$$u_t(t,x) - u_{xx}(t,x) = f(t,x,u(t,x),u_x(t,x)), \qquad x \in [c,d], t \in [a,b].$$

$\square$

The proof of the existence results are based on the following theorem.

Theorem 2.2 ([14]). *Let X be a nonempty closed convex subset of a Banach space Y. Suppose that T and S map X into Y such that*

(1) S *is continuous and* $S(X)$ *contained in a compact subset of* Y.

(2) $T : X \longmapsto Y$ *is expansive and onto.*

Then there exists a point $x^* \in X$ such that

$$Sx^* + Tx^* = x^*.$$

Definition 2.3. *Let (X,d) be a metric space and M be a subset of X. The mapping $T : M \longmapsto X$ is said to be expansive if there exists a constant $h > 1$ such that*

$$d(Tx,Ty) \geq hd(x,y)$$

for any $x, y \in M$.

3 Proof of Theorem 1.1

Let $B > \|\phi\|_{\mathcal{C}^2([0,1])}$ be arbitrarily chosen. Since $\phi \in \mathcal{C}([0,1])$, $f \in \mathcal{C}([0,1]\times[0,1]\times[-B,B]\times[-B,B])$ we have that there exists a constant $M_{11} > 0$ such that

$$\begin{aligned} |\phi(x)| &\leq M_{11} \quad \text{in} \quad [0,1], \\ |f(t,x,y,z)| &\leq M_{11} \quad \text{in} \quad [0,1]\times[0,1]\times[-B,B]\times[-B,B]. \end{aligned}$$

We take $l, m \in (0,1)$ so that

$$\begin{aligned} & lB + l(B + M_{11}) + 3lBm + lM_{11}m \leq B \\ & l(5B + 2M_{11}) \leq B. \end{aligned} \tag{3.1}$$

Let $E_{11} = \mathcal{C}^1([0,m],\mathcal{C}^2([0,1]))$ be endowed with the norm

$$\begin{aligned} \|u\| = \max\Big\{ & \max_{(t,x)\in[0,m]\times[0,1]} |u(t,x)|, \qquad \max_{(t,x)\in[0,m]\times[0,1]} |u_t(t,x)|, \\ & \max_{(t,x)\in[0,m]\times[0,1]} |u_x(t,x)|, \qquad \max_{(t,x)\in[0,m]\times[0,1]} |u_{xx}(t,x)| \Big\}. \end{aligned}$$

By $\tilde{K}_{11}$ we denote the set of all equi-continuous families in E_{11}, i.e., for every $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$ such that

$$|u(t_1, x_1) - u(t_2, x_2)| < \epsilon, \qquad |u_t(t_1, x_1) - u_t(t_2, x_2)| < \epsilon,$$

$$|u_x(t_1, x_1) - u_x(t_2, x_2)| < \epsilon, \qquad |u_{xx}(t_1, x_1) - u_{xx}(t_2, x_2)| < \epsilon$$

whenever $|t_1 - t_2| < \delta$, $|x_1 - x_2| < \delta$. Let also,

$$K'_{11} = \overline{\tilde{K}_{11}}, \quad K_{11} = \{u \in K'_{11} : ||u|| \leq B\}$$

and

$$L_{11} = \{u \in K'_{11} : ||u|| \leq (1 + l)B\}.$$

We note that K_{11} is a closed convex subset of L_{11}.

For $u \in L_{11}$ we define the operators

$$\begin{aligned} T_{11}(u)(t, x) &= (1 + l)u(t, x), \\ S_{11}(u)(t, x) &= -lu(t, x) + l \int_0^x \int_0^y (u(t, z) - \phi(z))dzdy \\ &\quad -l \int_0^t (u(\tau, x) - u(\tau, 0) - xu_x(\tau, 0))d\tau \\ &\quad -l \int_0^t \int_0^x \int_0^y f(\tau, z, u(\tau, z), u_x(\tau, z))dzdyd\tau. \end{aligned}$$

We will prove that the problem

$$u_t - u_{xx} = f(t, x, u_x) \quad \text{in} \quad [0, m] \times [0, 1], \tag{3.2}$$

$$u(0, x) = \phi(x) \quad \text{in} \quad [0, 1], \tag{3.3}$$

has a solution $u \in \mathcal{C}^1([0, m], \mathcal{C}^2([0, 1]))$.

a)$S_{11} : K_{11} \longmapsto K_{11}$**.** Let $u \in K_{11}$. Then $S_{11}(u) \in \mathcal{C}^1([0, m], \mathcal{C}^2([0, 1]))$ and for $(t, x) \in [0, m] \times$

$[0,1]$, using the first inequality of (3.1), we get

$$
\begin{aligned}
|S_{11}(u)(t,x)| &= \Big| -lu(t,x) + l\int_0^x\int_0^y (u(t,z)-\phi(z))dzdy \\
&\quad -l\int_0^t (u(\tau,x)-u(\tau,0)-xu_x(\tau,0))d\tau \\
&\quad -l\int_0^t\int_0^x\int_0^y f(\tau,z,u(\tau,z),u_x(\tau,z))dzdyd\tau \Big| \\
&\le l|u(t,x)| + l\int_0^x\int_0^y (|u(t,z)|+|\phi(z)|)\,dzdy \\
&\quad +l\int_0^t (|u(\tau,x)|+|u(\tau,0)|+x|u_x(\tau,0)|)\,d\tau \\
&\quad +l\int_0^t\int_0^x\int_0^y |f(\tau,z,u(\tau,z),u_x(\tau,z))|dzdyd\tau
\end{aligned}
$$

$$
\begin{aligned}
&\le lB + l(B+M_{11}) + 3lBm + lM_{11}m \\
&\le B.
\end{aligned}
$$

Note that

$$
\begin{aligned}
S_{11}(u)_t(t,x) &= -lu_t(t,x) + l\int_0^x\int_0^y u_t(t,z)dzdy \\
&\quad -l(u(t,x)-u(t,0)-xu_x(t,0)) \\
&\quad -l\int_0^x\int_0^y f(t,z,u(t,z),u_x(t,z))dzdy, \\
&\quad (t,x)\in[0,m]\times[0,1].
\end{aligned}
$$

Then, using the second inequality of (3.1), we obtain

$$
\begin{aligned}
|S_{11}(u)_t(t,x)| &= \Big| -lu_t(t,x) + l\int_0^x\int_0^y u_t(t,z)dzdy \\
&\quad -l(u(t,x)-u(t,0)-xu_x(t,0)) \\
&\quad -l\int_0^x\int_0^y f(t,z,u(t,z),u_x(t,z))dzdy\Big| \\
&\le l|u_t(t,x)| + l\int_0^x\int_0^y |u_t(t,z)|dzdy \\
&\quad +l\left(|u(t,x)|+|u(t,0)|+x|u_x(t,0)|\right) \\
&\quad +l\int_0^x\int_0^y |f(t,z,u(t,z),u_x(t,z))|dzdy \\
&\le lB + lB + 3lB + lM_{11} \\
&= l(5B+M_{11}) \\
&\le B, \quad (t,x)\in[0,m]\times[0,1].
\end{aligned}
$$

Also,

$$
\begin{aligned}
S_{11}(u)_x(t,x) &= -lu_x(t,x) + l\int_0^x (u(t,z)-\phi(z))dz \\
&\quad -l\int_0^t (u_x(\tau,x)-u_x(\tau,0))d\tau \\
&\quad -l\int_0^t\int_0^x f(\tau,z,u(\tau,z),u_x(\tau,z))dzd\tau, \\
&\quad (t,x)\in[0,m]\times[0,1].
\end{aligned}
$$

Hence, using the first inequality of (3.1),

$$
\begin{aligned}
|S_{11}(u)_x(t,x)| &= \Big| -lu_x(t,x) + l\int_0^x (u(t,z)-\phi(z))dz \\
&\quad -l\int_0^t (u_x(\tau,x)-u_x(\tau,0))d\tau \\
&\quad -l\int_0^t\int_0^x f(\tau,z,u(\tau,z),u_x(\tau,z))dzd\tau \Big| \\
&\le l|u_x(t,x)| + l\int_0^x (|u(t,z)|+|\phi(z)|)\,dz \\
&\quad +l\int_0^t (|u_x(\tau,x)|+|u_x(\tau,0)|)\,d\tau \\
&\quad +l\int_0^t\int_0^x |f(\tau,z,u(\tau,z),u_x(\tau,z))|dzd\tau \\
&\le lB + l(B+M_{11}) + 2lBm + lM_{11}m \\
&\le B, \quad (t,x)\in[0,m]\times[0,1].
\end{aligned}
$$

For $(t,x)\in[0,m]\times[0,1]$ we have

$$
\begin{aligned}
S_{11}(u)_{xx}(t,x) &= -lu_{xx}(t,x) + l(u(t,x)-\phi(x)) \\
&\quad -l\int_0^t u_{xx}(\tau,x)d\tau \\
&\quad -l\int_0^t f(\tau,x,u(\tau,x),u_x(\tau,x))d\tau,
\end{aligned}
$$

from where, using the first inequality of (3.1),

$$\begin{aligned}
|S_{11}(u)_{xx}(t,x)| &= \Big| -lu_{xx}(t,x) + l(u(t,x) - \phi(x)) \\
&\quad -l\int_0^t u_{xx}(\tau,x)d\tau \\
&\quad -l\int_0^t f(\tau,x,u(\tau,x),u_x(\tau,x))d\tau \Big| \\
&\le l|u_{xx}(t,x)| + l\left(|u(t,x)| + |\phi(x)|\right) \\
&\quad +l\int_0^t |u_{xx}(\tau,x)|d\tau \\
&\quad +l\int_0^t |f(\tau,x,u(\tau,x),u_x(\tau,x))|d\tau \\
&\le lB + l(B+M_{11}) + lBm + lM_{11}m \\
&\le B.
\end{aligned}$$

We note that $\{S_{11}(u) : u \in K_{11}\}$ is an equi-continuous family in E_{11}. Consequently $S_{11} : K_{11} \longmapsto K_{11}$. Also, $S_{11}(K_{11}) \subset K_{11} \subset L_{11}$, i.e., $S_{11}(K_{11})$ resides in a compact subset of L_{11}.

b) $S_{11} : K_{11} \longmapsto K_{11}$ **is a continuous operator.** We note that if $\{u_n\}_{n=1}^{\infty}$ be a sequence of elements of K_{11} such that $u_n \longrightarrow u$ in K_{11} as $n \longrightarrow \infty$, then $S_{11}(u_n) \longrightarrow S_{11}(u)$ in K_{11} as $n \longrightarrow \infty$. Therefore $S_{11} : K_{11} \longmapsto K_{11}$ is a continuous operator.

c) $T_{11} : K_{11} \longmapsto L_{11}$ **is an expansive operator and onto.** For $u, v \in K_{11}$ we have that

$$||T_{11}(u) - T_{11}(v)|| = (1+l)||u - v||,$$

i.e., $T_{11} : K_{11} \longmapsto L_{11}$ is an expansive operator with constant $1+l$.

Let $v \in L_{11}$. Then $\frac{v}{1+l} \in K_{11}$ and

$$T_{11}\left(\frac{v}{1+l}\right) = v,$$

i.e., $T_{11} : K_{11} \longmapsto L_{11}$ is onto.

From a), b), c) and from Theorem 2.2, it follows that there is $u_{11} \in K_{11}$ such that

$$T_{11}u_{11} + S_{11}u_{11} = u_{11}$$

or

$$
\begin{aligned}
&(1+l)u_{11}(t,x) - lu_{11}(t,x) + l\int_0^x \int_0^y (u_{11}(t,z) - \phi(z))dzdy \\
&\quad -l\int_0^t (u_{11}(\tau,x) - u_{11}(\tau,0) - xu_{11x}(\tau,0))d\tau \\
&\quad -l\int_0^t \int_0^x \int_0^y f(\tau,z,u_{11}(\tau,z),u_{11x}(\tau,z))dzdyd\tau \\
&= u_{11}(t,x),
\end{aligned}
$$

or

$$
\begin{aligned}
&\int_0^x \int_0^y (u_{11}(t,z) - \phi(z))dzdy - \int_0^t (u_{11}(\tau,x) - u_{11}(\tau,0) - xu_{11x}(\tau,0))d\tau \\
&\quad -\int_0^t \int_0^x \int_0^y f(\tau,z,u_{11}(\tau,z),u_{11x}(\tau,z))dzdyd\tau \\
&= 0, \quad (t,x) \in [0,m] \times [0,1],
\end{aligned}
$$

whereupon, using Lemma 2.1, we conclude that $u_{11} \in \mathcal{C}^1([0,1],\mathcal{C}^2([0,1]))$ is a solution to the problem (3.2), (3.3).

4 Proof of Theorem 1.2

Now we consider the problem

$$
u_t - u_{xx} = f(t,x,u(t,x),u_x(t,x)) \quad \text{in} \quad (0,m] \times [1,2], \tag{4.1}
$$

$$
u(0,x) = \phi(x) \quad \text{in} \quad [1,2]. \tag{4.2}
$$

Let $E_{12} = \mathcal{C}^1([0,m],\mathcal{C}^2([1,2]))$ be endowed with the norm

$$
\begin{aligned}
\|u\| &= \max\Big\{ \max_{(t,x)\in[0,m]\times[1,2]} |u(t,x)|, \quad \max_{(t,x)\in[0,m]\times[1,2]} |u_t(t,x)|, \\
&\quad \max_{(t,x)\in[0,m]\times[1,2]} |u_x(t,x)|, \quad \max_{(t,x)\in[0,m]\times[1,2]} |u_{xx}(t,x)| \Big\}.
\end{aligned}
$$

By $\tilde{K}_{12}$ we denote the set of all equi-continuous families in E_{12}.

Let $K'_{12} = \overline{\tilde{K}_{12}}$,

$$
K_{12} = \{u \in K'_{12} : \|u\| \le B\}.
$$

Since $\phi \in \mathcal{C}([1,2])$, $f \in \mathcal{C}([0,m] \times [1,2] \times [-B,B] \times [-B,B])$ we have that there exists a constant $M_{12} > 0$ such that

$$
\begin{aligned}
|\phi(x)| &\leq M_{12} \quad \text{in} \quad [1,2], \\
|f(t,x,y,z)| &\leq M_{12} \quad \text{in} \quad [0,m] \times [1,2] \times [-B,B] \times [-B,B].
\end{aligned}
$$

Let $l_1 > 0$ be chosen so that

$$l_1(5B + 2M_{12}) \leq B$$

$$l_1 B + l_1(B + M_{12}) + 3l_1 Bm + l_1 M_{12} m \leq B$$

Let also,

$$L_{12} = \{u \in K'_{12} : \|u\| \leq (1 + l_1)B\}.$$

We note that K_{12} is a closed convex subset of L_{12}.

For $u \in L_{12}$ we define the operators

$$
\begin{aligned}
T_{12}(u)(t,x) &= (1 + l_1)u(t,x), \\
S_{12}(u)(t,x) &= -l_1 u(t,x) + l_1 \int_1^x \int_1^y (u(t,z) - \phi(z)) dz dy \\
&\quad - l_1 \int_0^t (u(\tau,x) - u_{11}(\tau,1) - (x-1)u_{11x}(\tau,1)) d\tau \\
&\quad - l_1 \int_0^t \int_1^x \int_1^y f(\tau,z,u(\tau,z),u_x(\tau,z)) dz dy d\tau.
\end{aligned}
$$

As in the previous section one can prove that there is $u_{12} \in \mathcal{C}^1([0,1], \mathcal{C}^2([1,2]))$ which is a solution to the problem (4.1), (4.2). This solution u_{12} satisfies the integral equation

$$
\begin{aligned}
&\int_1^x \int_1^y (u_{12}(t,z) - \phi(z)) dz dy \\
&\quad - \int_0^t (u_{12}(\tau,x) - u_{11}(\tau,1) - (x-1)u_{11x}(\tau,1)) d\tau \\
&\quad - \int_0^t \int_1^x \int_1^y f(\tau,z,u_{12}(\tau,z),u_{12x}(\tau,z)) dz dy d\tau \\
&= 0, \quad (t,x) \in [0,m] \times [1,2].
\end{aligned}
\tag{4.3}
$$

Now we put $x = 1$ in (4.3) and we find

$$\int_0^t (u_{12}(\tau,1) - u_{11}(\tau,1)) d\tau = 0,$$

which we differentiate with respect to t and we get

$$u_{12}(t,1) = u_{11}(t,1) \quad \text{in} \quad [0,m]. \tag{4.4}$$

Now we differentiate (4.3) with respect to x and we find

$$\int_1^x (u_{12}(t,z) - \phi(z))dz - \int_0^t (u_{12x}(\tau,x) - u_{11x}(\tau,1))d\tau$$

$$- \int_0^t \int_1^x f(\tau,z,u_{12}(\tau,z),u_{12x}(\tau,z))dzd\tau = 0, \quad (t,x) \in [0,m] \times [1,2].$$

In the last equation we put $x = 1$ and we become

$$\int_0^t (u_{12x}(\tau,x) - u_{11x}(\tau,1))d\tau = 0, \quad (t,x) \in [0,m] \times [1,2],$$

which we differentiate with respect to t and we find

$$u_{12x}(t,1) = u_{11x}(t,1) \quad \text{in} \quad [0,m]. \tag{4.5}$$

Now we differentiate (4.4) with respect to t and we get

$$u_{12t}(t,1) = u_{11t}(t,1) \quad \text{in} \quad [0,m].$$

Hence, (4.4), (4.5) and

$$f(t,1,u_{11}(t,1),u_{11x}(t,1)) = f(t,1,u_{12}(t,1),u_{12x}(t,1)),$$

we find

$$\begin{aligned} u_{12xx}(t,1) &= u_{12t}(t,1) - f(t,1,u_{12}(t,1),u_{12x}(t,1)) \\ &= u_{11t}(t,1) - f(t,1,u_{11}(t,1),u_{11x}(t,1)) \\ &= u_{11xx}(t,1) \quad \text{in} \quad [0,m]. \end{aligned}$$

Consequently the function

$$u(t,x) = \begin{cases} u_{11}(t,x) & \text{in} \quad [0,m] \times [0,1] \\ u_{12}(t,x) & \text{in} \quad [0,m] \times [1,2], \end{cases}$$

is a $\mathcal{C}^1([0,m],\mathcal{C}^2([0,2]))$-solution to the problem

$$\begin{aligned} u_t - u_{xx} &= f(t,x,u(t,x),u_x(t,x)) \quad \text{in} \quad (0,m] \times [0,2], \\ u(0,x) &= \phi(x) \quad \text{in} \quad [0,2]. \end{aligned}$$

Then we consider the problem

$$\begin{aligned} u_t - u_{xx} &= f(t, x, u(t, x), u_x(t, x)) \quad \text{in} \quad (0, m] \times [2, 3] \\ u(0, x) &= \phi(x) \quad \text{in} \quad [2, 3]. \end{aligned} \tag{4.6}$$

As in above there is $u_{13} \in \mathcal{C}^1([0, m], \mathcal{C}^2([2, 3]))$ which is a solution to the problem (4.6) and satisfies the integral equation

$$\begin{aligned} &\int_2^x \int_2^y (u_{13}(t, z) - \phi(z)) dz dy \\ &\qquad - \int_0^t (u_{13}(\tau, x) - u_{12}(\tau, 2) - (x - 2) u_{12x}(\tau, 2)) d\tau \\ &\qquad - \int_0^t \int_2^x \int_2^y f(\tau, z, u_{13}(\tau, z), u_{13x}(\tau, z)) dz dy d\tau \\ &= 0, \quad t \in [0, m], \quad x \in [2, 3]. \end{aligned}$$

The function

$$u(t, x) = \begin{cases} u_{11}(t, x) & \text{in} \quad [0, m] \times [0, 1] \\ u_{12}(t, x) & \text{in} \quad [0, m] \times [1, 2] \\ u_{13}(t, x) & \text{in} \quad [0, m] \times [2, 3] \end{cases}$$

is a $\mathcal{C}^1([0, m], \mathcal{C}^2([0, 3]))$-solution to the problem

$$u_t - u_{xx} = f(t, x, u(t, x), u_x(t, x)) \quad \text{in} \quad [0, m] \times [0, 3],$$

$$u(0, x) = \phi(x) \quad \text{in} \quad [0, 3].$$

An so on. We construct a solution $u_1 \in \mathcal{C}^1([0, m], \mathcal{C}^2(\mathbb{R}))$ which is a solution to the problem

$$u_t - u_{xx} = f(t, x, u(t, x), u_x(t, x)) \quad \text{in} \quad (0, m] \times \mathbb{R},$$

$$u(0, x) = \phi(x) \quad \text{in} \quad \mathbb{R}.$$

5 Proof of Theorem 1.4

We have that the solution $u(t, x, \phi)$ satisfies the following integral equation

$$\begin{aligned}
Q(\phi) &= \int_c^x \int_c^y (u(t, z, \phi(z)) - \phi(z))dzdy \\
&\quad - \int_0^t (u(\tau, x, \phi(x)) - u(\tau, c, \phi(c)) - (x - c)u_x(\tau, c, \phi(c)))d\tau \\
&\quad - \int_0^t \int_c^x \int_c^y f(\tau, z, u(\tau, z, \phi(z)), u_x(\tau, z, \phi(z)))dz \\
&= 0, \quad t \in [0, m], \quad x \in [c, d].
\end{aligned}$$

Then

$$\begin{aligned}
Q(\phi) - Q(\phi_1) &= \int_c^x \int_c^y (u(t, z, \phi(z)) - u(t, z, \phi_1(z)) - (\phi(z) - \phi_1(z)))dzdy \\
&\quad - \int_0^t (u(\tau, x, \phi(x)) - u(\tau, x, \phi_1(x)))d\tau \\
&\quad + \int_0^t (u(\tau, c, \phi(c)) - u(\tau, c, \phi_1(c)))d\tau \\
&\quad + \int_0^t (x - c)(u_x(\tau, c, \phi(c)) - u_x(\tau, c, \phi_1(c)))d\tau \\
&\quad - \int_0^t \int_c^x \int_c^y \Big(f(\tau, z, u(\tau, z, \phi(z)), u_x(\tau, z, \phi(z))) \\
&\quad - f(\tau, z, u(\tau, z, \phi_1(z)), u_x(\tau, z, \phi_1(z))) \Big) dzdyd\tau \\
&= \int_c^x \int_c^y \left(\frac{\partial u}{\partial \phi}(t, z, \phi(z)) - 1 \right) dzdy
\end{aligned}$$

$$\begin{aligned}
&- \int_0^t \frac{\partial u}{\partial \phi}(\tau, x, \phi(x))d\tau + \int_0^t \frac{\partial u}{\partial \phi}(\tau, c, \phi(c))d\tau + \int_0^t (x - c) \left(\frac{\partial u}{\partial \phi} \right)_x (\tau, c, \phi(c))d\tau \\
&- \int_0^t \int_c^x \int_c^y \frac{\partial f}{\partial u}(\tau, z, u(\tau, z, \phi(z)), u_x(\tau, z, \phi(z))) \frac{\partial u}{\partial \phi}(\tau, z, \phi(z))dzdyd\tau \\
&- \int_0^t \int_c^x \int_c^y \frac{\partial f}{\partial u_x}(\tau, z, u(\tau, z, \phi(z)), u_x(\tau, z, \phi(z))) \left(\frac{\partial u}{\partial \phi} \right)_x (\tau, z, \phi(z))dzdyd\tau \\
&+ \delta\{\phi, \phi_1\},
\end{aligned}$$

where $\delta\{\phi, \phi_1\} \longrightarrow 0$ as $\phi(x) \longrightarrow \phi_1(x)$ for every $x \in [c, d]$. Hence, when $\phi(x) \longrightarrow \phi_1(x)$ for every $x \in [c, d]$, we get

$$\begin{aligned} 0 &= \int_c^x \int_c^y (v(t,z) - 1) dz dy - \int_0^t v(\tau, x) d\tau \\ &+ \int_0^t v(\tau, c) d\tau + \int_0^t x v_x(\tau, c) d\tau \\ &- \int_0^t \int_c^x \int_c^y \frac{\partial f}{\partial u}(\tau, z, u(\tau, z, \phi(z)), u_x(\tau, z, \phi(z))) v(\tau, z) dz dy d\tau \\ &- \int_0^t \int_c^x \int_c^y \frac{\partial f}{\partial u_x}(\tau, z, u(\tau, z, \phi(z)), u_x(\tau, z, \phi(z))) v_x(\tau, z) dz dy d\tau, \end{aligned} \tag{5.1}$$

which we differentiate twice in x and once in t and we get that v satisfies (1.3). Now we put $t = 0$ in (5.1) and then we differentiate twice in x, and we find that v satisfies (1.4).

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On New Types of Sets Via γ-open Sets in $(\mathfrak{a})$Topological Spaces

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ABSTRACT

In this paper, we introduced the notion of γ-semi-open sets and γ-P-semi-open sets in $(\mathfrak{a})$topological spaces which is a set equipped with countable number of topologies. Several properties of these notions are discussed.

RESUMEN

En este artículo, introducimos la noción de conjuntos γ-semi-abiertos y conjuntos γ-P-semi-abiertos en espacios $(\mathfrak{a})$topológicos, el cual es un conjunto dotado con una cantidad numerable de topologías. Discutimos diversas propiedades de estas nociones.

Keywords and Phrases: $(\mathfrak{a})$topological spaces, $(\mathfrak{a})$-γ-semi-open sets, $(\mathfrak{a})$-γ-P-semi-open sets.

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1 Introduction

The notion of bitopological space (X, τ_1, τ_2) (a non empty set X endowed with two topologies τ_1 and τ_2) is introduced by Kelly [5]. Kovár [7, 8] also studied the properties of a non empty set equipped with three topologies. Many authors studied a countable number of topologies in (ω)topological spaces and $(\aleph_0)$topological spaces in [1, 2, 3, 4]. Ogata [9] defined an operation γ on a topological space (X, τ) as a mapping from τ into the power set $P(X)$ of X such that $U \subseteq \gamma(U)$ for each $U \in \tau$, where $\gamma(U)$ denotes the value of γ at U. A susbet A of X is said to be γ-open if for each $x \in A$, there exists an open set U containing x such that $\gamma(U) \subseteq A$. In topological spaces, γ-P-open set are defined by Khalaf and Ibrahim [6]. The main purpose of this paper is to introduce the concept of γ-P-semi-open sets and γ-semi-open sets in (a)topological spaces. We give some properties related to these sets and introduce some separation axioms in (a)topological spaces. Further we define new types of functions in (a)topological spaces, namely (a)-γ-semi-continuous and (a)-γ-P-semi-continuous. An operation γ on (a)topological space $(X, \{\tau_n\})$ is a mapping $\gamma\colon \bigcup \tau_n \to P(X)$ such that $U \subseteq \gamma(U)$ for each $U \in \bigcup \tau_n$.
Throughout the paper, $\mathbb{N}$ denotes the set of natural numbers. The elements of $\mathbb{N}$ are denoted by i, m, n etc. μ stands for the discrete topology. The (τ_n)-closure (resp. (τ_n)-interior) of a set A is denoted by τ_n-cl(A) (resp. τ_n-Int(A)). By $\tau_{m\gamma}$-Int(A) and $\tau_{m\gamma}$-cl(A), we denote the $\tau_{m\gamma}$-interior of A and $\tau_{m\gamma}$-closure of A in $(X, \{\tau_n\})$, respectively. If there is no scope of confusion, we denote the (a)topological space $(X, \{\tau_n\})$ by X.

2 (a)topological spaces

Definition 2.1. *[10] If $\{\tau_n\}$ is a sequence of topologies on a set X, then the pair $(X, \{\tau_n\})$ is called an (a)topological space.*

Definition 2.2. *[9] A susbet A of X is said to be γ-open if for each $x \in A$, there exists an open set U containing x such that $\gamma(U) \subseteq A$.*

Definition 2.3. *Let X be an (a)topological space. A subset S of X is said to be:*

(i). (m, n)-*semi-open if* $S \subseteq \tau_m\text{-cl}(\tau_n\text{-Int}(S))$.

(ii). (m, n)-γ-*semi-open if* $S \subseteq \tau_{m\gamma}\text{-cl}(\tau_{n\gamma}\text{-Int}(S))$.

(iii). (m, n)-γ-P-*semi-open if* $S \subseteq \tau_m\text{-cl}(\tau_{n\gamma}\text{-Int}(S))$.

The complements of (m, n)-semi-open set, (m, n)-γ-semi-open set and (m, n)-γ-P-semi-open set are (m, n)-semi-closed, (m, n)-γ-semi-closed and (m, n)-γ-P-semi-closed, respectively.

Definition 2.4. *Let X be an (a)topological space. A subset S of X is said to be:*

(i). (a)-*semi-open if* S *is* (m, n)-*semi-open for all* $m \neq n$.

(ii). (a)-γ-*semi-open if* S *is* (m,n)-γ-*semi-open for all* $m \neq n$.

(iii). (a)-γ-P-*semi-open if* S *is* (m,n)-γ-P-*semi-open for all* $m \neq n$.

The complements of (a)-semi-open set, (a)-γ-semi-open set and (a)-γ-P-semi-open set are (a)-semi-closed, (a)-γ-semi-closed and (a)-γ-P-semi-closed, respectively.

By $SO(X)$, $\gamma SO(X)$ and $\gamma PSO(X)$, we denote the family of all (a)-semi-open sets, (a)-γ-semi-open sets and (a)-γ-P-semi-open sets in X, respectively.

Theorem 2.1. *Every* (a)-γ-P-*semi-open set is* (a)-γ-*semi-open.*

Proof. Let S be an (a)-γ-P-semi-open set. Then S is (m,n)-γ-P-semi-open for all $m \neq n$. So $S \subseteq \tau_m\text{-}cl(\tau_{n\gamma}\text{-}Int(S)) \subseteq \tau_{m\gamma}\text{-}cl(\tau_{n\gamma}\text{-}Int(S))$ for all $m \neq n$. This implies that S is (m,n)-γ-semi-open for all $m \neq n$. Thus, S is (a)-γ-semi-open. □ □

The following example shows that the converse of the above theorem is not true generally.

Example 2.5. *Consider* $X = \{a,b,c,d\}$ *with topologies* $\tau_1 = \{X,\emptyset,\{b\},\{d\},\{b,d\},\{a,b,c\}\}$, $\tau_2 = \{X,\emptyset,\{a\},\{d\},\{a,d\},\{a,b\},\{a,b,d\}\}$ *and* $\tau_i = \mu$ *for* $i \neq 1,2$. *Let* γ *be an operation on* $\bigcup \tau_n$ *defined as follows* :

$$\gamma(U) = \begin{cases} U, & \text{if } U = \{d\} \\ X, & \text{if } U \neq \{d\} \end{cases}$$

Then $\{b,c,d\}$ *is* (a)-γ-*semi-open but it is not* (a)-γ-P-*semi-open.*

Theorem 2.2. *Every* (a)-γ-P-*semi-open set is* (a)-*semi-open.*

Proof. Let S be an (a)-γ-P-semi-open set. Then S is (m,n)-γ-P-semi-open for all $m \neq n$. So $S \subseteq \tau_m\text{-}cl(\tau_{n\gamma}\text{-}Int(S)) \subseteq \tau_m\text{-}cl(\tau_n\text{-}Int(S))$ for all $m \neq n$. This implies that S is (m,n)-semi-open for all $m \neq n$. Thus, S is (a)-semi-open. □ □

The following example shows that the converse of the above theorem is not true generally.

Example 2.6. *Let* X, τ_1 *and* γ *be as in Example* 2.6. *and let* $\tau_i = \tau_2$ *for all* $i \neq 1$.
Then $\{a,b,c\}$ *is* (a)-*semi-open but not* (a)-γ-P-*semi-open.*

Following example shows that there is no relation between (a)-semi-open sets and (a)-γ-semi-open sets.

Example 2.7. *Let* $(X,\{\tau_n\})$ *and* γ *be as in Example* 2.8.
Then $\{a,b,c\}$ *is* (a)-*semi-open but not* (a)-γ-*semi-open and* $\{b,d\}$ *is* (a)-γ-*semi-open but not* (a)-*semi-open.*

Following example shows that (a)-γ-P-semi-open set need not be τ_i-open set.

Example 2.8. *Consider* $X = \{a, b, c, d\}$ *with topologies* $\tau_1 = \{X, \emptyset, \{a\}, \{d\}, \{a, d\}, \{a, b\}, \{a, b, d\}\}$, $\tau_i = \{X, \emptyset, \{b\}, \{d\}, \{b, d\}\}$ *for all* $i \neq 1$. *Let* γ *be an operation on* $\bigcup \tau_n$ *defined as follows*:

$$\gamma(U) = \begin{cases} U, & \text{if } U = \{d\} \\ X, & \text{if } U \neq \{d\} \end{cases}$$

Then $\{c, d\}$ *is* (a)-γ-P-*semi-open but is not* τ_i-*open.*

Following example shows that (a)-γ-P-semi-open set need not be γ_i-open set.

Example 2.9. *Let* $(X, \{\tau_n\})$ *and* γ *be as in Example* 2.10.
Then $\{c, d\}$ *is* (a)-γ-P-*semi-open but not* γ_i-*open* .

Theorem 2.3. *Let* $\{S_\alpha \colon \alpha \in \Lambda\}$ *be a class of* (a)-γ-P-*semi-open sets. Then* $\bigcup_{\alpha\in\Lambda} S_\alpha$ *is also an* (a)-γ-P-*semi-open set.*

Proof. Since each S_α is an (a)-γ-P-semi-open set, S_α is (m, n)-γ-P-semi-open for all $\alpha \in \Lambda$ and for all $m \neq n$. We have $S_\alpha \subseteq \tau_m\text{-}cl(\tau_{n\gamma}\text{-}Int(S_\alpha))$ for all $\alpha \in \Lambda$ and for all $m \neq n$. Hence, it is obtained

$$\begin{aligned} \bigcup_{\alpha\in\Lambda} S_\alpha &\subseteq \bigcup_{\alpha\in\Lambda} \tau_m\text{-}cl(\tau_{n\gamma}\text{-}Int(S_\alpha)) \\ &\subseteq \tau_m\text{-}cl(\bigcup_{\alpha\in\Lambda} \tau_{n\gamma}\text{-}Int(S_\alpha)) \\ &\subseteq \tau_m\text{-}cl(\tau_{n\gamma}\text{-}Int(\bigcup_{\alpha\in\Lambda} S_\alpha)). \end{aligned}$$

Therefore, $\bigcup_{\alpha\in\Lambda} S_\alpha$ is also an (a)-γ-P-semi-open set. □ □

Following example shows that the intersection of two (a)-γ-P-semi-open sets need not be again (a)-γ-P-semi-open.

Example 2.10. *Consider* $X = \{a, b, c, d\}$ *with topologies* $\tau_1 = \{X, \emptyset, \{c\}, \{d\}, \{c, d\}\}$, $\tau_i = \{X, \emptyset, \{c\}, \{d\}, \{c, d\}, \{b, c, d\}\}$ *for all* $i \neq 1$. *Let* γ *be an operation on* $\bigcup \tau_n$ *defined as follows* :

$$\gamma(U) = \begin{cases} U, & \text{if } U \in \{\{c\}, \{d\}\} \\ X, & \text{if } U \notin \{\{c\}, \{d\}\} \end{cases}$$

Then $\{b, c\}$ *and* $\{b, d\}$ *are* (a)-γ-P-*semi-open but their intersection* $\{b\}$ *is not* (a)-γ-P-*semi-open.*

Theorem 2.4. *A subset* F *is* (a)-γ-P-*semi-closed in* (a)*topological space* $(X, \{\tau_n\})$ *if and only if* $\tau_m\text{-}Int(\tau_{n\gamma}\text{-}cl(F)) \subseteq F$ *for all* $m \neq n$.

Proof. Let F be an (a)-γ-P-semi-closed set in X. Then $X \backslash F$ is (a)-γ-P-semi-open, so $X \backslash F \subseteq \tau_m$-$cl(\tau_{n\gamma}\text{-}Int(X \backslash F))$ for all $m \neq n$.

It follows that

$$\begin{aligned} F &\supseteq X\backslash\tau_m\text{-}cl(\tau_{n\gamma}\text{-}Int(X\backslash F)) \\ &= \tau_m\text{-}Int(X\backslash\tau_{n\gamma}\text{-}Int(X\backslash F)) \\ &= \tau_m\text{-}Int(\tau_{n\gamma}\text{-}cl(F)). \end{aligned}$$

Conversely, for all $m \neq n$, we obtain

$$\begin{aligned} X\backslash F &\subseteq X\backslash\tau_m\text{-}Int(\tau_{n\gamma}\text{-}cl(F)) \\ &= \tau_m\text{-}cl(X\backslash\tau_{n\gamma}\text{-}cl(F)) \\ &= \tau_m\text{-}cl(\tau_{n\gamma}\text{-}Int(X\backslash F)). \end{aligned}$$

which completes the proof. □ □

Theorem 2.5. *Let $\{F_\alpha\colon \alpha \in \Lambda\}$ be a class of (a)-γ-P-semi-closed sets. Then $\bigcap_{\alpha\in\Lambda} F_\alpha$ is also an (a)-γ-P-semi-closed.*

Proof. For each $\alpha \in \Lambda$, F_α is an (a)-γ-P-semi-closed set. This implies that $X\backslash F_\alpha$ is an (a)-γ-P-semi open set. By Theorem 2.12., $\bigcup_{\alpha\in\Lambda} X\backslash F_\alpha$ is an (a)-γ-P-semi open set. By De Morgan's Law, $X\backslash\bigcap_{\alpha\in\Lambda} F_\alpha$ is an (a)-γ-P-semi open set. Thus, $\bigcap_{\alpha\in\Lambda} F_\alpha$ is an (a)-γ-P-semi-closed set. □ □

Following example shows that the union of two (a)-γ-P-semi-closed sets need not be (a)-γ-P-semi-closed.

Example 2.11. *Let $(X, \{\tau_n\})$ and γ be as in Example 2.13.*
Then $\{a, c\}$ and $\{a, d\}$ are (a)-γ-P-semi-closed but their union $\{a, c, d\}$ is not (a)-γ-P-semi-closed.

Definition 2.12. *In an (a)topological space X, a point x of X is said to be (a)-γ-P-semi interior ((a)-γ-semi interior) point of S if there exists an (a)-γ-P-semi-open ((a)-γ-semi-open) set V such that $x \in V \subseteq S$.*

By (a)-γ-PS-$Int(A)$ (resp.(a)-γ-S-$Int(A)$), we denote the (a)-γ-PS-interior (resp.(a)-γ-S-interior) of A consisting of all (a)-γ-P-semi interior ((a)-γ-semi interior) points of A.

Theorem 2.6. *The following properties hold for any subset A of (a)topological space X :*

(i). (a)-γ-PS-$Int(A)$ *is the union of all (a)-γ-P-semi-open sets (the largest (a)-γ-P-semi-open set) contained in A.*

(ii). (a)-γ-PS-$Int(A)$ *is an (a)-γ-P-semi-open set.*

(iii). A *is (a)-γ-P-semi-open if and only if $A = (a)$-γ-PS-$Int(A)$.*

Proof. The proof follows from definitions. □ □

Theorem 2.7. *The following properties hold for any subsets A_1, A_2 and any class of subsets $\{A_\alpha : \alpha \in \Lambda\}$ of (a)topological space X :*

(i). *If $A_1 \subseteq A_2$, then (a)-γ-PS-$\mathrm{Int}(A_1) \subseteq (a)$-$\gamma$-$\mathrm{PS}$-$\mathrm{Int}(A_2)$.*

(ii). $\bigcup_{\alpha\in\Lambda}(a)$-$\gamma$-$\mathrm{PS}$-$\mathrm{Int}(A_\alpha) \subseteq (a)$-$\gamma$-$\mathrm{PS}$-$\mathrm{Int}(\bigcup_{\alpha\in\Lambda} A_\alpha)$.

(iii). (a)-γ-PS-$\mathrm{Int}(\bigcap_{\alpha\in\Lambda} A_\alpha) \subseteq \bigcap_{\alpha\in\Lambda}(a)$-$\gamma$-$\mathrm{PS}$-$\mathrm{Int}(A_\alpha)$.

Proof. (i). Since $A_1 \subseteq A_2$, (a)-γ-PS-$\mathrm{Int}(A_1)$ is an (a)-γ-P-semi-open set contained in A_2. But (a)-γ-PS-$\mathrm{Int}(A_2)$ is the largest (a)-γ-P-semi-open set contained in A_2. So (a)-γ-PS-$\mathrm{Int}(A_1) \subseteq (a)$-$\gamma$-$\mathrm{PS}$-$\mathrm{Int}(A_2)$.

(ii). From (i), we have (a)-γ-PS-$\mathrm{Int}(A_\alpha) \subseteq (a)$-$\gamma$-$\mathrm{PS}$-$\mathrm{Int}(\bigcup_{\alpha\in\Lambda} A_\alpha)$ for all $\alpha \in \Lambda$. Hence, $\bigcup_{\alpha\in\Lambda}(a)$-$\gamma$-$\mathrm{PS}$-$\mathrm{Int}(A_\alpha) \subseteq (a)$-$\gamma$-$\mathrm{PS}$-$\mathrm{Int}(\bigcup_{\alpha\in\Lambda} A_\alpha)$.

(iii). From (i), (a)-γ-PS-$\mathrm{Int}(\bigcap_{\alpha\in\Lambda} A_\alpha) \subseteq (a)$-$\gamma$-$\mathrm{PS}$-$\mathrm{Int}(A_\alpha)$ for all $\alpha \in \Lambda$. Hence, (a)-γ-PS-$\mathrm{Int}(\bigcap_{\alpha\in\Lambda} A_\alpha) \subseteq \bigcap_{\alpha\in\Lambda}(a)$-$\gamma$-$\mathrm{PS}$-$\mathrm{Int}(A_\alpha)$. □

□

The reverse inclusion in (ii) and (iii) of Theorem 2.19. may not be applicable as shown in the following examples.

Example 2.13. *Consider $X = \{a, b, c\}$ with topologies $\tau_1 = \{X, \emptyset, \{a\}, \{b, c\}\}$, $\tau_i = \{X, \emptyset, \{b\}\}$ for all $i \neq 1$. Let γ be an operation on $\bigcup \tau_n$ defined as follows :*

$$\gamma(U) = \begin{cases} U, & \text{if } U = \{c\} \\ X, & \text{if } U \neq \{c\} \end{cases}$$

$\{a, b, c\} = (a)$-γ-PS-$\mathrm{Int}\{a, b, c\} \nsubseteq (a)$-$\gamma$-$\mathrm{PS}$-$\mathrm{Int}\{a\} \cup (a)$-$\gamma$-$\mathrm{PS}$-$\mathrm{Int}\{b, c\} = \emptyset$.

Example 2.14. *Let $(X, \{\tau_n\})$ and γ be as in Example 2.13.*
$\{b\} = (a)$-γ-PS-$\mathrm{Int}\{b, c\} \cap (a)$-$\gamma$-$\mathrm{PS}$-$\mathrm{Int}\{b, d\} \nsubseteq (a)$-$\gamma$-$\mathrm{PS}$-$\mathrm{Int}\{b\} = \emptyset$.

Definition 2.15. *In an (a)topological space X, a point x of X is said to be (a)-γ-P-semi cluster ((a)-γ-semi cluster) point of a subset $A \subset X$ if $A \cap V \neq \emptyset$ for every (a)-γ-P-semi-open ((a)-γ-semi-open set) containing x.*

By (a)-γ-PS-$\mathrm{cl}(A)$ (resp.(a)-γ-S-$\mathrm{cl}(A)$), we denote the (a)-γ-PS-closure (resp.(a)-γ-S-closure) of A consisting of all (a)-γ-P-semi cluster ((a)-γ-semi cluster) points of A.

Theorem 2.8. *The following properties hold for any subset A of an (a)topological space X :*

(i). *(a)-γ-PS-$\mathrm{cl}(A)$ is the intersection of all (a)-γ-P-semi-closed sets (the smallest (a)-γ-P-semi-closed set) containing A.*

(ii). (a)-γ-PS-$cl(A)$ *is an* (a)-γ-P-*semi-closed set.*

(iii). A *is* (a)-γ-P-*semi-closed if and only if* $A = (a)$-γ-PS-$cl(A)$.

Proof. The proof follows from definitions. ☐ ☐

Theorem 2.9. *The following properties hold for any subsets* A_1, A_2 *and any class of subsets* $\{A_\alpha : \alpha \in \Lambda\}$ *of an* (a)*topological space* X:

(i). *If* $A_1 \subseteq A_2$, *then* (a)-γ-PS-$cl(A_1) \subseteq (a)$-γ-PS-$cl(A_2)$.

(ii). $\bigcup_{\alpha\in\Lambda}(a)$-$\gamma$-PS-$cl(A_\alpha) \subseteq (a)$-$\gamma$-PS-$cl(\bigcup_{\alpha\in\Lambda} A_\alpha)$.

(iii). (a)-γ-PS-$cl(\bigcap_{\alpha\in\Lambda} A_\alpha) \subseteq \bigcap_{\alpha\in\Lambda}(a)$-$\gamma$-PS-$cl(A_\alpha)$.

Proof. (i). Since $A_1 \subseteq A_2$, (a)-γ-PS-$cl(A_2)$ is an (a)-γ-P-semi-closed set containing A_1. But (a)-γ-PS-$cl(A_1)$ is the smallest (a)-γ-P-semi-closed set containing A_1. so (a)-γ-PS-$cl(A_1) \subseteq (a)$-γ-PS-$cl(A_2)$.

(ii). From (i), (a)-γ-PS-$cl(A_\alpha) \subseteq (a)$-$\gamma$-PS-$cl(\bigcup_{\alpha\in\Lambda} A_\alpha)$ for all $\alpha \in \Lambda$. Hence, $\bigcup_{\alpha\in\Lambda}(a)$-$\gamma$-PS-$cl(A_\alpha) \subseteq (a)$-$\gamma$-PS-$cl(\bigcup_{\alpha\in\Lambda} A_\alpha)$.

(iii). From (i), (a)-γ-PS-$cl(\bigcap_{\alpha\in\Lambda} A_\alpha) \subseteq (a)$-$\gamma$-PS-$cl(A_\alpha)$ for all $\alpha \in \Lambda$. Hence, (a)-γ-PS-$cl(\bigcap_{\alpha\in\Lambda} A_\alpha) \subseteq \bigcap_{\alpha\in\Lambda}(a)$-$\gamma$-PS-$cl(A_\alpha)$. ☐

☐

The reverse inclusion in (ii) and (iii) of Theorem 2.24 may not be applicable as shown in the following examples.

Example 2.16. *Let* $(X, \{\tau_n\})$ *and* γ *be as in Example* 2.13.
$\{a, b, c, d\} = (a)$-γ-PS-$cl\{a, c, d\} \nsubseteq (a)$-$\gamma$-PS-$cl\{a, c\} \cup (a)$-$\gamma$-PS-$cl\{a, d\} = \{a\}$.

Example 2.17. *Consider* $X = \{a, b, c\}$ *with topologies* $\tau_1 = \{X, \emptyset, \{a\}, \{a, b\}\}$ *and* $\tau_i = \{X, \emptyset, \{b\}, \{a, b\}\}$ *for all* $i \neq 1$. *Let* γ *be an operation on* $\bigcup \tau_n$ *defined as follows* :

$$\gamma(U) = \begin{cases} U, & \textit{if } U = \{a, b\} \\ X, & \textit{if } U \neq \{a, b\} \end{cases}$$

$\{a, b, c\} = (a)$-γ-PS-$cl\{a, c\} \cap (a)$-γ-PS-$cl\{b, c\} \nsubseteq (a)$-$\gamma$-PS-$cl\{c\} = \{c\}$.

Theorem 2.10. *The following properties hold for a subset* A *of an* (a)*topological space* X:

(i). (a)-γ-PS-$Int(X\backslash A) = X\backslash(a)$-$\gamma$-PS-$cl(A)$.

(ii). (a)-γ-PS-$cl(X\backslash A) = X\backslash(a)$-$\gamma$-PS-$Int(A)$.

Proof. 1. By part (i). of Theorem 2.18., we have

$$\begin{aligned}
(a)\text{-}\gamma\text{-PS-Int}(X\backslash A) &= \bigcup\{S \subset X\colon S \text{ is } (a)\text{-}\gamma\text{-P-semi-open and } S \subset X\backslash A\}\\
&= \bigcup\{X\backslash(X\backslash S) \subset X\colon X\backslash S \text{ is } (a)\text{-}\gamma\text{-P-semi-closed and } A \subset X\backslash S\}\\
&= X\backslash \bigcap\{X\backslash S \subset X\colon X\backslash S \text{ is } (a)\text{-}\gamma\text{-P-semi-closed and } A \subset X\backslash S\}\\
&= X\backslash \bigcap\{F \subset X\colon F \text{ is } (a)\text{-}\gamma\text{-P-semi-closed and } A \subset F\}\\
&= X\backslash (a)\text{-}\gamma\text{-PS-cl}(A).
\end{aligned}$$

2. By part (i). of Theorem 2.23., we have

$$\begin{aligned}
(a)\text{-}\gamma\text{-PS-cl}(X\backslash A) &= \bigcap\{S \subset X\colon S \text{ is } (a)\text{-}\gamma\text{-P-semi-closed and } X\backslash A \subset S\}\\
&= \bigcap\{X\backslash(X\backslash S) \subset X\colon X\backslash S \text{ is } (a)\text{-}\gamma\text{-P-semi-open and } X\backslash S \subset A\}\\
&= X\backslash \bigcup\{X\backslash S \subset X\colon X\backslash S \text{ is } (a)\text{-}\gamma\text{-P-semi-open and } X\backslash S \subset A\}\\
&= X\backslash \bigcup\{F \subset X\colon X\backslash F \text{ is } (a)\text{-}\gamma\text{-P-semi-open and } F \subset A\}\\
&= X\backslash (a)\text{-}\gamma\text{-PS-Int}(A).
\end{aligned}$$

□

Definition 2.18. *A set A is said to be (a)-γ-P-semi neighborhood of a point x in an (a)topological space X if there exists an (a)-γ-P-semi-open set U such that $x \in U \subseteq A$.*

Theorem 2.11. *A subset of an (a)topological space X is (a)-γ-P-semi-open if and only if it is (a)-γ-P-semi neighborhood of each of its points.*

Proof. The proof follows from definition 2.28. □ □

Definition 2.19. *An (a)topological space X is said to be (a)-γ-PS-T_0 if for every distinct points x and y of X, there exists an (a)-γ-P-semi-open set U such that $x \in U$ but $y \notin U$ or vice versa.*

Theorem 2.12. *An (a)topological space X is (a)-γ-PS-T_0 if and only if for each distinct points x and y of X (a)-γ-PS-$cl\{x\} \neq (a)$-γ-PS-$cl\{y\}$.*

Proof. Let x and y be any two distinct points of X. Then there exists an (a)-γ-P-semi-open set U such that $x \in U$ but $y \notin U$ or vice versa. Without loss of generality, assume that U containing x but not y. Then we have $\{y\} \cap U = \emptyset$ which implies $x \notin (a)$-γ-PS-$cl\{y\}$. Hence, (a)-γ-PS-$cl\{x\} \neq (a)$-γ-PS-$cl\{y\}$.

Conversely, let x and y be any two distinct points of X. Then we have (a)-γ-PS-$cl\{x\} \neq (a)$-γ-PS-$cl\{y\}$. Without loss of generality let $z \in (a)$-γ-PS-$cl\{y\}$ but $z \notin (a)$-γ-PS-$cl\{x\}$. Then $\{y\} \cap U \neq \emptyset$ for every (a)-γ-P-semi-open set U containing z and $\{x\} \cap U = \emptyset$ for atleast one (a)-γ-P-semi-open set U containing z. Thus, $y \in U$ and $x \notin U$. Hence, X is (a)-γ-PS-T_0. □ □

Definition 2.20. *An* (a)*topological space* $(X, \{\tau_n\})$ *is said to be* (a)-γ-PS-T_1 *if for every distinct points* x *and* y *of* X*, there exist two* (a)-γ-P-*semi-open sets which one of them contains* x *but not* y *and the other one contains* y *but not* x*.*

Theorem 2.13. *An* (a)*topological space* X *is* (a)-γ-PS-T_1 *if and only if for each point* x *of* X (a)-γ-PS-$\mathsf{cl}\{x\} = \{x\}$*.*

Proof. Since $\{x\} \subseteq (a)$-γ-PS-$\mathsf{cl}\{x\}$, Let $y \in (a)$-γ-PS-$\mathsf{cl}\{x\}$ be arbitrary. On contrary suppose that $y \notin \{x\}$. Then there exists an (a)-γ-P-semi-open set U such that $y \in U$ but $x \notin U$. Then we have $\{x\} \cap U = \emptyset$ which implies $y \notin (a)$-γ-PS-$\mathsf{cl}\{x\}$. Hence, contradiction.
Conversely, let $x \neq y$ for $x, y \in X$. Since $x \notin (a)$-γ-PS-$\mathsf{cl}\{y\}$ and $y \notin (a)$-γ-PS-$\mathsf{cl}\{x\}$, there exist (a)-γ-P-semi-open sets U and V containing x and y, respectively such that $\{y\} \cap U = \emptyset$ and $\{x\} \cap V = \emptyset$. Thus, we have $x \in U$, $y \notin U$ and $y \in V$, $x \notin V$. Hence, X is (a)-γ-PS-T_1. □ □

Definition 2.21. *An* (a)*topological space* X *is said to be* (a)-γ-PS-T_2 *if for every distinct points* x *and* y *of* X*, there exist two disjoint* (a)-γ-P-*semi-open sets* U *and* V *containing* x *and* y*, respectively.*

Theorem 2.14. *An* (a)*topological space* X *is* (a)-γ-PS-T_2 *if and only if for each distinct points* x *and* y *of* X *there exists an* (a)-γ-P-*semi-open set* U *containing* x *such that* $y \notin (a)$-γ-PS-$\mathsf{cl}(U)$*.*

Proof. Let X be an (a)-γ-PS-T_2 space. On contrary suppose that $y \in (a)$-γ-PS-$\mathsf{cl}(U)$ for all (a)-γ-P-semi-open set U containing x. Then $U \cap V \neq \emptyset$ for every (a)-γ-P-semi-open set V containing y and (a)-γ-P-semi-open set U containing x. Thus, contradiction.
Conversely, let x and y be any two distinct point of X. Then there exist two disjoint (a)-γ-P-semi-open sets U and V containing x and y, respectively. This implies that $\{y\} \cap U = \emptyset$. Hence, $y \notin (a)$-γ-PS-$\mathsf{cl}(U)$. □ □

Theorem 2.15. *An* (a)*topological space* X *is* (a)-γ-PS-T_2 *if and only if the intersection of all* (a)-γ-PS-*closed neighborhood of each point of* X *consists of only that point.*

Proof. Let $x \in X$ be arbitrary and $y \in X$ such that $y \neq x$. Then there exist disjoint (a)-γ-P-semi-open sets U_y and V_y containing x and y, respectively. Since $U_y \subseteq X \backslash V_y$, $X \backslash V_y$ is an (a)-γ-PS-closed neighborhood of x which does not contain y. Hence, $\cap\{X \backslash V_y : y \in X, y \neq x\} = \{x\}$.
Conversely, let x and y be any two distinct points of X. Since $\{x\} = \cap\{S \subset X\colon S$ is (a)-γ-PS-closed neighborhood of $x\}$. This implies that there exists an (a)-γ-PS-closed neighborhood U of x not containing y. Then, $y \in X \backslash U$ and $X \backslash U$ is (a)-γ-P-semi-open. Since, U is an (a)-γ-PS-neighborhood of x, then there exists an (a)-γ-P-semi-open set V containing x such that $V \subseteq U$. Clearly, V and $X \backslash U$ are disjoint. Hence, $(X, \{\tau_n\})$ is (a)-γ-PS-T_2. □ □

Remark 2.22. (i). *Every* (a)-γ-PS-T_2 (a)*topological space is* (a)-γ-PS-T_1*.*

(ii). *Every* (a)-γ-PS-T_1 (a)*topological space is* (a)-γ-PS-T_0*.*

Following examples shows that converse of above remark need not be true.

Example 2.23. *Let* $X = \{a, b, c, d\}$ *with topologies* $\tau_1 = \{\emptyset, X, \{c\}, \{d\}, \{c, d\}\{a, b, c\}\}$ *and* $\tau_i = \{\emptyset, X, \{c\}, \{d\}, \{c, d\}\}$ *for all* $i \neq 1$.

$$\gamma(U) = \begin{cases} U, & \text{if } U \in \{\{c\}, \{d\}, \{a, b, c\}\} \\ X, & \text{if } U \notin \{\{c\}, \{d\}, \{a, b, c\}\} \end{cases}$$

Then $\tau_{n\gamma} = \tau_n$ *for all* $n \in \mathbb{N}$ *and* (a)-γ-$PSO = \{\emptyset, X, \{c\}, \{d\}, \{c, d\}\{a, b, c\}, \{a, c\}, \{b, c\}\}$.
Clearly, $(X, \{\tau_n\})$ *is* (a)-γ-PS-T_0 *but not* (a)-γ-PS-T_1.

Example 2.24. *Let* $X = \{a, b, c\}$ *with topologies* $\tau_n = \mu$ *for all* n.

$$\gamma(U) = \begin{cases} U, & \text{if } U \in \{\{a, b\}, \{a, c\}, \{b, c\}\} \\ X, & \text{if } U \notin \{\{a, b\}, \{a, c\}, \{b, c\}\} \end{cases}$$

Then $\tau_{n\gamma} = \{\emptyset, X, \{a, b\}, \{a, c\}, \{b, c\}\}$ *for all* $n \in \mathbb{N}$ *and* (a)-γ-$PSO = \{\emptyset, X, \{a, b\}, \{a, c\}, \{b, c\}\}$.
Clearly $(X, \{\tau_n\})$ *is* (a)-γ-PS-T_1 *but not* (a)-γ-PS-T_2.

Example 2.25. *Let* $X = \{a, b, c\}$ *with topologies* $\tau_n = \mu$ *for all* n.

$$\gamma(U) = \begin{cases} U, & \text{if } U \in \{\{a\}, \{b\}, \{c\}\} \\ X, & \text{if } U \notin \{\{a\}, \{b\}, \{c\}\} \end{cases}$$

Then $\tau_{n\gamma} = \mu$ *for all* $n \in \mathbb{N}$ *and* (a)-γ-$PSO = \mu$
Clearly X *is* (a)-γ-PS-T_2 *space.*

Definition 2.26. *Let* $f\colon (X, \{\tau_n\}) \to (Y, \{\zeta_n\})$ *be a function and* x *be any point of* X. f *is said to be* (a)-γ-P-*semi continuous* (*resp.*(a)-γ-*semi continuous*) *at* x *if for every* ζ_n *open subset* O *of* Y *containing* $f(x)$ *there exists an* (a)-γ-P-*semi-open* (*resp.* (a)-γ-*semi-open*) *set* G *of* X *containing* x *such that* $f(G) \subseteq O$.

Theorem 2.16. *For a function* $f\colon (X, \{\tau_n\}) \to (Y, \{\zeta_n\})$, *the followings statements are equivalent* :

(i). f *is* (a)-γ-P-*semi continuous* (*resp.*(a)-γ-*semi continuous*).

(ii). *For every* ζ_n *open subset* O *of* Y, $f^{-1}(O)$ *is an* (a)-γ-P-*semi-open* (*resp.*(a)-γ-*semi-open*) *set in* X.

(iii). *For every* ζ_n *closed subset* F *of* Y, $f^{-1}(F)$ *is an* (a)-γ-P-*semi-closed* (*resp.*(a)-γ-*semi-closed*) *set in* X.

(iv). *For every subset* T *of* X, $f((a)\text{-}\gamma\text{-PS-}cl(T)) \subseteq \zeta_n\text{-}cl(f(T))$ (*resp.* $f((a)\text{-}\gamma\text{-S-}cl(T)) \subseteq \zeta_n\text{-}cl(f(T))$).

(v). *For every subset* F *of* Y, $(a)\text{-}\gamma\text{-PS-}cl(f^{-1}F) \subseteq f^{-1}(\zeta_n\text{-}cl(F))$(*resp.* $(a)\text{-}\gamma\text{-S-}cl(f^{-1}F) \subseteq f^{-1}(\zeta_n\text{-}cl(F))$).

Proof. $(1). \Longrightarrow (ii)$. Let O be ζ_n open in Y and $x \in f^{-1}(O)$ be arbitrary. Since f is (a)-γ-P-semi continuous on X, there exists an (a)-γ-P-semi-open set G of X containing x such that $f(G) \subseteq O$. Thus, we have $G \subseteq f^{-1}(O)$. Hence, $f^{-1}(O)$ is an (a)-γ-P-semi-open set in X.
$(ii). \Longrightarrow (i)$. Let x be any point of X and H be a ζ_n open set containing $f(x)$. We get $f^{-1}(H)$ is (a)-γ-P-semi-open and $x \in f^{-1}(H)$. Take $G = f^{-1}(H)$, we have $f(G) \subseteq H$. Hence, f is (a)-γ-P-semi continuous.
$(ii) \Longleftrightarrow (iii)$. Obviously.
$(i). \Longrightarrow (iv)$. Let T be a subset of X and $f(x) \in f((a)$-γ-PS-cl$(T))$, for $x \in (a)$-γ-PS-cl(T). Let H be any ζ_n open set of Y containing $f(x)$. By hypothesis there exists an (a)-γ-P-semi-open set G of X containing x such that $f(G) \subseteq H$. Since $G \cap T \neq \emptyset$, $H \cap f(T) \neq \emptyset$. This implies that $f(x) \in \zeta_n$-cl$(f(T))$. Hence, (a)-γ-PS-cl$(f^{-1}F) \subseteq f^{-1}(\zeta_n$-cl$(F))$.
$(iv). \Longrightarrow (v)$. Let F be a subset of Y. By hypothesis, we have $f((a)$-γ-PS-cl$(F)) \subseteq \zeta_n$-cl$(f(F))$. Taking the pre-image on both sides, we get (a)-γ-PS-cl$(f^{-1}F) \subseteq f^{-1}(\zeta_n$-cl$(F))$.
$(v). \Longrightarrow (iii)$. Let F be ζ_n-closed in Y. By hypothesis, we have (a)-γ-PS-cl$(f^{-1}F) \subseteq f^{-1}(F)$. Hence, $f^{-1}(F)$ is (a)-γ-P-semi-closed in X. □ □

Corolary 1. (i). *Every* (a)-γ-P-*semi continuous function is* (a)-γ-*semi continuous.*

(ii). *Every* (a)-γ-P-*semi continuous function is* (a)-*semi continuous.*

Following example shows that (a)-γ-semi continuous function need not be (a)-γ-P-semi continuous.

Example 2.27. *Consider* $X = \{a, b, c, d\}$ *with topologies* $\tau_1 = \{X, \emptyset, \{b\}, \{d\}, \{b, d\}\}$, $\tau_i = \{X, \emptyset, \{a\}, \{d\}, \{a, d\}, \{a, b\}, \{a, b, d\}\}$ *for all* $i \neq 1$. *Let* γ *be an operation on* $\bigcup \tau_n$ *defined as follows* :

$$\gamma(U) = \begin{cases} U, & \text{if } U = \{d\} \\ X, & \text{if } U \neq \{d\} \end{cases}$$

Define $f\colon (X, \{\tau_n\}) \to (X, \{\tau_n\})$ *as* $f\{a, b, d\} = d$, $f(c) = c$. *Then* f *is* (a)-γ-*semi continuous function but not* (a)-γ-*P-semi continuous as* $\{a, b, d\}$ *is not* (a)-γ-*P-semi-open.*

Example 2.28. *Consider* $X = \{a, b, c, d\}$ *with topologies* $\tau_1 = \{X, \emptyset, \{b\}, \{d\}, \{b, d\}, \{a, b, c\}\}$, $\tau_i = \{X, \emptyset, \{a\}, \{d\}, \{a, d\}, \{a, b\}, \{a, b, d\}\}$ *for all* $i \neq 1$. *Let* γ *be an operation on* $\bigcup \tau_n$ *defined as follows* :

$$\gamma(U) = \begin{cases} U, & \text{if } U = \{a\}, \{b\} \\ X, & \text{if } U \neq \{a\}, \{b\} \end{cases}$$

Define $f\colon (X, \{\tau_n\}) \to (X, \{\tau_n\})$ *as* $f\{a, b, c\} = d$, $f(d) = c$. *Then* f *is* (a)-*semi continuous function but not* (a)-γ-*P-semi continuous as* $\{d\}$ *is not* (a)-γ-*P-semi-open.*

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Optimal control of a SIR epidemic model with general incidence function and a time delays

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ABSTRACT

In this paper, we introduce an optimal control for a SIR model governed by an ODE system with time delay. We extend the stability studies of model (2.2) in section 2, by incorporating suitable controls. We consider two control strategies in the optimal control model, namely: the vaccination and treatment strategies. The model has a time delays that represent the incubation period. We derive the first-order necessary conditions for the optimal control and perform numerical simulations to show the effectiveness as well as the applicability of the model for different values of the time delays. These numerical simulations show that the model is sensitive to the delays representing the incubation period.

RESUMEN

En este artículo, introducimos un control óptimo para un modelo SIR gobernado por un sistema de EDOs con retardo temporal. Extendemos los estudios de estabilidad del modelo (2) en la sección 2, incorporando controles apropiados. Consideramos dos estrategias de control en el modelo de control óptimo, llámense: las estrategias de vacunación y tratamiento. El modelo tiene un retardo en el tiempo que representa el período de incubación. Derivamos las condiciones necesarias de primer orden para el control óptimo y realizamos simulaciones numéricas para mostrar la efectividad y también la aplicabilidad del modelo para diferentes valores de los retardos temporales. Estas simulaciones numéricas muestran que el modelo es sensible a los retardos que representan el período de incubación.

Keywords and Phrases: SIR, general incidence, delays, optimal control, epidemic models, Hamiltonian.

2010 AMS Mathematics Subject Classification: 34K19, 34K20, 49K25, 49K30, 65N06, 90C90.

1 Introduction

Mathematical modeling of population are often used to describe the dynamics of epidemic diseases. This is a fast growing research area and has been paying important roles in discovering relations between species and their interactions. There have been many variations such as classical epidemiological models [11]. These models are based on the standard Susceptible-Infectious-Susceptible (SIS), Susceptible-Infectious-Recovered (SIR) and Susceptible-Exposed-Infectious-Recovered (SEIR) models, which are determined according to the difference on the method of transmission, nature of the disease, those with short/long incubation period, killer/curable diseases, etc, and the response of the individuals to it, for instance, gaining transient/permanent immunity, dying from the disease, etc.[6, 16]. The main purpose of formulating a such epidemiological model is to understand the long-term behavior of the epidemic disease and to determine the possible strategies to control it.
Differential equations, whether there are ordinary, delay, partial or stochastic are one of the main mathematical tools being used to formulate many epidemiological models. The focus in such epidemiological models has been on the general incidence at which people move from the class of susceptible individuals to the class of infective individuals.these general incidence have been modeled mostly by using bilinear and Holling type of functional responses [10, 12].
On the other hand, optimal control has extensively been used a strategy to control the epidemic outbreaks [8]. The main idea behind using the optimal control in epidemics is to search for, among the available strategies, the most effective strategy that reduces the infection rate to a minimum level while optimizing the cost of deploying a therapy or preventive vaccine that is used for controlling the disease progression [18]. In terms of epidemic diseases, such strategies can include therapies, vaccines, isolation and educational campaigns [3, 5].
Mathematical models have become important tools in analyzing the spread and control of infectious diseases. The model formulation process clarifies assumptions, variables parameters. There have been many studies that have mathematically analyzed infectious diseases [4, 7, 15]. Recently, many control optimal models pertaining to epidemic disease to epidemic diseases have appeared in the literature. They include, but not limited to, delayed SIRS epidemic model [13], delayed SIR model [1], tuberculosis model [17], HIV model [9] and dengue fever [2].
In this paper, we consider an optimal control problem governed by a system of delay differential equations with general incidence function and time delays. The governing state equations of the optimal control are described in a SIR framework with a general incidence function and a time delays representing the incubation period. Then we derive first-order necessary conditions for existence of the optimal control and develop a numerical method to solve them.
The rest of this paper is organized as follows. In section 2, we give the statement of the optimal control problem. We derive the necessary conditions for existence of the optimal control in section 3. In section 4, we describe the numerical method and present the resulting numerical simulations. Finally, we discuss these results in section 5 along with some concluding remarks.

2 Statement of the optimal control problem

Compute the optimal pair of vaccination and treatment strategies (u_1, u_2) that would maximize the recovered population and minimize both the infected and susceptible population, and at the same time minimize the costs of applying the vaccination and treatment strategies.
So we consider the optimal control problem of the form (see Eihab B. M. et al):

$$\min_{(u_1,u_2)\in \mathbb{U}} J(u_1(t), u_2(t)) = \begin{cases} S(T) + I(T) - R(T) \\ + \displaystyle\int_0^T (c_1 u_1^2(t) + c_2 u_2^2(t) + S(t) + I(t) - R(t))dt, \end{cases} \tag{2.1}$$

subject to the quation

$$\begin{cases} \dot{S} = B - \mu_1 S - f(S, I_\tau) - u_1 S, \\ \dot{I} = f(S, I_\tau) - (\mu_2 + \gamma)I - u_2 I, \\ \dot{R} = \gamma I - \mu_3 R. \end{cases} \tag{2.2}$$

The two functions $u_1(t)$ and $u_2(t)$ represent vaccination and treatment strategies. These control functions are assumed to be $L^\infty(0, T)$ functions belonging to a set of admissible controls

$$\mathbb{U} = \{(u_1, u_2) \in (L^\infty(0, T))^2 : u_{1min} \le u_1(t) \le u_{1max}, u_{2min} \le u_2(t) \le u_{2max}\},$$

where $0 \le u_{1min} < u_{1max} \le 1$ and $0 \le u_{2min} < u_{2max} \le 1$. The two constants c_1 and c_2 are weighted cost associated with the use of the controls $u_1(t)$ and $u_2(t)$, respectively. The state equations are formulated from an SIR model with general incidence model, where $S(t)$, $I(t)$, $R(t)$ are the numbers of susceptible, infected and recovered individuals at time t, respectively. The parameters B is the recruitment rate, the death rates for the classes are μ_1, μ_2 and μ_3, respectively. The average time spent in class I before recovery is $1/\gamma$. For biological reasons, we assume that $\mu_1 \le \mu_2 + \gamma$; that is, removal of infectives is at least as fast as removal of susceptibles. The time delays τ represents the incubation period. That is to say, only susceptible individuals who got infected a time $t - \tau$ are able to communicate the disease at time t.
As general as possible, the incidence function f must satisfy technical conditions. Thus, we assume that
H1 f is non-negative C^1 functions on the non-negative quadrant,
H2 for all $(S, I) \in \mathbb{R}_+^2$, $f(S, 0) = f(0, I) = 0$.
Let us denote by f_1 and f_2 the partial derivatives of f with respect to the first and to the second variable

The differential equation model described by (2.2) without controls ($u_1 = u_2 = 0$) has two equilibrium points: a disease-free equilibrium E_0 given by

$$E_0 = \left(\frac{B}{\mu_1}, 0, 0\right)$$

and an endemic equilibrium $E^* = (S^*, I^*, R^*)$ where,

$$\begin{aligned} S^* &= \frac{B - (\mu_2 + \gamma)I^*}{\mu_1} \\ I^* &= I^* \\ R^* &= \frac{\gamma}{\mu_3} I^* \end{aligned}$$

The basic reproduction number of (2.2) without controls is given by

$$R_0 = \frac{f_2(S^0, 0)}{\mu_2 + \gamma}$$

It was proven that if $R_0 < 1$, then the disease-free equilibrium is asymptotically stable and if $R_0 > 1$ then it is unstable.
On the other-hand, when the controls is not null ($u_1 \neq 0$ and or $u_2 \neq 0$), we have the SIR model (2.2).

The disease-free equilibrium for system (2.2) is given by

$$E_0^c = \left(\frac{B}{\mu_1 + u_1}, 0, 0 \right) \tag{2.3}$$

whereas the endemic equilibrium E_c^* is given by

$$E_c^* = \left(\frac{B - (\mu_2 + \gamma + u_2)I^*}{\mu_1 + u_1}, I^*, \frac{\gamma}{\mu_3} I^* \right)$$

The basic reproduction number R_c of system (2.2) is given by

$$R_c = \frac{f_2(S^0, 0)}{\mu_2 + \gamma + u_2}$$

and it is clear that when $u_1 \to 0$ and $u_2 \to 0$ then $R_c \to R_0$

3 Existence and characterization of the optimal control

In this section, we discuss the existence of the optimal control and then construct the Hamiltonian of the optimal control problem to derive the first order necessary conditions for the optimal control.

3.1 Existence of optimal control

To show the existence of the optimal control for the problem under consideration, we notice that the set of admissible controls $\mathbb{U}$ is, by definition, closed and bounded. It is also convex because $[u_{1min}, u_{1max}] \times [u_{2min}, u_{2max}]$ is convex in $\mathbb{R}^2$. It is obvious that there is an admissible pair $((u_1(t), u_2(t)))$ for the problem. Hence, the existence of the optimal control comes as a direct result from the Filippove-Cesari theorem [14]. We therefore, have the following result:

Theorem 3.1. *Consider the optimal control problem* (2.1) *subject to* (2.2). *Then there exists an optimal pair of controls* (u_1^*, u_2^*) *and a corresponding optimal states* (S^*, I^*, R^*) *that minimizes the objective function* $J(u_1, u_2)$ *over set of admissible controls* $\mathbb{U}$.

Proof. To prove the existence of an optimal control pair, it is important to verify the following assertion.

(1) The set of controls and corresponding state variables is nonempty.

(2) The admissible set $\mathbb{U}$ is convex and closed.

(3) The right-hand side of the state system (2.2) is bounded by a linear function in the state and control variables.

(4) The integrand of the objective functional $L_{S,I,R}(u_1, u_2)$ is convex on the set $\mathbb{U}$. The hessian matrix of $L_{S,I,R}(u_1, u_2)$ on $\mathbb{U}$ is done by :

$$M = \begin{pmatrix} 2c_1 & 0 \\ 0 & 2c_2 \end{pmatrix},$$

$Sp(M) = \{2c_1,\ 2c_2\} \subset \mathbb{R}_+^*$,
then, $L_{S,I,R}(u_1, u_2)$ is strictly convex in $\mathbb{U}$.

(5) There exists constants $\omega_1 > 0$, ω_2 and $\rho > 1$ such that the integrand $L_{S,I,R}(u_1, u_2)$ of the objective functional satisfies

$$L_{S,I,R}(u_1, u_2) \geq \omega_1 |(u_1, u_2)|^\rho - \omega_2.$$

$$\begin{aligned} L_{S,I,R}(u_1, u_2) &= c_1 u_1^2(t) + c_2 u_2^2(t) + S(t) + I(t) - R(t) \\ &\geq \min(c_1, c_2)(u_1^2(t) + u_2^2(t)) - R(t) \end{aligned}$$

$R(t)$ is bounded because $N = S + I + R$
i.e

$$\exists \alpha, \beta,\ \alpha < R(t) < \beta, \forall t$$

Let $\omega_1 = \min(c_1, c_2)$ and $\omega_2 = \beta$. We have,

$$L_{S,I,R}(u_1, u_2) \geq \omega_1 \|(u_1; u_2)\|^2 - \omega_2.$$

□

3.2 Characterization of optimal control

In this subsection, we derive the first order necessary conditions for the existence of optimal control, by constructing the Hamiltonian H and applying the Pontryagin's maximum principle.
To simplify the notations, we write $x(t) = [S(t), I(t), R(t)]^T$, $u(t) = [u_1(t), u_2(t)]^T$ and $\lambda(t) = [\lambda_1(t), \lambda_2(t), \lambda_3(t)]$. We denote by $g(u(t), x(t))$ the integrand part of the objective function (2.1).With these notations and terminologies, the Hamiltonian is given by
$H = H(u(t), x(t), \lambda(t))$
$H = g(u(t), x(t)) + \lambda^T(t).\dot{x}(t)$

$$\begin{aligned} &= c_1 u_1^2 + c_2 u_2^2 + S + I - R + \lambda_1\Big(B - \mu_1 S - f(S, I_\tau) - u_1 S\Big) \\ &\quad + \lambda_2\Big(f(S, I_\tau) - (\mu_2 + \gamma)I - u_2 I\Big) + \lambda_3\Big(\gamma I - \mu_3 R\Big). \end{aligned} \tag{3.1}$$

Let $\chi_{[a,b]}(t)$ be the characteristic function defined by

$$\chi_{[a,b]}(t) = \begin{cases} 1, & \text{if } \ t \in [a, b], \\ 0, & \text{otherwise.} \end{cases} \tag{3.2}$$

Let $u^* = [u_1^*, u_2^*]^T$ be the optimal control and $x^*(t) = [S^*(t), I^*(t), R^*(t)]^T$ be the corresponding optimal trajectory. Then there exists $\lambda(t) \in \mathbb{R}^3$ such that the first order necessary conditions for the existence of optimal control are given by the equations

$$\frac{\partial H}{\partial u}(t) = 0, \tag{3.3}$$

$$\frac{dx}{dt}(t) = \frac{\partial H}{\partial \lambda}, \tag{3.4}$$

$$\frac{d\lambda}{dt}(t) = -\frac{\partial H}{\partial x}. \tag{3.5}$$

The optimality conditions:

$$\left[\frac{\partial H}{\partial u_1}(t)\right]_{u(t)=u^*(t)} = 0, \tag{3.6}$$

$$\left[\frac{\partial H}{\partial u_2}(t)\right]_{u(t)=u^*(t)} = 0. \tag{3.7}$$

Simplifying (3.5) and (3.6), we obtain

$$2c_1 u_1^* - S\lambda_1 = 0, \tag{3.8}$$

$$2c_2 u_2^* - \lambda_2 I = 0. \tag{3.9}$$

Further simplification of (3.8) and (3.9) yields

$$u_1^*(t) = \min\left\{u_{1max}; \max\left\{0; \frac{S(t)\lambda_1(t)}{2c_1}\right\}\right\} \tag{3.10}$$

and

$$u_2^*(t) = \min\left\{u_{1max}; \max\left\{0; \frac{I(t)\lambda_2(t)}{2c_2}\right\}\right\}. \tag{3.11}$$

The state equations: given by the forms (2.2)
The co-state equations:

$$\begin{aligned}
\frac{d\lambda_1}{dt}(t) &= -\frac{\partial H}{\partial S}(t),\\
\frac{d\lambda_2}{dt}(t) &= -\left[\frac{\partial H}{\partial I}(t) + \chi_{[0,T-\tau]}\frac{\partial H}{\partial I_\tau}(t+\tau)\right],\\
\frac{d\lambda_3}{dt}(t) &= -\frac{\partial H}{\partial R}(t),
\end{aligned}$$

which when simplified, lead to

$$\begin{aligned}
\frac{d\lambda_1}{dt} &= -1 + (\lambda_1(t) - \lambda_2(t))f_1(S,I) + (\mu_1 + u_1)\lambda_1(t),\\
\frac{d\lambda_2}{dt} &= -1 + (\lambda_1(t+\tau) - \lambda_2(t+\tau))\chi_{[0,T-\tau]}(t)f_2(S,I) + (\mu_2 + \gamma + u_2)\lambda_2(t) - \gamma\lambda_3(t),\\
\frac{d\lambda_3}{dt} &= 1 + \mu_3\lambda_3(t).
\end{aligned}$$

The transversality conditions:

$$\begin{aligned}
\lambda_1(T) &= 1,\\
\lambda_2(T) &= 1,\\
\lambda_3(T) &= -1.
\end{aligned}$$

Remark 3.2. *It is noting that*

1. *The Hamiltonian function* H *is strongly convex in the control variables.*
2. *The right-hand sides of the state and co-state equations are Lipschitz continuous.*
3. *The set of the admissible controls* $\mathbb{U}$ *is convex*

4 Numerical simulations

In this section, we apply the above optimal control theory with consideration of its applicability. we discuss the discretization of the optimal control problem described and present the numerical results obtained through our simulations. The algorithm describing the approximation method to obtain the optimal control is the following algorithm inspired from [13]. The Algorithm used here is a numerical variation of forward Euler method with a step size h. We explicitly write the forward Euler method for the state and the adjoint.

step1: $\text{for} \;\; i=-m,...,0, \text{do}:$

$S_i = S_0; I_i = I_0; R_i = R_0; u_1^i = 0; u_2^i = 0$

$\text{end} \;\; \text{for}$

$\text{for} \;\; i=n,...,n+m$

$\lambda_1^i = 1; \lambda_2^i = 1; \lambda_3^i = -1$

$\text{end} \;\; \text{for}$

step2 $:\text{for} \;\; i=0,...,n-1$

$S_{i+1} = S_i + h(B - \mu_1 S_i - \beta S_i I_i - u_1^i S_i)$

$I_{i+1} = I_i + h(\beta S_i I_i - (\mu_2+\gamma) I_i - u_2^i I_i)$

$R_{i+1} = R_i + h(\gamma I_i - \mu_3 R_i)$

$\lambda_1^{n-i-1} = \lambda_1^{n-i} - h(-1 + (\lambda_1^{n-i} - \lambda_2^{n-i})\beta I_{i+1} + (\mu_1 + u_1^i)\lambda_1^{n-i}$

$$\begin{aligned} \lambda_2^{n-i-1} &= \lambda_2^{n-i} - h(-1 + (\lambda_1^{n+m-i} - \lambda_2^{n+m-i})\chi_{[0,T-\tau]}(t_{n-i})\beta S_{i+1} \\ &\quad + (\mu_2 + \gamma + u_2^i)\lambda_2^{n-i} - \gamma \lambda_3^{n-i}) \end{aligned}$$

$\lambda_3^{n-i-1} = \lambda_3^{n-i} - h(1 + \mu_3 \lambda_3^{n-i})$

$u_1^{i+1} = S_{i+1}\lambda_1^{n-i}/2c_1$

$u_2^{i+1} = I_{i+1}\lambda_2^{n-i}/2c_2$

$\text{end} \;\; \text{for}$

step3 $:\text{for} \;\; i=1,...,n, \;\; \textit{write}$

$S^*(t_i) = S_i, \;\; I^*(t_i) = I_i \;\; R^*(t_i) = R_i \;\; u_1^*(t_i) = u_1^i$ and $u_2^*(t_i) = u_2^i.$

Comments

Fig 1. represent the different dynamics of the susceptible population for different aspect of control. The Orange color represent the population when there is treatment but not vaccination ($u_1 = 0$ and $u_2 \neq 0$). The blue curve represent the population when there are vaccination and treatment ($u_1 \neq 0$ and $u_2 \neq 0$). The green curve show the evolution of the susceptible population when there is just treatment but not vaccination ($u_1 = 0$ and $u_2 \neq 0$). This show that, without vaccination so many people are exposed to disease.

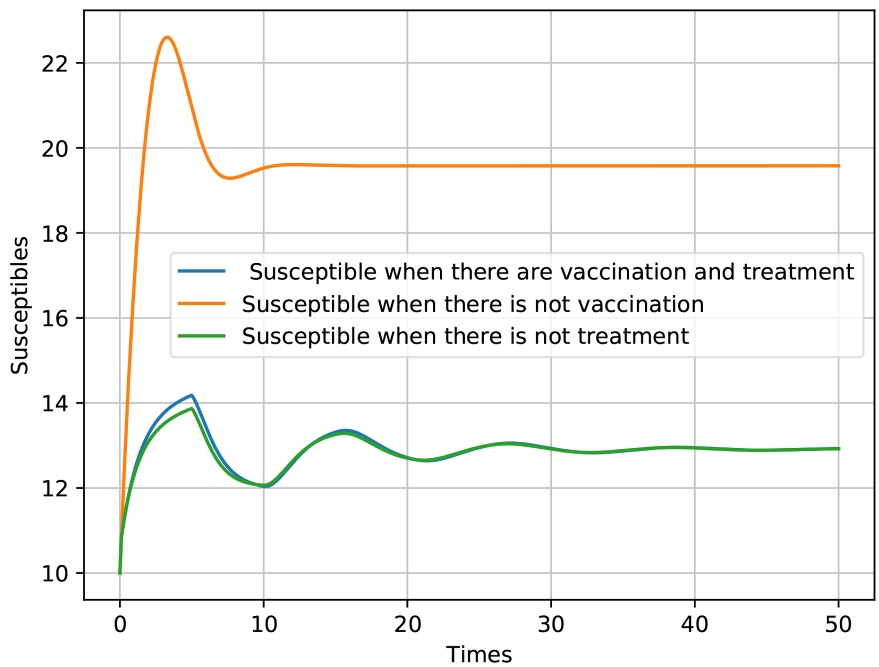

Fig 1. Dynamic of susceptible population with different aspect of control.

Fig 2. represent the different dynamics of the infected population for different aspect of control. The Orange color present the evolution of the infected population when there is treatment but not vaccination ($u_1 = 0$ and $u_2 \neq 0$). The blue curve represent the population when there are vaccination and treatment ($u_1 \neq 0$ and $u_2 \neq 0$). The green curve show the evolution of the infected population when there is just treatment but not vaccination ($u_1 = 0$ and $u_2 \neq 0$). Naturally, as many people are exposed to the disease without vaccination, we see the growth of the infected population.

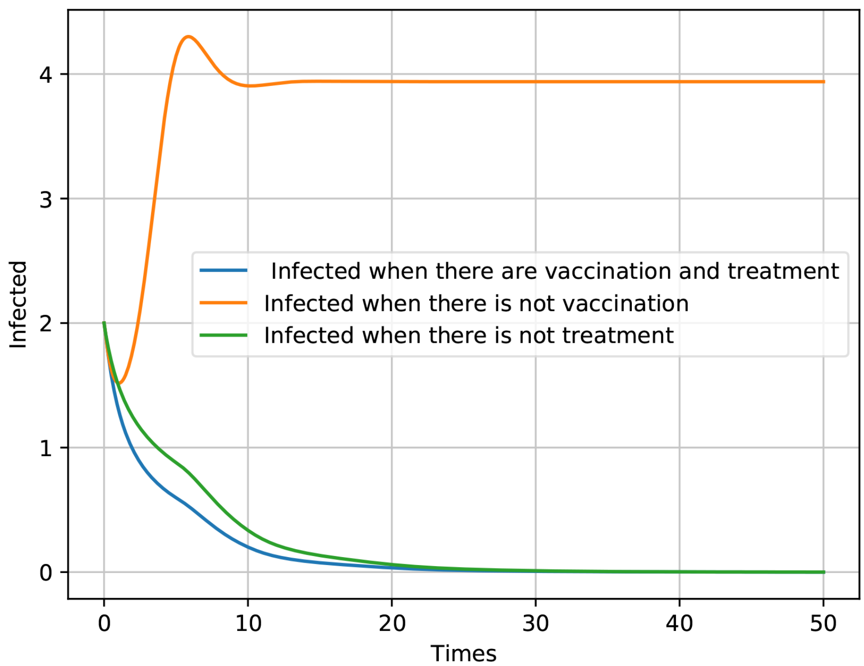

Fig 2. Dynamic of infected population with different aspect of control.

Fig 3. represent the different dynamics of the infected population for different aspect of control. The Orange color present the evolution of the recovered population when there is treatment but not vaccination ($u_1 = 0$ and $u_2 \neq 0$). The blue curve represent the population when there are vaccination and treatment ($u_1 \neq 0$ and $u_2 \neq 0$). The green curve show the evolution of the recovered population when there is just treatment but not vaccination ($u_1 = 0$ and $u_2 \neq 0$). As many people are exposed to the disease without vaccination, indeed we see the growth of the recovered population.

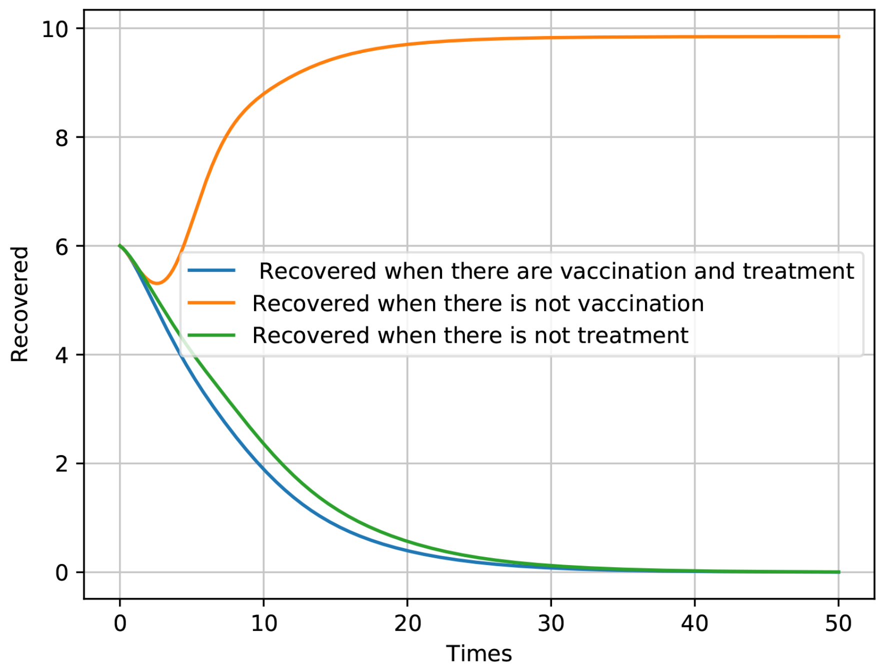

Fig 3. Dynamic of recovered population with different aspect of control.

In conclusion, observing the figures, we can deduce that the strategy leading to the vaccination alone ($u_1 \neq 0$ and $u_2 = 0$) should be preferable to the joint use of vaccination ($u_1 \neq 0$) and treatment ($u_2 \neq 0$). The optimal control strategy here shows that prevention is more effective for the eradication of the disease.

5 Conclusion

In this paper, we considered an optimal control problem for a SIR model with time delay (representing the incubation period) and general incidence function. The main idea developed here is the optimal control in epidemics in order to search among the available strategies, the most efficience one that reduce the infection rate to a minimum level while optimizing the cost deploying a therapy and preventive vaccine that is used to control the disease progression. The two control functions $u_1(t)$ and $u_2(t)$, which represent the vaccination and the treatment strategies are subject to time delays before being effective. Then we formulated the objective function of the optimal control problem. We discussed the existence of the optimal control and then derived the first order necessary conditions for the optimal control through constructing the Hamiltonian and using the Pontryagin's maximum principle to achieve our aim. Finally, to end our study, we do a numerical simulation to corroborate the theoretical results obtained.

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Competing interests

The authors declare that they have no competing interests.

Author's contribution

Aboudramane Guiro provide the subject, wrote the introduction and the conclusion and verified some calculation. Moussa BARRO conceived the study and computed the equilibria and their local stabilities. Dramane OUEDRAOGO wrote mathematical formula, he bring up the control strategy and did all the calculus with the second author. All the authors read and approved the final manuscript.

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Some remarks on the non-real roots of polynomials

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ABSTRACT

Let $f \in \mathbb{R}(t)[x]$ be given by $f(t,x) = x^n + t \cdot g(x)$ and $\beta_1 < \cdots < \beta_m$ the distinct real roots of the discriminant $\Delta_{(f,x)}(t)$ of $f(t,x)$ with respect to x. Let γ be the number of real roots of $g(x) = \sum_{k=0}^{s} t_{s-k} x^{s-k}$. For any $\xi > |\beta_m|$, if $n-s$ is odd then the number of real roots of $f(\xi,x)$ is $\gamma+1$, and if $n-s$ is even then the number of real roots of $f(\xi,x)$ is γ, $\gamma+2$ if $t_s > 0$ or $t_s < 0$ respectively. A special case of the above result is constructing a family of degree $n \geq 3$ irreducible polynomials over $\mathbb{Q}$ with many non-real roots and automorphism group S_n.

RESUMEN

Sea $f \in \mathbb{R}(t)[x]$ dada por $f(t,x) = x^n + t \cdot g(x)$ y $\beta_1 < \cdots < \beta_m$ las diferentes raíces reales del discriminante $\Delta_{(f,x)}(t)$ de $f(t,x)$ con respecto de x. Sea γ el número de raíces reales de $g(x) = \sum_{k=0}^{s} t_{s-k} x^{s-k}$. Para todo $\xi > |\beta_m|$, si $n-s$ es impar entonces el número de raíces reales de $f(\xi,x)$ es $\gamma+1$, y si $n-s$ es par entonces el número de raíces reales de $f(\xi,x)$ es γ, $\gamma+2$ si $t_s > 0$ o $t_s < 0$, respectivamente. Un caso especial del resultado anterior es construyendo una familia de polinomios irreducibles sobre $\mathbb{Q}$ de grado $n \geq 3$ con muchas raíces no-reales y grupo de automorfismos S_n

Keywords and Phrases: Polynomials, non-real roots, discriminant, Bezoutian, Galois groups.

2010 AMS Mathematics Subject Classification: 12D10, 12F10, 26C10.

1 Introduction

Let $f(x) \in \mathbb{Q}[x]$ be an irreducible polynomial of degree $n \geq 2$ and Gal (f) its Galois group over $\mathbb{Q}$. Let us assume that over $\mathbb{R}$, $f(x)$ is factored as

$$f(x) = a \prod_{j=1}^{r} (x - \alpha_j) \prod_{i=1}^{s} (x^2 + a_i x + b_i),$$

where $a_i^2 < 4b_i$, for all $i = 1, \ldots, s$. The pair (r, s) is called the *signature* of $f(x)$. Obviously $\deg f = 2s + r$. If $s = 0$ then $f(x)$ is called *totally real* and if $r = 0$ it is called *totally complex*. Equivalently the above terminology can be defined for binary forms $f(x, z)$. By a reordering of the roots we may assume that if $f(x)$ has $2s$ non-real roots then

$$\alpha := (1, 2)(3, 4) \cdots (2s - 1, 2s) \in \mathrm{Gal}(f).$$

In [?b-sh] it is proved that if $\deg f = p$, for a prime p, and s satisfies

$$s\,(s \log s + 2 \log s + 3) \leq p$$

then $\mathrm{Gal}(f) = A_p, S_p$. Moreover, a list of all possible groups for various values of r is given for $p \leq 29$; see [?b-sh, Thm. 2]. There are some follow up papers to [?b-sh].

In [?shimol] the author proves that if $p \geq 4s + 1$, then the Galois group is either S_p or A_p. This improves the bound given in [?b-sh]. The author also studies when polynomials with non-real roots are solvable by radicals, which are consequences of Table 2 and Theorem 2 in [?b-sh]. In [?otake] the author uses Bezoutians of a polynomial and its derivative to construct polynomials with real coefficients where the number of real roots can be counted explicitly. Thereby, irreducible polynomials in $\mathbb{Q}[x]$ of prime degree p are constructed for which the Galois group is either S_p or A_p.

In this paper we study a family of polynomials with non-real roots whose degree is not necessarily prime. Given a polynomial $g(x) = \sum_{i=0}^{s} t_i x^i$ and with γ number of non-real roots we construct a polynomial $f(t, x) = x^n + t\, g(x)$ which has $\gamma, \gamma + 1, \gamma + 2$ non-real roots for certain values of $t \in \mathbb{R}$; see Theorem **??**. The values of $t \in \mathbb{R}$ are given in terms of the Bezoutian matrix of polynomials or equivalently the discriminant of $f(t, x)$ with respect to x. This is the focus of Section **??** in the paper.

While most of the efforts have been focusing on the case of irreducible polynomials over $\mathbb{Q}$ which have real roots, the case of polynomials with no real roots is equally interesting. How should an irreducible polynomial over $\mathbb{Q}$ with all non-real roots must look like? What can be said about the Galois group of such totally complex polynomials? In [?e-sh] is developed a reduction theory for such polynomials via the hyperbolic center of mass. A special case of Theorem **??** provides a class of totally complex polynomials.

Notation For any polynomial $f(x)$ we denote by $\Delta_{(f,x)}$ its discriminant with respect to x. If f is a univariate polynomial then Δ_f is used and the leading coefficient is denoted by $\mathrm{led}(f)$. Throughout this paper the ground field is a field of characteristic zero.

2 Preliminaries

Let $f_1(x)$, $f_2(x)$ be polynomials over a field F of characteristic zero and, let n be an integer which is greater than or equal to $\max\{\deg f_1, \deg f_2\}$. Then, we put

$$\begin{aligned} B_n(f_1, f_2) &:= \frac{f_1(x)f_2(y) - f_1(y)f_2(x)}{x-y} = \sum_{i,j=1}^{n} \alpha_{ij} x^{n-i} y^{n-j} \in F[x, y], \\ M_n(f_1, f_2) &:= (\alpha_{ij})_{1 \le i,j \le n}. \end{aligned}$$

The matrix $M_n(f_1, f_2)$ is called the *Bezoutian* of f_1 and f_2. Clearly, $B_n(f_1, f_1) = 0$ and hence $M_n(f_1, f_1)$ is the zero matrix. The following properties hold true; see [?fuh, Theorem 8.25] for details.

Proposition 1. *The following are true:*

(1) $M_n(f_1, f_2)$ *is an* $n \times n$ *symmetric matrix over* F.

(2) $B_n(f_1, f_2)$ *is linear in* f_1 *and* f_2, *separately.*

(3) $B_n(f_1, f_2) = -B_n(f_2, f_1)$.

When $f_2 = f_1'$, the formal derivative of f_1 (with respect to the indeterminate x), we often write $B_n(f_1) := B_n(f_1, f_1')$. From now on, for any degree $n \ge 2$ polynomial $f(x) \in \mathbb{R}[x]$ we will denote by $M_n(f) := M_n(f, f')$ as above. The matrix $M_n(f)$ is called the **Bezoutian matrix** of f.

Remark 2.1. *It is often the case that the matrix* $M'_n(f_1, f_2) = (\alpha'_{ij})_{1 \le i,j \le n}$ *defined by the generating function*

$$B'_n(f_1, f_2) := \frac{f_1(x)f_2(y) - f_1(y)f_2(x)}{x-y} = \sum_{i,j=1}^{n} \alpha'_{ij} x^{i-1} y^{j-1} \in F[x, y]$$

is called the Bezoutian of f_1 *and* f_2. *But no difference can be seen between these two definitions as far as we consider the corresponding quadratic forms*

$$\sum_{i,j=1}^{n} \alpha_{ij} x_i x_j \quad \text{and} \quad \sum_{i,j=1}^{n} \alpha'_{ij} x_i x_j.$$

In fact, these two quadratic forms are equivalent over the prime field $\mathbb{Q}$ $(\subset F)$ *since we have* $M'_n(f_1, f_2) = {}^t J_n M_n(f_1, f_2) J_n$, *where*

$$J_n = \begin{bmatrix} 0 & & & 1 \\ & & 1 & \\ & \iddots & & \\ 1 & & & 0 \end{bmatrix}$$

is an $n \times n$ *anti-identity matrix. This implies that above two quadratic forms are equivalent over* $\mathbb{Q}$ *or more precisely, over the ring of rational integers* $\mathbb{Z}$.

Let $f(x) \in \mathbb{R}[x]$ be a degree $n \geq 2$ polynomial which is given by

$$f(x) = a_0 + a_1 x + \cdots + a_n x^n$$

Then over $\mathbb{R}$ this polynomial is factored as

$$f(x) = a \prod_{j=1}^{r} (x - \alpha_j) \prod_{i=1}^{s} (x^2 + a_i x + b_i)$$

for some $\alpha_1, \ldots, \alpha_r \in \mathbb{R}$ and $a_i, b_i, a \in \mathbb{R}$, where $a_i^2 < 4b_i$, for all $i = 1, \cdots, s$.

Throughout this paper, for a univariate polynomial f, its discriminant will be denoted by Δ_f. For any two polynomials $f_1(x)$, $f_2(x)$ the resultant with respect to x will be denoted by $\mathrm{Res}(f_1, f_2, x)$. We notice the following elementary fact, its proof is elementary and we skip the details.

Remark 2.2. *For any polynomial* $f(x)$*, the determinant of the Bezoutian is the same as the discriminant up to a multiplication by a constant. More precisely,*

$$\Delta_f = \frac{1}{led(f)^2} \det M_n(f),$$

where $led(f)$ *is the leading coefficient of* $f(x)$.

If $f(x) \in \mathbb{Q}[x]$ is irreducible and its degree is a prime number, say $\deg f = p$, then there is enough known for the Galois group of polynomials with some non-real roots; see [?b-sh], [?shimol], [?otake] for details. If the number of non-real roots is "small" enough with respect to the prime degree $\deg f = p$ of the polynomial, then the Galois group is A_p or S_p. Furthermore, using the classification of finite simple groups one can provide a complete list of possible Galois groups for every polynomial of prime degree p which has non-real roots; see [?b-sh] for details.

On the other extreme are the polynomials which have all roots non-real. We called them above, totally complex polynomials. We have the following:

Lemma 2.1. *The followings are equivalent:*

i) $f(x) \in \mathbb{R}[x]$ *is totally complex*

ii) $f(x)$ *can be written as*

$$f(x) = a \prod_{i=1}^{n} f_i$$

where $f_i = x^2 + a_i x + b_i$*, for* $i = 1, \ldots, n$ *and* $a_i, b_i, a \in \mathbb{R}$*, where* $a_i^2 < 4b_i$*, for all* $i = 1, \ldots, n$. *Moreover, the determinant of the Bezoutian* $M_n(f)$ *is given by*

$$\Delta_f = \frac{1}{led(f)^2} \det M_n(f) = \prod_{i=1}^{n} \Delta_{f_i} \cdot \prod_{i,j,i \neq j}^{n} (Res(f_i, f_j, x))^2$$

where led(f) *is the leading coefficient of* $f(x)$.

ii) the index of inertia of Bezoutian $M(f)$ *is 0*

iii) if $\Delta_f \neq 0$ *then the equivalence class of* $M(f)$ *in the Witt ring* $W(R)$ *is 0.*

Proof. The equivalence between i), ii), and iii) can be found in [?fuh]. □

It is not clear when such polynomials are irreducible over $\mathbb{Q}$. If that's the case, what is the Galois group Gal (f)? Clearly the group generated by the involution $(1,2)(3,4)\cdots(2n-1,2n)$ is embedded in Gal (f). Is Gal (f) larger in general?

3 On the number of real roots of polynomials

For any degree $n \geq 2$ polynomial $f(x) \in \mathbb{R}[x]$ and any symmetric matrix $M := M_n(f)$ with real entries, let N_f be the **number of distinct real roots** of f and $\sigma(M)$ be the index of inertia of M, respectively. The next result plays a fundamental role throughout this section ([?fuh, Theorem 9.2]).

Proposition 2. *For any real polynomial* $f \in \mathbb{R}[x]$, *the number* N_f *of its distinct real roots is the index of inertia of the Bezoutian matrix* $M_n(f)$. *In other words,*

$$N_f = \sigma\left(M_n(f)\right).$$

Let us cite one more result which says that the roots of a polynomial depend continuously on its coefficients ([?mar, Theorem 1.4], [?rs, Theorem 1.3.1]).

Proposition 3. *Let be given a polynomial*

$$f(x) = \sum_{l=0}^{n} a_l x^l \in \mathbb{C}[x],$$

with distinct roots $\alpha_1, \ldots, \alpha_k$ *of multiplicities* $m_1, \ldots, m_k$ *respectively. Then, for any given a positive*

$$\varepsilon < \min_{1\leq i<j\leq k} \left\{ \frac{|\alpha_i - \alpha_j|}{2} \right\},$$

there exists a real number $\delta > 0$ *such that any monic polynomial* $g(x) = \sum_{l=0}^{n} b_l x^l \in \mathbb{C}[x]$ *whose coefficients satisfy*

$$|b_l - a_l| < \delta,$$

for $l = 0, \cdots, n-1$, *has exactly* m_j *roots in the disk*

$$\mathcal{D}(\alpha_j; \varepsilon) = \{z \in \mathbb{C} \mid |z - \alpha_j| < \epsilon\} \ (j = 1, \cdots, k).$$

Let n, s be positive integers such that $n > s$ and let

$$g(t_0, \cdots, t_s; x) = \sum_{k=0}^{s} t_{s-k} x^{s-k},$$
$$f^{(n)}(t_0, \cdots, t_s, t; x) = x^n + t \cdot g(t_0, \cdots, t_s; x) \tag{3.1}$$

be polynomials in x over $E_1 = \mathbb{R}(t_0, \cdots, t_s)$, $E_2 = \mathbb{R}(t_0, \cdots, t_s, t)$, respectively. Here, E_1 (resp., E_2) is a rational function field with $s+1$ (resp., $(s+2)$) variables $t_0, \cdots, t_s$ (resp., $(t_0, \cdots, t_s, t)$). To ease notation, let us put

$$g(x) = g(t_0, \cdots, t_s; x), \ \ f(t; x) = f^{(n)}(t_0, \cdots, t_s, t; x)$$

and for any real vector $\nu = (\nu_0, \cdots, \nu_s) \in \mathbb{R}^{s+1}$, we put

$$g_\nu(x) = g(\nu_0, \cdots, \nu_s; x), \ \ f_\nu(t; x) = f^{(n)}(\nu_0, \cdots, \nu_s, t; x). \tag{3.2}$$

By using Proposition **??**, we can prove the next theorem ([?otake, Main Theorem 1.3]).

Theorem 3.1. *Let $r = (r_0, \cdots, r_s) \in \mathbb{R}^{s+1}$ be a vector such that $N_{g_r} = s$. Let us consider $f_r(t; x) = f^{(n)}(r_0, \cdots, r_s, t; x)$ as a polynomial over $\mathbb{R}(t)$ in x and put*

$$P_r(t) = \det M_n(f_r(t; x)) = \det M_n(f_r(t; x), f_r'(t; x)),$$

where $f_r'(t; x)$ is a derivative of $f_r(t; x)$ with respect to x. Then, for any real number $\xi > \alpha_r = \max\{\alpha \in \mathbb{R} \mid P_r(\alpha) = 0\}$, we have

$$N_{f_r(\xi; x)} = \begin{cases} s+1 & \textit{if } n-s : \textit{odd} \\ s & \textit{if } n-s : \textit{even}, \ r_s > 0 \\ s+2 & \textit{if } n-s : \textit{even}, \ r_s < 0. \end{cases}$$

By this theorem and a theorem of Oz Ben-Shimol [?shimol, Theorem 2.6], we can obtain an algorithm to construct prime degree p polynomials with given number of real roots, and whose Galois groups are isomorphic to the symmetric group S_p or the alternating group A_p ([?otake, Corollary 1.6]).

In this section, we extend this theorem as follows;

Theorem 3.2. *Let $r = (r_0, \cdots, r_s) \in \mathbb{R}^{s+1}$ be a vector such that $g_r(x)$ is a degree s separable polynomial satisfying $N_{g_r(x)} = \gamma$ $(0 \leq \gamma \leq s)$. Let us consider $f_r(t; x) = f^{(n)}(r_0, \cdots, r_s, t; x)$ as a polynomial over $\mathbb{R}(t)$ in x and put*

$$P_r(t) = \det M_n(f_r(t; x)) = \det M_n(f_r(t; x), f_r'(t; x)),$$

where $f_r'(t; x)$ is a derivative of $f_r(t; x)$ with respect to x. Then, for any real number $\xi > \alpha_r = \max\{\alpha \in \mathbb{R} \mid P_r(\alpha) = 0\}$, we have

$$N_{f_r(\xi; x)} = \begin{cases} \gamma+1 & \textit{if } n-s : \textit{odd} \\ \gamma & \textit{if } n-s : \textit{even}, \ r_s > 0 \\ \gamma+2 & \textit{if } n-s : \textit{even}, \ r_s < 0. \end{cases} \tag{3.3}$$

The above theorem can be restated as follows:

Corolary 1. *Let* $f \in \mathbb{R}(t)[x]$ *be given by*

$$f(t,x) = x^n + t \cdot \sum_{k=0}^{s} t_{s-k} x^{s-k}$$

and $\beta_1 < \cdots < \beta_m$ *the distinct real roots of the degree* s *polynomial*

$$P(t) := \frac{1}{t^{n-1}} \Delta_{(f,x)}(t).$$

For any $\xi > |\beta_m|$, *the number of real roots of* $f(\xi, x)$ *is*

$$N_{f(\xi,x)} = \begin{cases} \gamma + 1 & \text{if } n-s : \text{odd} \\ \gamma & \text{if } n-s : \text{even, } t_s > 0 \\ \gamma + 2 & \text{if } n-s : \text{even, } t_s < 0. \end{cases}$$

where γ *is the number or real roots of* $g(x) = \frac{f(x)-x^n}{t} \in \mathbb{R}[x]$.

The rest of the section is concerned with proving Thm. **??**.

3.1 The Bezoutian of $f(t;x)$

First, let us put

$$\begin{aligned} A(t_0, \cdots, t_s, t) &= (a_{ij}(t_0, \cdots, t_s, t))_{1 \le i,j \le n} = M_n(f(t;x)) \in \mathrm{Sym}_n(E_2), \\ B(t_0, \cdots, t_s) &= (b_{ij}(t_0, \cdots, t_s))_{1 \le i,j \le s} = M_s(g(x)) \in \mathrm{Sym}_s(E_1). \end{aligned}$$

For ease of notation, we also write

$$A(t_0, \cdots, t_s, t) = A(t) = (a_{ij}(t))_{1 \le i,j \le n}, \; B(t_0, \cdots, t_s) = B = (b_{ij})_{1 \le i,j \le s}$$

and we put $B(t) = (b_{ij}(t))_{1 \le i,j \le s} = t^2 B$. Then, by Proposition **??**, we have

$$\begin{aligned} A(t) &= M_n(x^n + tg(x), nx^{n-1} + tg'(x)) \\ &= nM_n(x^n, x^{n-1}) - ntM_n(x^{n-1}, g(x)) + tM_n(x^n, g'(x)) + t^2 M_n(g(x), g'(x)) \\ &= nM_n(x^n, x^{n-1}) - nt \sum_{k=0}^{s} t_{s-k} M_n(x^{n-1}, x^{s-k}) \\ &\qquad + t \sum_{k=0}^{s-1} (s-k) t_{s-k} M_n(x^n, x^{s-k-1}) + t^2 M_n(g(x), g'(x)). \end{aligned}$$

Lemma 3.1. *Let* λ, μ, ν *be integers such that* $\lambda \ge \mu > \nu \ge 0$. *Then* $M_\lambda(x^\mu, x^\nu) = (m_{ij})_{1 \le i,j \le \lambda}$, *where*

$$m_{ij} = \begin{cases} 1 & i + j = 2\lambda - (\mu + \nu) + 1 \quad (\lambda - \mu + 1 \le i, j \le \lambda - \nu), \\ 0 & \text{otherwise.} \end{cases}$$

Proof. By definition, we have

$$\begin{aligned}B_\lambda(x^\mu, x^\nu) &= \frac{x^\mu y^\nu - x^\nu y^\mu}{x-y}\\ &= \sum_{k=1}^{\mu-\nu} x^{\mu-k}y^{\nu+k-1} = \sum_{k=1}^{\mu-\nu} x^{\lambda-(\lambda-\mu+k)}y^{\lambda-(\lambda-\nu-k+1)},\end{aligned}$$

which implies

$$\begin{aligned}m_{ij} &= \begin{cases}1 & (i,j) = (\lambda-\mu+k, \lambda-\nu-k+1) \quad (1 \le k \le \mu-\nu)\\ 0 & \text{otherwise}\end{cases}\\ &= \begin{cases}1 & i+j = 2\lambda-(\mu+\nu)+1 \quad (\lambda-\mu+1 \le i,j \le \lambda-\nu),\\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

This completes the proof. ∎

Here, let us divide $A(t)$ into two parts $\hat{A}(t)$ and $\tilde{A}(t)$, where

$$\begin{aligned}\hat{A}(t) = (\hat{a}_{ij}(t))_{1\le i,j\le n} &= nM_n(x^n, x^{n-1}) - nt\sum_{k=0}^{s} t_{s-k}M_n(x^{n-1}, x^{s-k})\\ &\quad + t\sum_{k=0}^{s-1}(s-k)t_{s-k}M_n(x^n, x^{s-k-1}),\\ \tilde{A}(t) = (\tilde{a}_{ij}(t))_{1\le i,j\le n} &= t^2 M_n(g(x), g'(x))\end{aligned}$$

and put $l_k = n-s+k+2$ $(= 2n-(n+s-k-1)+1)$. Then, by lemma **??**, we have

$$\begin{cases}\hat{a}_{11}(t) = n\\ \hat{a}_{1,l_k-1}(t) = \hat{a}_{l_k-1,1}(t) = (s-k)t_{s-k}t \ (0 \le k \le s-1).\end{cases}$$

Moreover, when $i+j = l_k$, we have

$$\hat{a}_{ij}(t) = -ntt_{s-k} + t(s-k)t_{s-k} = -(l_k-2)t_{s-k}t \quad (2 \le i,j \le l_k-2,\ 0 \le k \le s). \tag{3.4}$$

Remark 3.3. *Note that, if* $s = n-1$*, we have*

$$-nt\sum_{k=0}^{s} t_{s-k}M_n(x^{n-1}, x^{s-k}) = -nt\sum_{k=1}^{s} t_{s-k}M_n(x^{n-1}, x^{s-k}),$$

Thus, when $i+j = l_k$*, equation* (**??**) *should be modified by*

$$\hat{a}_{ij}(t) = -ntt_{s-k} + t(s-k)t_{s-k} = -(l_k-2)t_{s-k}t \quad (2 \le i,j \le l_k-2,\ 1 \le k \le s).$$

We avoid this minor defect by considering that there is no entries satisfying $2 \le i,j \le l_0-2$ *when* $s = n-1$ *since* $l_0 - 2 = n-s = 1$.

Proposition 4. *Put* $l_k = n - s + k + 2$. *Then*

$$\hat{a}_{ij}(t) = \begin{cases} n & (i,j) = (1,1) \\ (s-k)t_{s-k}t & (i,j) = (1, l_k - 1) \text{ or } (l_k - 1, 1) \quad (0 \le k \le s-1) \\ -(l_k - 2)t_{s-k}t & i + j = l_k,\ 2 \le i, j \le l_k - 2,\ (0 \le k \le s) \\ 0 & \text{otherwise.} \end{cases}$$

$$\tilde{a}_{ij}(t) = \begin{cases} b_{i-(n-s), j-(n-s)} t^2 & n - s + 1 \le i, j \le n \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The statement for $\hat{a}_{ij}(t)$ has just been proved. For $\tilde{a}_{ij}(t)$, it is enough to see that we can denote

$$M_s(g(x)) = \sum_{\ell=0}^{s} \sum_{m=1}^{s} m t_\ell t_m M_s(x^\ell, x^{m-1}),$$
$$M_n(g(x)) = \sum_{\ell=0}^{s} \sum_{m=1}^{s} m t_\ell t_m M_n(x^\ell, x^{m-1}),$$

that is, we can obtain $M_n(g(x))$ from $M_s(g(x))$ by just replacing s with n for all $M_s(x^\ell, x^m)$, which, by Lemma **??**, means that $s \times s$ matrix $M_s(g(x))$ occupies the part $\{b^\dagger_{ij} \mid n - s + 1 \le i, j \le n\}$ of the matrix $M_n(g(x)) = (b^\dagger_{ij})_{1 \le i,j \le n}$. $\square$

By Proposition **??**, we can express the matrix $A(t)$ as follows;

$$A(t) = \left[\begin{array}{cccc|cccc} n & 0 & \dots & 0 & st_s t & (s-1)t_{s-1}t & \dots & t_1 t \\ 0 & & & -(n-s)t_s t & -(n-s+1)t_{s-1}t & \dots & -(n-1)t_1 t & -nt_0 t \\ \vdots & & \iddots & \iddots & & \iddots & \iddots & 0 \\ 0 & -(n-s)t_s t & \iddots & & & \iddots & 0 & 0 \\ \hline st_s t & -(n-s+1)t_{s-1}t & & & & & & \\ (s-1)t_{s-1}t & \vdots & \iddots & \iddots & & & & \\ \vdots & -(n-1)t_1 t & \iddots & 0 & & C(t) & & \\ t_1 t & -nt_0 t & 0 & 0 & & & & \end{array}\right]. \tag{3.5}$$

Here, $C(t) = (c_{ij}(t))_{1 \le i,j \le s} = C(t_0, \cdots, t_s, t) = (c_{ij}(t_0, \cdots, t_s, t))_{1 \le i,j \le s}$ is an $s \times s$ symmetric matrix whose entries are of the form

$$\begin{aligned} c_{ij}(t_0, \cdots, t_s, t) &= b_{ij} t^2 + \lambda_{ij} t \\ &= b_{ij}(t_0, \cdots, t_s) t^2 + \lambda_{ij}(t_0, \cdots, t_s) t \quad (\lambda_{ij} = \lambda_{ij}(t_0, \cdots, t_s) \in E_1). \end{aligned}$$

Next, let $A(t)_1 = (a_{ij}(t)_1)_{1 \le i,j \le n} = A(t_0, \cdots, t_s, t)_1 = (a_{ij}(t_0, \cdots, t_s, t)_1)_{1 \le i,j \le n}$ be the $n \times n$ symmetric matrix obtained from $A(t)$ by multiplying the first row and the first column by $1/\sqrt{n}$ and then sweeping out the entries of the first row and the first column by the $(1,1)$ entry 1. Here, let $Q_m(k;c) = (q_{ij})_{1 \le i,j \le m}$ and $R_m(k,l;c) = (r_{ij})_{1 \le i,j \le m}$ be $m \times m$ elementary matrices such that

$$Q_m(k;c)=\begin{bmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & c & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{bmatrix},\ R_m(k,l;c)=\begin{bmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & c & & \\ & & & \ddots & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{bmatrix},$$

where $q_{kk} = c$ and $r_{kl} = c$. Moreover, for any $m \times m$ matrices $M_1, M_2, \cdots, M_l$, put $\prod_{k=1}^{l} M_k = M_1M_2\cdots M_l$. Then, we have $A(t)_1 = {}^tS(t)_1A(t)S(t)_1$, where

$$S(t)_1 = Q_n(1;1/\sqrt{n})\prod_{k=0}^{s-1} R_n(1,l_k-1;-a_{1,l_k-1}(t)/\sqrt{n}).$$

The matrix $A(t)_1$ can be expressed as follows;

$$A(t)_1 = \left[\begin{array}{cccc|cccc} 1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & -(n-s)t_st & -(n-s+1)t_{s-1}t & \dots & -(n-1)t_1t & -nt_0t \\ \vdots & \vdots & \iddots & \iddots & & \iddots & \iddots & 0 \\ 0 & -(n-s)t_st & \iddots & & & \iddots & 0 & 0 \\ \hline 0 & -(n-s+1)t_{s-1}t & & & & & & \\ 0 & \vdots & \iddots & \iddots & & & & \\ \vdots & -(n-1)t_1t & \iddots & 0 & & C(t)_1 & & \\ 0 & -nt_0t & 0 & 0 & & & & \end{array}\right]. \tag{3.6}$$

Here, $C(t)_1 = (c_{ij}(t)_1)_{1\le i,j\le s} = C(t_0,\cdots,t_s,t)_1 = (c_{ij}(t_0,\cdots,t_s,t)_1)_{1\le i,j\le s}$ is an $s\times s$ symmetric matrix whose entries are of the form

$$c_{ij}(t_0,\cdots,t_s,t)_1 = \bar{b}_{ij}(t_0,\cdots,t_s)t^2 + \lambda_{ij}(t_0,\cdots,t_s)t \quad (\bar{b}_{ij}(t_0,\cdots,t_s) \in E_1),$$

where

$$\bar{b}_{ij}(t_0,\cdots,t_s) = b_{ij}(t_0,\cdots,t_s) - \frac{(s-i+1)(s-j+1)}{n}t_{s-i+1}t_{s-j+1} \tag{3.7}$$

for any i,j $(1 \le i,j \le s)$. We put $\bar{b}_{ij}(t_0,\cdots,t_s) = \bar{b}_{ij}$ and $\bar{B} = (\bar{b}_{ij})_{1\le i,j\le s}$.

3.2 Some results for the Bezoutian of $f_r(t;x)$

Let $r = (r_0,\cdots,r_s) \in \mathbb{R}^{s+1}$ be a vector as in Theorem **??**. We put

$$A_r(t) = (a_{ij}^{(r)}(t))_{1\le i,j\le n} = A(r_0,\cdots,r_s,t) \in \mathrm{Sym}_n(\mathbb{R}(t)),$$
$$B_r = (b_{ij}^{(r)})_{1\le i,j\le s} = B(r_0,\cdots,r_s) \in \mathrm{Sym}_s(\mathbb{R})$$

and $B_r(t) = t^2 B_r$. Let us also put $A_r(t)_1 = A(r_0, \cdots, r_s, t)_1$. By equation (**??**), the matrix $A_r(t)_1$ can be expressed as follows;

$$A_r(t)_1 = \left[\begin{array}{cccc|cccc}
1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & -(n-s)r_s t & -(n-s+1)r_{s-1}t & \cdots & -(n-1)r_1 t & -nr_0 t \\
\vdots & \vdots & \iddots & \iddots & & \iddots & \iddots & 0 \\
0 & -(n-s)r_s t & \iddots & & & \iddots & 0 & 0 \\
\hline
0 & -(n-s+1)r_{s-1}t & & & & & & \\
0 & \vdots & \iddots & \iddots & & & & \\
\vdots & -(n-1)r_1 t & \iddots & 0 & & C_r(t)_1 & & \\
0 & -nr_0 t & 0 & 0 & & & &
\end{array}\right].$$

Here, $C_r(t)_1 = (c_{ij}^{(r)}(t)_1)_{1\le i,j\le s} = C(r_0, \cdots, r_s, t)_1$ and

$$c_{ij}^{(r)}(t)_1 = \bar{b}_{ij}(r_0, \cdots, r_s)t^2 + \lambda_{ij}(r_0, \cdots, r_s)t \quad (\bar{b}_{ij}(r_0, \cdots, r_s), \lambda_{ij}(r_0, \cdots, r_s) \in \mathbb{R}).$$

Note that, by equation (**??**), we have

$$\bar{b}_{ij}(r_0, \cdots, r_s) = b_{ij}^{(r)} - \frac{(s-i+1)(s-j+1)}{n} r_{s-i+1} r_{s-j+1} \quad (1 \le i, j \le s).$$

To ease notation, we put $\bar{b}_{ij}(r_0, \cdots, r_s) = \bar{b}_{ij}^{(r)}$ and $\bar{B}_r = (\bar{b}_{ij}^{(r)})_{1\le i,j\le s}$.

In particular, since

$$\begin{aligned}
M_s(g_r) &= M_s\left(r_s x^s, \sum_{k=0}^{s-1}(s-k)r_{s-k}x^{s-k-1}\right) + M_s\left(\sum_{k=1}^{s} r_{s-k}x^{s-k}, g_r'\right) \\
&= \sum_{k=0}^{s-1}(s-k)r_s r_{s-k} M_s(x^s, x^{s-k-1}) + M_s\left(\sum_{k=1}^{s} r_{s-k}x^{s-k}, g_r'\right),
\end{aligned}$$

we have

$$b_{1,k+1}^{(r)} = b_{k+1,1}^{(r)} = (s-k)r_s r_{s-k} \ (0 \le k \le s-1) \tag{3.8}$$

by Lemma **??** and hence

$$\begin{aligned}
\bar{b}_{1j}^{(r)} &= (s-j+1)r_s r_{s-j+1} - \frac{s(s-j+1)}{n} r_s r_{s-j+1} \\
&= (s-j+1)\left(1 - \frac{s}{n}\right) r_s r_{s-j+1} \ (1 \le j \le s).
\end{aligned} \tag{3.9}$$

Lemma 3.2. *Put $\bar{B}_r(t) = t^2 \bar{B}_r$. Then, $B_r(\xi)$ and $\bar{B}_r(\xi)$ are equivalent over $\mathbb{R}$ for any real number ξ and we have $\sigma(\bar{B}_r(\xi)) = N_{g_r}$ for any non-zero real number ξ.*

Proof. Let us denote by $B_r^* = (b_{ij}^{(r,*)})_{1\le i,j\le s}$ $(\bar{B}_r^* = (\bar{b}_{ij}^{(r,*)})_{1\le i,j\le s})$ the matrix obtained from B_r $(\bar{B}_r)$ by multiplying the first row and the first column by $1\big/\pm\sqrt{b_{11}^{(r)}}$ $\left(1\big/\pm\sqrt{\bar{b}_{11}^{(r)}}\right)$ (the sign before $\sqrt{b_{11}^{(r)}}$ $\left(\sqrt{\bar{b}_{11}^{(r)}}\right)$ are the same as the sign of r_s; see the definition of d $(\bar{d})$ below) and then sweeping out the entries of the first row and the first column by the $(1,1)$ entry 1. Since $b_{11} = sr_s^2$ (>0) and $\bar{b}_{11} = s(1-s/n)r_s^2$ (>0) by (**??**) and (**??**), we have

$$B_r^* = {}^tTB_rT,\ \bar{B}_r^* = {}^t\bar{T}\bar{B}_r\bar{T}, \tag{3.10}$$

where

$$T = Q_s(1;1/d)\prod_{k=2}^{s} R_s(1,k;-b_{1k}^{(r)}/d)\ (d = \sqrt{s}\cdot r_s),$$
$$\bar{T} = Q_s(1;1/\bar{d})\prod_{k=2}^{s} R_s(1,k;-\bar{b}_{1k}^{(r)}/\bar{d})\ (\bar{d} = \sqrt{s(1-s/n)}\cdot r_s).$$

Note that in [?otake, Lemma 3.3], we have proved $b_{ij}^{(r,*)} = \bar{b}_{ij}^{(r,*)}$ $(1 \le i,j \le s)$ and hence $t^2B_r^* = t^2\bar{B}_r^*$, which, by (**??**), implies that symmetric matrices $B_r(\xi)$ and $\bar{B}_r(\xi)$ are equivalent over $\mathbb{R}$ for any real number ξ. Then, since $N_{g_r} = \sigma(B_r) = \sigma(B_r(\xi))$ for any $\xi \in \mathbb{R}\setminus\{0\}$, the latter half of the statement have also been proved. □

3.3 Nonvanishingness of some coefficients

In this subsection, we prove the next lemma.

Lemma 3.3. *Let*

$$\Phi(x) = \Phi(t_0,\cdots,t_s;x) = \sum_{k=0}^{s} h_{s-k}(t_0,\cdots,t_s)x^{s-k} \in E_1[x] \tag{3.11}$$

be the characteristic polynomial of $\bar{B}$. *Then,* $h_{s-k}(t_0,\cdots,t_s)$ *is a non-zero polynomial in* E_1 *for any* k $(1 \le k \le s)$.

Proof. Lemma **??** is clear for $s=1$, since we have

$$B = M_1(t_1x+t_0) = \left[\ t_1^2\ \right]$$

and hence, by equation (**??**),

$$\bar{B} = \left[\ t_1^2 - \frac{1}{n}t_1^2\ \right] = \left[\ \frac{n-1}{n}t_1^2\ \right].$$

Next, suppose $s \ge 2$. Then, by equation (**??**) and the definition of the Bezoutian, we have $h_{s-k}(t_0,\cdots,t_s) \in \mathbb{R}[t_0,\cdots,t_s]$ for any k $(1 \le k \le s)$. Thus, we have only to prove that $h_{s-k}(t_0,\cdots,t_s) \ne 0$ for any k $(1 \le k \le s)$, which is clear from the next Lemma **??**. □

Lemma 3.4. *Suppose* $s \ge 2$ *and put* $u_0 = u_s = 1$, $u_1 = t_1$ *and* $u_k = 0$ $(2 \le k \le s-1)$. *Then,* $h_{s-k}(u_0,\cdots,u_s)$ *is a non-constant polynomial in* $\mathbb{R}(t_1)$ *for any* k $(1 \le k \le s)$, *i.e.,* $h_{s-k}(u_0,\cdots,u_s) \in \mathbb{R}[t_1]\setminus\mathbb{R}$ $(1 \le k \le s)$.

To prove lemma ??, let us put $u = (u_0, \cdots, u_s)$ and

$$
\begin{aligned}
&g_u(x) = g(u_0, \cdots, u_s; x) = x^s + t_1 x + 1 \in \mathbb{R}(t_1)[x],\\
&f_u(t; x) = x^n + t g_u(x) \in \mathbb{R}(t_1, t)[x] \ (n > s),\\
&A_u(t) = (a_{ij}^{(u)}(t))_{1\le i,j\le n} = A(u_0, \cdots, u_s, t) \in \mathrm{Sym}_n(\mathbb{R}(t_1, t)),\\
&B_u = (b_{ij}^{(u)})_{1\le i,j\le s} = B(t_0, \cdots, u_s) \in \mathrm{Sym}_s(\mathbb{R}(t_1)),\ B_u(t) = t^2 B_u.
\end{aligned}
$$

Then, by equation (??), we have

$$
A_u(t) = \left[\begin{array}{cccc|cccc}
n & 0 & \dots & 0 & st & 0 & \dots & t_1 t\\
0 & & & -(n-s)t & 0 & \dots & -(n-1)t_1 t & -nt\\
\vdots & & \iddots & \iddots & & \iddots & \iddots & 0\\
0 & -(n-s)t & \iddots & & & \iddots & 0 & 0\\
\hline
st & 0 & & & & & & \\
0 & \vdots & \iddots & \iddots & & & & \\
\vdots & -(n-1)t_1 t & \iddots & 0 & & & C_u(t) & \\
t_1 t & -nt & 0 & 0 & & & &
\end{array}\right],
$$

where $C_u(t) = (c_{ij}^{(u)}(t))_{1\le i,j\le s} = C(u_0, \cdots, u_s, t)$ and

$$
c_{ij}^{(u)}(t) = b_{ij}(u_0, \cdots, u_s)t^2 + \lambda_{ij}(u_0, \cdots, u_s)t \quad (\lambda_{ij}(u_0, \cdots, u_s) \in \mathbb{R}(t_1)).
$$

Moreover, by equation (??), we also have

$$
A_u(t)_1 = \left[\begin{array}{cccc|cccc}
1 & 0 & \dots & 0 & 0 & 0 & \dots & 0\\
0 & 0 & \dots & -(n-s)t & 0 & \dots & -(n-1)t_1 t & -nt\\
\vdots & \vdots & \iddots & \iddots & & \iddots & \iddots & 0\\
0 & -(n-s)t & \iddots & & & \iddots & 0 & 0\\
\hline
0 & 0 & & & & & & \\
0 & \vdots & \iddots & \iddots & & & & \\
\vdots & -(n-1)t_1 t & \iddots & 0 & & & C_u(t)_1 & \\
0 & -nt & 0 & 0 & & & &
\end{array}\right].
$$

Here, $C_u(t)_1 = (c_{ij}^{(u)}(t)_1)_{1\le i,j\le s} = C(u_0, \cdots, u_s, t)_1$ and

$$
c_{ij}^{(u)}(t)_1 = \bar{b}_{ij}(u_0, \cdots, u_s)t^2 + \lambda_{ij}(u_0, \cdots, u_s)t \quad (\bar{b}_{ij}(u_0, \cdots, u_s) \in \mathbb{R}).
$$

Note that, by equation (??), we have

$$
\bar{b}_{ij}^{(u)} = \begin{cases}
b_{11}^{(u)} - (s^2/n) & (i,j) = (1,1)\\
b_{1s}^{(u)} - (s/n)t_1 & (i,j) = (1,s) \text{ or } (s,1)\\
b_{ss}^{(u)} - (1/n)t_1^2 & (i,j) = (s,s)\\
b_{ij}^{(u)} & \text{otherwise.}
\end{cases} \tag{3.12}
$$

Let us put $\bar{B}_u = (\bar{b}_{ij}^{(u)})_{1\le i,j\le s}$ and $\bar{B}_u(t) = t^2\bar{B}_u$. Then, since

$$\begin{aligned} M_s(g_u) &= M_s(x^s + t_1x + 1, sx^{s-1} + t_1) \\ &= sM_s(x^s, x^{s-1}) + t_1M_s(x^s, 1) - st_1M_s(x^{s-1}, x) - sM_s(x^{s-1}, 1) \\ &\qquad + t_1^2M_s(x, 1) + t_1M_s(1, 1), \end{aligned}$$

we have

(a) if $s = 2$,

$$B_u = \begin{bmatrix} 2 & t_1 \\ t_1 & t_1^2 - 2 \end{bmatrix},$$

(b) if $s \ge 3$,

$$b_{ij}^{(u)} = \begin{cases} s & (i,j) = (1,1) \\ t_1 & (i,j) = (1,s) \text{ or } (s,1) \\ (1-s)t_1 & i+j = s+1,\ 2 \le i,j \le s-1 \\ -s & i+j = s+2 \\ t_1^2 & (i,j) = (s,s), \\ 0 & \text{otherwise,} \end{cases}$$

which, by equation (**??**), implies

(a′) if $s = 2$,

$$\bar{B}_u = \begin{bmatrix} 2(n-2)/n & (n-2)t_1/n \\ (n-2)t_1/n & (n-1)t_1^2/n - 2 \end{bmatrix},$$

(b′) if $s \ge 3$,

$$\bar{b}_{ij}^{(u)} = \begin{cases} s(n-s)/n & (i,j) = (1,1) \\ (n-s)t_1/n & (i,j) = (1,s) \text{ or } (s,1) \\ (1-s)t_1 & i+j = s+1,\ 2 \le i,j \le s-1 \\ -s & i+j = s+2 \\ (n-1)t_1^2/n & (i,j) = (s,s), \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, if $s \ge 3$, the matrix $\bar{B}_u = (\bar{b}_{ij}^{(u)})_{1\le i,j\le s}$ has the expression of the form

$$\begin{bmatrix} s(n-s)/n & 0 & 0 & 0 & \cdots & 0 & (n-s)t_1/n \\ 0 & 0 & 0 & \cdots & 0 & (1-s)t_1 & -s \\ 0 & 0 & & \iddots & (1-s)t_1 & -s & 0 \\ 0 & \vdots & \iddots & \iddots & \iddots & \iddots & \vdots \\ \vdots & 0 & (1-s)t_1 & \iddots & \iddots & & 0 \\ 0 & (1-s)t_1 & -s & \iddots & & & 0 \\ (n-s)t_1/n & -s & 0 & \cdots & 0 & 0 & (n-1)t_1^2/n \end{bmatrix}.$$

Here, let us denote by

$$\Phi_u(x) = \sum_{k=0}^{s} h_{s-k}^{(u)} x^{s-k} = \Phi(u_0, \cdots, u_s; x) \ \left(= \sum_{k=0}^{s} h_{s-k}(u_0, \cdots, u_s) x^{s-k} \right)$$

the characteristic polynomial of $\bar{B}_u$. Note that since we have $h_{s-k}^{(u)} \in \mathbb{R}[t_1]$ by the proof of Lemma **??**, we have only to prove $h_{s-k}^{(u)}$ is non-constant for any k $(1 \leq k \leq s)$.

By the above expression of $\bar{B}_u$, we have
(a'') if $s = 2$,

$$\Phi_u(x) = x^2 - \frac{(n-1)t_1^2 - 4}{n} x + \frac{(n-2)t_1^2 - 4n + 8}{n},$$

(b'') if $s \geq 3$,

$$\Phi_u(x) = \begin{cases} \begin{vmatrix} x - s(n-s)/n & & & & & & & & -(n-s)t_1/n \\ & x & & & & & (s-1)t_1 & s & \\ & & \ddots & & & \iddots & s & & \\ & & & \ddots & \iddots & \iddots & & & \\ & & & & x + (s-1)t_1 & s & & & \\ & & & \iddots & & & & & \\ & & & & s & x & & & \\ & & \iddots & \iddots & & & \ddots & & \\ & (s-1)t_1 & s & & & & & x & \\ -(n-s)t_1/n & s & & & & & & & x - (n-1)t_1^2/n \end{vmatrix} & (s \text{ is odd}), \\ \begin{vmatrix} x - s(n-s)/n & & & & & & & -(n-s)t_1/n \\ & x & & & & & (s-1)t_1 & s \\ & & \ddots & & & \iddots & s & \\ & & & x & (s-1)t_1 & \iddots & & \\ & & & (s-1)t_1 & x + s & & & \\ & & \iddots & \iddots & & \ddots & & \\ & (s-1)t_1 & s & & & & x & \\ -(n-s)t_1/n & s & & & & & & x - (n-1)t_1^2/n \end{vmatrix} & (s \text{ is even}). \end{cases}$$

Example 3.1. (1) *Put* $s = 7$ *and* $n = 10$. *Then, we have*

$$g_u(x) = x^7 + t_1 x + 1, \quad f_u(t; x) = x^{10} + t(x^7 + t_1 x + 1),$$

$$\Phi_u(x) = \begin{vmatrix} x-21/10 & 0 & 0 & 0 & 0 & 0 & -3t_1/10 \\ 0 & x & 0 & 0 & 0 & 6t_1 & 7 \\ 0 & 0 & x & 0 & 6t_1 & 7 & 0 \\ 0 & 0 & 0 & x+6t_1 & 7 & 0 & 0 \\ 0 & 0 & 6t_1 & 7 & x & 0 & 0 \\ 0 & 6t_1 & 7 & 0 & 0 & x & 0 \\ -3t_1/10 & 7 & 0 & 0 & 0 & 0 & x-9t_1^2/10 \end{vmatrix}$$

$$\begin{aligned}
=& x^7 + \left(-\frac{9}{10}t_1^2 + 6t_1 - \frac{21}{10}\right)x^6 + \left(-\frac{27}{5}t_1^3 - \frac{351}{5}t_1^2 - \frac{63}{5}t_1 - 147\right)x^5 \\
&+ \left(\frac{324}{5}t_1^4 - \frac{2106}{5}t_1^3 + \frac{1197}{5}t_1^2 - 588t_1 + \frac{3087}{10}\right)x^4 \\
&+ \left(\frac{1944}{5}t_1^5 + \frac{5832}{5}t_1^4 + \frac{5859}{5}t_1^3 + \frac{16758}{5}t_1^2 + \frac{6174}{5}t_1 + 7203\right)x^3 \\
&+ \left(-\frac{5832}{5}t_1^6 + \frac{34992}{5}t_1^5 - \frac{21546}{5}t_1^4 + \frac{50274}{5}t_1^3 - \frac{95697}{10}t_1^2 + 14406t_1 - \frac{151263}{10}\right)x^2
\end{aligned}$$

$$\begin{aligned}
&+ \left(-\frac{34992}{5}t_1^7 + \frac{11664}{5}t_1^6 - \frac{81648}{5}t_1^5 + \frac{15876}{5}t_1^4 - \frac{111132}{5}t_1^3 + \frac{21609}{5}t_1^2 - \frac{151263}{5}t_1\right. \\
&\left. - 117649\right)x + \frac{69984}{5}t_1^7 + \frac{2470629}{10}.
\end{aligned}$$

(2) *Put* $s = 8$ *and* $n = 12$. *Then, we have*

$$g_u(x) = x^8 + t_1 x + 1, \ \ f_u(t; x) = x^{12} + t(x^8 + t_1 x + 1)$$

and

$$\Phi_u(x) = \begin{vmatrix} x-8/3 & 0 & 0 & 0 & 0 & 0 & 0 & -t_1/3 \\ 0 & x & 0 & 0 & 0 & 0 & 7t_1 & 8 \\ 0 & 0 & x & 0 & 0 & 7t_1 & 8 & 0 \\ 0 & 0 & 0 & x & 7t_1 & 8 & 0 & 0 \\ 0 & 0 & 0 & 7t_1 & x+8 & 0 & 0 & 0 \\ 0 & 0 & 7t_1 & 8 & 0 & x & 0 & 0 \\ 0 & 7t_1 & 8 & 0 & 0 & 0 & x & 0 \\ -t_1/3 & 8 & 0 & 0 & 0 & 0 & 0 & x-11t_1^2/12 \end{vmatrix}$$

$$\begin{aligned}
=& x^8 + \left(-\frac{11}{12}t_1^2 + \frac{16}{3}\right)x^7 + \left(-152t_1^2 - \frac{640}{3}\right)x^6 + \left(\frac{539}{4}t_1^4 - 256t_1^2 - 1024\right)x^5 \\
&+ \left(\frac{22736}{3}t_1^4 + \frac{45824}{3}t_1^2 + 16384\right)x^4 + \left(-\frac{26411}{4}t_1^6 - \frac{22736}{3}t_1^4 + \frac{31744}{3}t_1^2 + 65536\right)x^3 \\
&+ \left(-\frac{355348}{3}t_1^6 - 213248t_1^4 - \frac{1064960}{3}t_1^2 - 524288\right)x^2 + \left(\frac{1294139}{12}t_1^8 + \frac{1075648}{3}t_1^6\right. \\
&\left. + \frac{1404928}{3}t_1^4 + \frac{1835008}{3}t_1^2 - \frac{4194304}{3}\right)x - \frac{823543}{3}t_1^8 + \frac{16777216}{3}.
\end{aligned}$$

Proof of Lemma **??**. To prove Lemma **??**, it is enough to prove $\deg h_{s-k}^{(u)} \geq 1$ for any k $(1 \leq k \leq s)$. This is clear for $s = 2$ by (a'') and we suppose $s \geq 3$ hereafter. To prove $\deg h_{s-k}^{(u)} \geq 1$ $(1 \leq k \leq s)$, let us compute the leading term of $h^{(u)}_{s-k}$ ($\in \mathbb{R}[t_1]$). Then, since $h^{(u)}_{s-k}$ is the coefficient of the term $h^{(u)}_{s-k}x^{s-k}$ of the characteristic polynomial $\Phi_u(x)$, we need to maximize the degree in t_1 when we take '$s-k$' x and the remaining k elements from $\mathbb{R}[t_1]$.

(a) Suppose s is odd. Let us divide the case into three other sub-cases.

$(a1)$ Suppose k is odd and $1 \le k \le s-2$.
In this case, the degree of the leading term of $h^{(u)}_{s-k}$ is $k+1$. In fact, it is obtained by taking

$(a11)$ $-(n-1)t_1^2/n$ from the (s,s) entry $x-(n-1)t_1^2/n$,

$(a12)$ '$k-1$' $(s-1)t_1$ from entries of the form $(i, s+1-i)$ $(2 \le i \le s-1)$.

First, suppose we take the (s,s) entry $x-(n-1)t_1^2/n$ from the s-th row. Then we must take the $(1,1)$ entry from the first row. Next, let us proceed to the $(s-1)$-th row. If we take the $(s-1,s-1)$ entry x from the $(s-1)$-th row, then we must also take x from the second row, while if we take $(s-1)t_1$ from the $(s-1)$-th row, then we must also take $(s-1)t_1$ from the second row. The situation is the same for the $(s-2)$-th row, the $(s-3)$-th row ... and so on, which implies that $(s-1)t_1$ must occur in pair.

Hence, the leading term of $h^{(u)}_{s-k}$ is

$$-\frac{n-1}{n}t_1^2 \cdot \binom{(s-3)/2}{(k-1)/2}\{(-1)\cdot(s-1)^2t_1^2\}^{(k-1)/2} \qquad \left(\binom{n}{0}=1 \ (n \ge 0)\right)$$

and the degree of this term is $k+1$ (≥ 2).
$(a2)$ Suppose k is odd and $k=s$.
If $k=s$, $h^{(u)}_{s-k}=h^{(u)}_0$ is the constant term of $\Phi_u(x)$. In this case, the degree of the leading term of $h^{(u)}_0$ is s. In fact, it is obtained by taking

$(a21)$ $-(n-1)t_1^2/n$ from the (s,s) entry $x-(n-1)t_1^2/n$,

$(a22)$ If $s \ge 5$ ($\Leftrightarrow (s,k) \ne (3,3)$), '$(s-3)/2$' pairs of $(s-1)t_1$ from entries of the form $(i, s+1-i)$ $(2 \le i \le (s-1)/2,\ (s+3)/2 \le i \le s-1)$,

$(a23)$ $(s-1)t_1$ from the $((s+1)/2,(s+1)/2)$ entry $x+(s-1)t_1$,

$(a24)$ $-s(n-s)/n$ from the $(1,1)$ entry $x-s(n-s)/n$

or by taking

$(a25)$ all anti-diagonal entries.

Therefore, the leading term of $h_0^{(u)}$ is

$$\begin{aligned}
&-\frac{n-1}{n}t_1^2 \cdot \{(-1)\cdot(s-1)^2t_1^2\}^{(s-3)/2}\cdot(s-1)t_1\cdot\left(-\frac{s(n-s)}{n}\right)\\
&\qquad\qquad + (-1)\cdot\left(-\frac{n-s}{n}t_1\right)^2\cdot\{(-1)\cdot(s-1)^2t_1^2\}^{(s-3)/2}\cdot(s-1)t_1\\
&= \frac{(n-s)(s-1)}{n}\cdot(-1)^{(s-3)/2}(s-1)^{s-2}t_1^s\\
&= (-1)^{(s-3)/2}\frac{(n-s)(s-1)^{s-1}}{n}t_1^s
\end{aligned}$$

for any s ($s \geq 3$) and the degree of this term is s.

$(a3)$ Suppose k is even.

In this case, we have $2 \leq k \leq s-1$ and the degree of the leading term of $h_{s-k}^{(u)}$ is $k+1$. In fact, it is obtained by taking

$(a31)$ $-(n-1)t_1^2/n$ from the (s,s) entry $x-(n-1)t_1^2/n$,

$(a32)$ If $s \geq 5$ ($\Leftrightarrow (s,k) \neq (3,2)$), '$(k-2)/2$' pairs of $(s-1)t_1$ from entries of the form $(i, s+1-i)$ $(2 \leq i \leq (s-1)/2,\ (s+3)/2 \leq i \leq s-1)$,

$(a33)$ $(s-1)t_1$ from the $((s+1)/2,(s+1)/2)$ entry $x+(s-1)t_1$.

Therefore, the leading term of $h_{s-k}^{(u)}$ is

$$-\frac{n-1}{n}t_1^2\cdot\binom{(s-3)/2}{(k-2)/2}\{(-1)\cdot(s-1)^2t_1^2\}^{(k-2)/2}\cdot(s-1)t_1$$

for any s ($s \geq 3$) and the degree of this term is $k+1$ (≥ 3).

(b) Suppose s is even ($s \geq 4$). We also divide this case into three other sub-cases.

$(b1)$ Suppose k is odd.

In this case, we have $1 \leq k \leq s-1$ and the degree of the leading term of $h_{s-k}^{(u)}$ is $k+1$. In fact, it is obtained by taking

$(b11)$ $-(n-1)t_1^2/n$ from the (s,s) entry $x-(n-1)t_1^2/n$,

$(b12)$ '$(k-1)/2$' pairs of $(s-1)t_1$ from entries of the form $(i, s+1-i)$ $(2 \leq i \leq s-1)$.

Therefore, the leading term of $h_{s-k}^{(u)}$ is

$$-\frac{n-1}{n}t_1^2\cdot\binom{(s-2)/2}{(k-1)/2}\{(-1)\cdot(s-1)^2t_1^2\}^{(k-1)/2}$$

and the degree of this term is $k+1$ (≥ 2).

$(b2)$ Suppose k is even and $2 \leq k \leq s-2$.

In this case, the degree of the leading term of $h_{s-k}^{(u)}$ is k. In fact, it is obtained by taking (b21) $-(n-1)t_1^2/n$ from the (s,s) entry $x-(n-1)t_1^2/n$,

(b22) '$(k-2)/2$' pairs of $(s-1)t_1$ from entries of the form $(i, s+1-i)$ $(2 \le i \le s-1)$,

(b23) $-s(n-s)/n$ from the $(1,1)$ entry $x-s(n-s)/n$

or by taking

(b24) $-(n-1)t_1^2/n$ from the (s,s) entry $x-(n-1)t_1^2/n$,

(b25) I f $s \ge 6$ ($\Leftrightarrow (s,k) \ne (4.2)$), '$(k-2)/2$' pairs of $(s-1)t_1$ from entries of the form $(i, s+1-i)$ $(2 \le i \le (s-2)/2,\ (s+4)/2 \le i \le s-1)$,

(b26) s from the $((s+2)/2, (s+2)/2)$ entry $x+s$

or by taking

(b27) '$k/2$' pairs of $(s-1)t_1$ from entries of the form $(i, s+1-i)$ $(2 \le i \le s-1)$

or by taking

(b28) One pair of $-(n-s)t_1/n$ from the $(1,s)$ and the $(s,1)$ entry,

(b29) '$(k-2)/2$' pairs of $(s-1)t_1$ from entries of the form $(i, s+1-i)$ $(2 \le i \le s-1)$.

Here, note that if we take the $(s,1)$ entry $-(n-s)t_1/n$ from the s-th row, we must also take the $(1,s)$ entry $-(n-s)t_1/n$ from the first row.

Therefore, the leading term of $h_{s-k}^{(u)}$ is

$$
\begin{aligned}
&-\frac{n-1}{n}t_1^2 \cdot \binom{(s-2)/2}{(k-2)/2}\{(-1)\cdot(s-1)^2t_1^2\}^{(k-2)/2} \cdot \left(-\frac{s(n-s)}{n}\right) \\
&\quad -\frac{n-1}{n}t_1^2 \cdot \binom{(s-4)/2}{(k-2)/2}\{(-1)\cdot(s-1)^2t_1^2\}^{(k-2)/2} \cdot s + \binom{(s-2)/2}{k/2}\{(-1)\cdot(s-1)^2t_1^2\}^{k/2} \\
&\quad + \left((-1)\cdot\frac{\{-(n-s)\}^2}{n^2}t_1^2\right)\cdot\binom{(s-2)/2}{(k-2)/2}\{(-1)\cdot(s-1)^2t_1^2\}^{(k-2)/2} \\
&= \left(\frac{s(n-s)(n-1)}{n^2}\binom{(s-2)/2}{(k-2)/2} - \frac{s(n-1)}{n}\binom{(s-4)/2}{(k-2)/2}\right. \\
&\qquad\qquad \left. -(s-1)^2\binom{(s-2)/2}{k/2} - \frac{(n-s)^2}{n^2}\binom{(s-2)/2}{(k-2)/2}\right)\{(-1)\cdot(s-1)^2t_1^2\}^{(k-2)/2}t_1^2.
\end{aligned}
$$

for any s $(s \ge 4)$. Then, since

$$
\binom{(s-4)/2}{(k-2)/2} = \frac{s-k}{s-2}\binom{(s-2)/2}{(k-2)/2}, \quad \binom{(s-2)/2}{k/2} = \frac{s-k}{k}\binom{(s-2)/2}{(k-2)/2},
$$

we have

$$\begin{aligned}
&\frac{s(n-s)(n-1)}{n^2}\binom{(s-2)/2}{(k-2)/2}-\frac{s(n-1)}{n}\binom{(s-4)/2}{(k-2)/2} \\
&\qquad\qquad -(s-1)^2\binom{(s-2)/2}{k/2}-\frac{(n-s)^2}{n^2}\binom{(s-2)/2}{(k-2)/2} \\
&=\left(\frac{s(n-s)(n-1)}{n^2}-\frac{s(s-k)(n-1)}{n(s-2)}-\frac{(s-1)^2(s-k)}{k}-\frac{(n-s)^2}{n^2}\right)\binom{(s-2)/2}{(k-2)/2} \\
&=\frac{s\{\big(k(k+s^2-4s+2)-s^3+4s^2-5s+2\big)n-k(k+s^2-4s+2)\}}{nk(s-2)}\binom{(s-2)/2}{(k-2)/2}.
\end{aligned} \tag{3.13}$$

Hence, if the above value becomes zero, we have

$$\big(k(k+s^2-4s+2)-s^3+4s^2-5s+2\big)\,n-k(k+s^2-4s+2)=0,$$

which implies

$$k(k+s^2-4s+2)=0,\ -s^3+4s^2-5s+2=0 \tag{3.14}$$

or

$$n=\frac{k(k+s^2-4s+2)}{k(k+s^2-4s+2)-s^3+4s^2-5s+2}. \tag{3.15}$$

Here, (**??**) is impossible since $-s^3+4s^2-5s+2=-(s-1)^2(s-2)$ and $s\geq 4$. Also, (**??**) is impossible since, for any $s\geq 4$ and $2\leq k\leq s-2$, we have

$$k(k+s^2-4s+2)\geq 2(2+s^2-4s+2)\geq 2(s-2)^2>0$$

and

$$\begin{aligned}
&k(k+s^2-4s+2)-s^3+4s^2-5s+2\\
&\leq (s-2)\{(s-2)+s^2-4s+2\}-s^3+4s^2-5s+2\\
&=-s^2+s+2\\
&=-(s+1)(s-2)<0,
\end{aligned}$$

which implies $n<0$, a contradiction. Thus, the above value (**??**) is non-zero and the degree of the leading term of $h_{s-k}^{(u)}$ is k.

(b3) Suppose k is even and $k=s$.

If $k=s$, $h_{s-k}^{(u)}=h_0^{(u)}$ is the constant term of $\Phi_u(x)$. In this case, the degree of the leading term of $h_0^{(u)}$ is s. In fact, it is obtained by taking

(b31) $-(n-1)t_1^2/n$ from the (s,s) entry $x-(n-1)t_1^2/n$,

(b32) '$(s-2)/2$' pairs of $(s-1)t_1$ from entries of the form $(i,s+1-i)$ $(2\leq i\leq s-1)$,

(b33) $-s(n-s)/n$ from the $(1,1)$ entry $x-s(n-s)/n$

or by taking

(b34) all anti-diagonal entries.

Therefore, the leading term of $h_0^{(u)}$ is

$$
\begin{aligned}
-\frac{n-1}{n}t_1^2 &\cdot \{(-1)\cdot(s-1)^2t_1^2\}^{(s-2)/2}\cdot\left(-\frac{s(n-s)}{n}\right)\\
&\qquad + (-1)\cdot\left(-\frac{n-s}{n}t_1\right)^2\cdot\{(-1)\cdot(s-1)^2t_1^2\}^{(s-2)/2}\\
&= (-1)^{(s-2)/2}\frac{(n-s)(s-1)^{s-1}}{n}t_1^s
\end{aligned}
$$

and the degree of this term is s $(s \geq 4)$. $\square$

Lemma 3.5. *Let $v = (v_0, \cdots, v_s) \in \mathbb{R}^{s+1}$ be a real vector and n $(> s)$ be an integer. Put*

$$P_v(t) = \det M_n(f_v(t;x)) = \det M_n(f^{(n)}(v_0, \cdots, v_s, t; x))$$

and $\alpha_v = \max\{\alpha \in \mathbb{R} \mid P_v(\alpha) = 0\}$. If there exists a real number ρ_0 $(> \alpha_v)$ such that $N_{f_v(\xi;x)} = \gamma_0$ for any $\xi > \rho_0$, we have $N_{f_v(\xi;x)} = \gamma_0$ for any $\xi > \alpha_v$.

Proof. Put $A_v(t) = M_n(f_v(t;x))$. Then, by Proposition ??, we have $\gamma_0 = \sigma(A_v(\xi))$ for any $\xi > \rho_0$. Let us also put

$$R = \{\rho \in \mathbb{R} \mid \rho > \alpha_v,\ \sigma(A_v(\xi)) = \gamma_0 \text{ for any } \xi > \rho\}.$$

Since R is a nonempty set $(\rho_0 \in R)$ having a lower bound α_v, R has the infimum ρ_v; $\rho_v = \inf R$. Then, it is enough to prove $\rho_v = \alpha_v$. Here, suppose to the contrary that $\rho_v > \alpha_v$ and we denote by

$$\Omega_v(t;x) = \sum_{k=0}^{n}\omega_k(t)x^k \in \mathbb{R}(t)[x]$$

the characteristic polynomial of $A_v(t)$. Note that $\omega_k(t) \in \mathbb{R}[t]$ $(0 \leq k \leq n)$ and for any $\xi > \alpha_v$, $\Omega_v(\xi;x)$ has n non-zero real roots (counted with multiplicity) since $A_v(\xi)$ is symmetric and $\det A_v(\xi) \neq 0$. Then, by Proposition ??, there exists a positive real number δ such that $\rho_v - \delta > \alpha_v$ and for any $\xi \in [\rho_v - \delta, \rho_v + \delta]$, $\Omega_v(\xi;x)$ has the same number of positive and hence negative real roots with $\Omega_v(\rho_v;x)$. On the other hand, since $\rho_v = \inf R$, there exist real numbers ξ_+ $(\rho_v < \xi_+ < \rho_v + \delta)$ and ξ_- $(\rho_v - \delta < \xi_- < \rho_v)$ such that $\sigma(A_v(\xi_+)) \neq \sigma(A_v(\xi_-))$, which implies $\Omega_v(\xi_+;x)$ and $\Omega_v(\xi_-;x)$ have different number of positive and hence negative real roots. This is a contradiction and we have $\rho_v = \alpha_v$. $\square$

3.4 Proof of Theorem ??

Let $r = (r_0, \cdots, r_s) \in \mathbb{R}^{s+1}$ be the vector as in Theorem ?? and put

$$n_0 = \begin{cases} (n-s+1)/2, & n-s-1 : \text{even} \\ (n-s+2)/2, & n-s-1 : \text{odd.} \end{cases}$$

When $n-s \geq 2$, we inductively define the matrix $A_r(t)_k = (a^{(r)}_{ij}(t)_k)_{1\leq i,j\leq n}$ $(2 \leq k \leq n-s)$ as the matrix obtained from $A_r(t)_{k-1}$ by sweeping out the entries of the k-th row (k-th column) by the (k, l_0-k) entry $-(n-s)r_s t$ ((l_0-k, k) entry $-(n-s)r_s t$). That is, we define $A_r(t)_k = {}^tS_r(t)_k A_r(t)_{k-1} S_r(t)_k$, where

$$S_r(t)_k = \begin{cases} \displaystyle\prod_{m=l_0-k+1}^{n} R_n\left(l_0-k, m; -\frac{a^{(r)}_{km}(t)_{k-1}}{-(n-s)r_s t}\right) & (2 \leq k \leq n_0) \\ \displaystyle R_n\left(l_0-k, k; -\frac{a^{(r)}_{kk}(t)_{k-1}}{-2(n-s)r_s t}\right)\prod_{m=k+1}^{n} R_n\left(l_0-k, m; -\frac{a^{(r)}_{km}(t)_{k-1}}{-(n-s)r_s t}\right) & \\ & (n_0 < k \leq n-s). \end{cases}$$

Then, if $n-s \geq 1$, we can express the matrix $A_r(t)_{n-s}$ as follows;

$$A_r(t)_{n-s} = \left[\begin{array}{cccc|c} 1 & 0 & \dots & 0 & \\ 0 & 0 & \dots & -(n-s)r_s t & \\ \vdots & \vdots & \iddots & 0 & O \\ 0 & -(n-s)r_s t & 0 & 0 & \\ \hline & & O & & C_r(t)_{n-s} \end{array}\right].$$

Note that $a^{(r)}_{km}(t)_{k-1}$ and $a^{(r)}_{kk}(t)_{k-1}$ appearing in $S_r(t)_k$ are degree 1 monomials in t and hence the numbers $-a^{(r)}_{km}(t)_{k-1}/(-(n-s)r_s t)$, $-a^{(r)}_{kk}(t)_{k-1}/(-2(n-s)r_s t)$ appearing in $S_r(t)_k$ are just real numbers. Therefore, the entries of the $s \times s$ symmetric matrix $C_r(t)_{n-s} = (c^{(r)}_{ij}(t)_{n-s})_{1\leq i,j\leq s}$ $(n-s \geq 1)$ are of the form

$$c^{(r)}_{ij}(t)_{n-s} = \bar{b}^{(r)}_{ij} t^2 + \bar{\lambda}^{(r)}_{ij} t \quad (\bar{\lambda}^{(r)}_{ij} \in \mathbb{R}). \tag{3.16}$$

Moreover, since the matrix

$$D_r(t)_{n-s} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & -(n-s)r_s t \\ \vdots & \vdots & \iddots & 0 \\ 0 & -(n-s)r_s t & 0 & 0 \end{bmatrix}$$

is equivalent to the matrix

$$\bar{D}_r(t)_{n-s} = \begin{cases} \left[\begin{array}{c|cc} 1 & & \\ \hline & \begin{array}{|cc|} 0 & -(n-s)r_s t \\ -(n-s)r_s t & 0 \end{array} & \\ & \ddots & \\ & & \begin{array}{|cc} 0 & -(n-s)r_s t \\ -(n-s)r_s t & 0 \end{array} \end{array}\right] & (n-s : \text{odd}) \\ \left[\begin{array}{c|ccc} 1 & & & \\ \hline & -(n-s)r_s t & & \\ & & \begin{array}{|cc|} 0 & -(n-s)r_s t \\ -(n-s)r_s t & 0 \end{array} & \\ & & \ddots & \\ & & & \begin{array}{|cc} 0 & -(n-s)r_s t \\ -(n-s)r_s t & 0 \end{array} \end{array}\right] & (n-s : \text{even}) \end{cases}$$

over $\mathbb{R}$, we have

$$\sigma(D_r(\xi)_{n-s}) = \sigma(\bar{D}_r(\xi)_{n-s}) = \begin{cases} 1 & n-s : \text{odd} \\ 0 & n-s : \text{even}, \ r_s > 0 \\ 2 & n-s : \text{even}, \ r_s < 0 \end{cases} \tag{3.17}$$

for any real number $\xi > \alpha_r$ (≥ 0). Here, note that since $P_r(0) = 0$, we have $\alpha_r \geq 0$.

Next, let $\Phi_r(t;x)$, $\Psi_r(t;x)$ be characteristic polynomials of $\bar{B}_r(t)$, $C_r(t)_{n-s}$, respectively. Then, by equations (**??**) and (**??**), we have

$$\begin{aligned} \Phi_r(t;x) &= x^s + h_{s-1}^{(r)} t^2 x^{s-1} + \cdots + h_1^{(r)} t^{2s-2} x + h_0^{(r)} t^{2s} \\ &\qquad \left(h_{s-k}^{(r)} = h_{s-k}(r_0, \cdots, r_s) \in \mathbb{R} \ (1 \leq k \leq s)\right), \\ \Psi_r(t;x) &= x^s + \left(h_{s-1}^{(r)} t^2 + \psi_{s-1}(t)\right) x^{s-1} + \cdots \\ &\qquad + \left(h_1^{(r)} t^{2s-2} + \psi_1(t)\right) x + \left(h_0^{(r)} t^{2s} + \psi_0(t)\right) \\ &\qquad (\psi_0(t), \cdots, \psi_{s-1}(t) \in \mathbb{R}[t], \deg \psi_{s-k}(t) < 2k \ (1 \leq k \leq s)). \end{aligned}$$

Here, let us divide the proof into next two cases.

(i) The case $h_0^{(r)} h_1^{(r)} \cdots h_{s-1}^{(r)} \neq 0$.

In this case, we have

$$\begin{aligned} \Psi_r(t;x) &= x^s + h_{s-1}^{(r)} t^2 \left(1 + \frac{\psi_{s-1}(t)}{h_{s-1}^{(r)} t^2}\right) x^{s-1} + \cdots \\ &\qquad + h_1^{(r)} t^{2s-2} \left(1 + \frac{\psi_1(t)}{h_1^{(r)} t^{2s-2}}\right) x + h_0^{(r)} t^{2s} \left(1 + \frac{\psi_0(t)}{h_0^{(r)} t^{2s}}\right) \end{aligned}$$

and $1+\psi_{s-k}(t)/h^{(r)}_{s-k}t^{2k} \to 1$ $(t \to \infty)$ for any k $(1 \le k \le s)$. Moreover, since $h^{(r)}_0 h^{(r)}_1 \cdots h^{(r)}_{s-1} \neq 0$, we have $h^{(r)}_0 \neq 0$, which implies that for any non-zero real number ξ, $\Phi_r(\xi; x)$ have s non-zero real roots (counted with multiplicity). Thus, there exists a real number ρ_0 $(> \alpha_r)$ such that for any real number $\xi > \rho_0$, $\Psi_r(\xi; x)$ have the same number of positive (hence also negative) real roots with $\Phi_r(\xi; x)$ by Proposition ??, which implies $\sigma(C_r(\xi)_{n-s}) = \sigma(\bar{B}_r(\xi))$ and hence $\sigma(C_r(\xi)_{n-s}) = N_{g_r} = \gamma$ $(\xi > \rho_0)$ by Lemma ??. Then, by the equation (??), we have

$$\sigma(A_r(\xi)_{n-s}) = \begin{cases} \gamma+1 & n-s : \text{odd} \\ \gamma & n-s : \text{even, } r_s > 0 \\ \gamma+2 & n-s : \text{even, } r_s < 0 \end{cases}$$

for any $\xi > \rho_0$, which implies

$$N_{f_r(\xi;x)} = \sigma(A_r(\xi)) = \begin{cases} \gamma+1 & n-s : \text{odd} \\ \gamma & n-s : \text{even, } r_s > 0 \\ \gamma+2 & n-s : \text{even, } r_s < 0 \end{cases}$$

for any $\xi > \rho_0$ since $A_r(\xi)$ and $A_r(\xi)_{n-s}$ are equivalent over $\mathbb{R}$. Hence, by Lemma ??, we have

$$N_{f_r(\xi;x)} = \begin{cases} \gamma+1 & n-s : \text{odd} \\ \gamma & n-s : \text{even, } r_s > 0 \\ \gamma+2 & n-s : \text{even, } r_s < 0 \end{cases}$$

for any $\xi > \alpha_r$.

(ii) General case.

Let ε_0 be a positive real number and for any vector $\nu \in \mathbb{R}^{s+1}$, set

$$\alpha'_\nu = \max\{|\alpha| \mid \alpha \in \mathbb{C}, P_\nu(\alpha) = 0\}.$$

Clearly, we have $\alpha'_\nu \ge \alpha_\nu$ for any $\nu \in \mathbb{R}^{s+1}$. Here, let us put $\rho'_0 = \alpha'_r + \varepsilon_0$. Then, by Lemma ??, it is enough to prove the next claim.

Claim 1. *For any real number $\xi > \rho'_0$, we have*

$$N_{f_r(\xi;x)} = \begin{cases} \gamma+1 & n-s : \textit{odd} \\ \gamma & n-s : \textit{even, } r_s > 0 \\ \gamma+2 & n-s : \textit{even, } r_s < 0. \end{cases}$$

Proof. By the assumption that $g_r(x)$ is a separable polynomial of degree s and the fact that the non-real roots must occur in pair with its complex conjugate, there exists a real number δ_0 such that for any vector $\nu = (\nu_0, \cdots, \nu_s) \in \mathbb{R}^{s+1}$ satisfying $|r-\nu|_0 = \max_{0\le k\le s}\{|r_k - \nu_k|\} < \delta_0$, $g_\nu(x)$ is also a degree s separable polynomial satisfying $N_{g_\nu} = N_{g_r} = \gamma$ by Proposition ??.

(S1) If a vector $\nu \in \mathbb{R}^{s+1}$ satisfies $|r-\nu|_0 < \delta_0$, then $g_\nu(x)$ is also a degree s separable polynomial satisfying $N_{g_\nu} = N_{g_r} = \gamma$.

Next, we put

$$P(t) = \sum_{k \geq 0} x_k(t_0, \cdots, t_s)t^k = \det A(t) \ (A(t) = A(t_0, \cdots, t_s, t))$$

and let us consider $P(t)$ as a polynomial over $E_1 = \mathbb{R}(t_0, \cdots, t_s)$ in t. Then, since $x_k(t_0, \cdots, t_s) \in \mathbb{R}[t_0, \cdots, t_s]$ for any $k \geq 0$, there exists a real number $\delta_1 > 0$ such that for any vector $v \in \mathbb{R}^{s+1}$ satisfying $|r - v|_0 < \delta_1$, we have $|\alpha'_r - \alpha'_v| < \varepsilon_0$ by Proposition **??**;

(S2) If a vector $v \in \mathbb{R}^{s+1}$ satisfies $|r - v|_0 < \delta_1$, we have $|\alpha'_r - \alpha'_v| < \varepsilon_0$.

Here, let ξ be any real number such that $\xi > \rho'_0 = \alpha'_r + \varepsilon_0$ and let

$$\Omega(t_0, \cdots, t_s, \xi; x) = \sum_{k=0}^{n} y_k(t_0, \cdots, t_s)x^k \in E_1[x]$$

be the characteristic polynomial of the Bezoutian

$$A(t_0, \cdots, t_s, \xi; x) = M_n(f^{(n)}(t_0, \cdots, t_s, \xi; x), f^{(n)}(t_0, \cdots, t_s, \xi; x)').$$

Here, $f^{(n)}(t_0, \cdots, t_s, \xi; x)'$ is the derivative of

$$f^{(n)}(t_0, \cdots, t_s, \xi; x) = \sum_{k=0}^{n} z_k(t_0, \cdots, t_s)x^k \in E_1[x]$$

with respect to x. Then, since $z_k(t_0, \cdots, t_s) \in \mathbb{R}[t_0, \cdots, t_s]$ $(0 \leq k \leq n)$, we also have $y_k(t_0, \cdots, t_s) \in \mathbb{R}[t_0, \cdots, t_s]$ $(0 \leq k \leq n)$. Moreover, since $\xi > \rho'_0 > \alpha_r$, we have $\det A_r(\xi) = \det A(r_0, \cdots, r_s, \xi) \neq 0$.

By these arguments, we can also deduce that there exists a positive real number δ_2 such that for any vector $v \in \mathbb{R}^{s+1}$ satisfying $|r - v|_0 < \delta_2$, the characteristic polynomial $\Omega_v(\xi; x)$ have the same number of positive and hence negative real roots with $\Omega_r(\xi; x)$ (counted with multiplicity), which implies $N_{f_r(\xi;x)} = \sigma(A_r(\xi)) = \sigma(A_v(\xi)) = N_{f_v(\xi;x)}$.

(S3) If a vector $v \in \mathbb{R}^{s+1}$ satisfies $|r - v|_0 < \delta_2$, we have $N_{f_r(\xi;x)} = N_{f_v(\xi;x)}$.

Put $\delta = \min\{\delta_0, \delta_1, \delta_2\} > 0$. Then, there exists a vector $w = (w_0, \cdots, w_s) \in \mathbb{R}^{s+1}$ such that

$$(a) \ |r - w|_0 < \delta, \ (b) \ h_0^{(w)} h_1^{(w)} \cdots h_{s-1}^{(w)} \neq 0.$$

Here, we put $h_{s-k}^{(w)} = h_{s-k}(w_0, \cdots, w_s)$ for any k $(1 \leq k \leq s)$. In fact, since $h_{s-k}(t_0, \cdots, t_s)$ is a non-zero polynomial for any k $(1 \leq k \leq s)$ by Lemma **??**, the product $\prod_{k=1}^{s} h_{s-k}(t_0, \cdots, t_s)$ is also non-zero, which implies that there exists a vector $w \in \mathbb{R}^{s+1}$ satisfying (a) and (b).

Let $w \in \mathbb{R}^{s+1}$ be the vector as above. Then, since $|r - w|_0 < \delta \leq \delta_0$, $g_w(x)$ is a degree s separable polynomial satisfying $N_{g_w} = \gamma$ by (S1) and also, by (S2), we have $\alpha_w \leq \alpha'_w < \alpha'_r + \varepsilon_0 = \rho'_0 < \xi$. Thus, by (b) and the case (i), we have

$$N_{f_w(\xi;x)} = \begin{cases} \gamma + 1 & n - s : \text{odd} \\ \gamma & n - s : \text{even}, \ r_s > 0 \\ \gamma + 2 & n - s : \text{even}, \ r_s < 0, \end{cases}$$

which, by (S3), implies

$$N_{f_r(\xi;x)} = \begin{cases} \gamma+1 & n-s : \text{odd} \\ \gamma & n-s : \text{even}, \ r_s > 0 \\ \gamma+2 & n-s : \text{even}, \ r_s < 0. \end{cases}$$

Since ξ is any real number such that $\xi > \rho_0'$, this completes the proof of Claim and hence the proof of Theorem **??**. ☐

Proposition 5. *Let* $g(x) = \sum_{i=0}^{s} a_i x^i$ *be a polynomial in* $\mathbb{R}[x]$ *such that* $\Delta_g \neq 0$ *and*

$$f(t, x) = x^n + t \cdot g(x) \tag{3.18}$$

If $g(x)$ *is totally complex,* $(n-s)$ *is even, and* $a_s > 0$ *then* $f(\beta, x)$ *is totally complex for all* $\beta > \max\{\alpha \,|\, \Delta_{(f,x)}(\alpha) = 0\}$.

Proof. We have to show that $f(\beta, x)$ has no real roots. Since $g(x)$ is totally complex we have that $\gamma = 0$. $N_{f(\beta,x)} = \gamma$ as $\beta > \max\{\alpha \,|\, \Delta_{(f,x)}(\alpha) = 0\}$ and $a_s > 0$, so $N_{f(\beta,x)} = \gamma = 0$. Hence, $f(\beta, x)$ is totally complex.

☐

Let $K := \mathbb{Q}(t, a_0, \ldots, a_s)$ be the field of transcendental degree $s+1$ and $g(x) = \sum_{i=0}^{s} a_i x^i$. Then we have the following.

Corolary 2. *Let* $K := \mathbb{Q}(t, a_0, \ldots, a_s)$ *be the field of transcendental degree* $s+1$, $g(x) = \sum_{i=0}^{s} a_i x^i$ *and*

$$f(t, x) = x^n + t \cdot g(x)$$

For any value of $(\lambda_0, \ldots, \lambda_s) \in \mathbb{Z}^{s+1}$, *if* $g(\lambda_0, \ldots, \lambda_s, x) \in \mathbb{Z}[x]$ *is irreducible and satisfies the conditions of the Eisenstein criteria, then* $f(x)$ *is irreducible, over* $\mathbb{Q}$.

We also note:

Remark 3.4. *It can be verified computationally by Maple that if* $n \leq 9$ *and* $1 \leq s < n$ *then the Galois group Gal* $_K(f, x)$ *is isomorphic to* S_n.

Remark 3.5. *Polynomials in Eq.* (**??**) *for* $s = 1$ *and* $t = 1$ *has been treated by Y. Zarhin in* [?zarhin] *while studying Mori trinomials. It is shown there that the Galois group of* $f(x)$ *over* $\mathbb{Q}$ *is isomorphic to* S_n*; see* [?zarhin, Cor. 3.5] *for details.*

In general, if we let $K := \mathbb{Q}(t, a_0, \ldots, a_s)$ be the field of transcendental degree $s+1$, for $1 \leq s < n$, then we expect that Gal $_K(f) \cong S_n$ for all $n \geq 1$. If true, this would generalize Zarhin's result to a more general class of polynomials.

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