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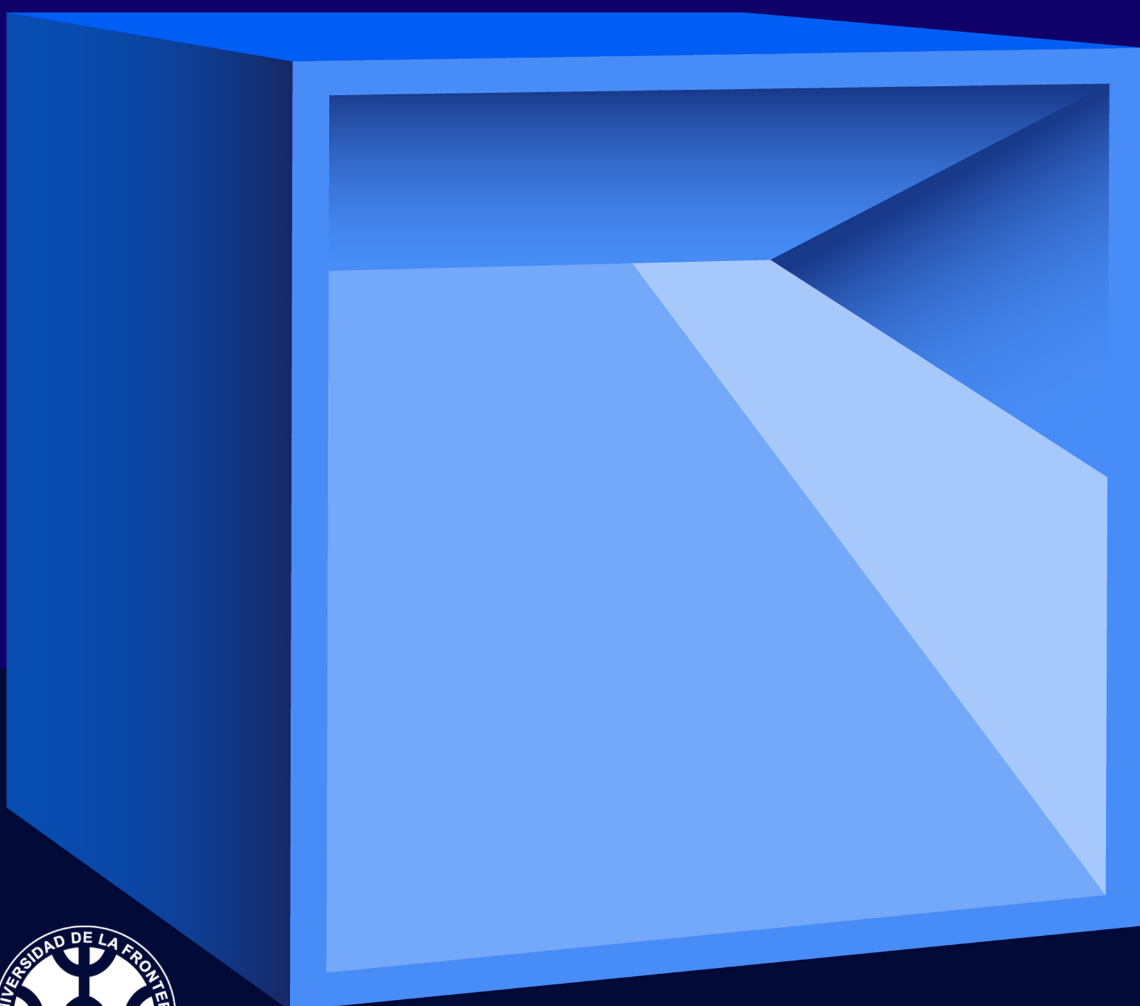

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CUBO
A Mathematical Journal

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SUMMARY

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Quantitative Approximation by a Kantorovich-Shilkret quasi-interpolation neural network operator

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ABSTRACT

In this article we present multivariate basic approximation by a Kantorovich-Shilkret type quasi-interpolation neural network operator with respect to supremum norm. This is done with rates using the multivariate modulus of continuity. We approximate continuous and bounded functions on $\mathbb{R}^{\mathsf{N}}$, $\mathsf{N} \in \mathbb{N}$. When they are additionally uniformly continuous we derive pointwise and uniform convergences.

RESUMEN

En este artículo presentamos un resultado de aproximación básico multivariado a través de un operador de cuasi-interpolación en red neuronal de tipo Kantorovich-Shilkret con respecto a la norma del supremo. Esto se realiza con tasas usando el módulo de continuidad multivariado. Aproximamos funciones continuas y acotadas en $\mathbb{R}^{\mathsf{N}}$, $\mathsf{N} \in \mathbb{N}$. Cuando ellas son adicionalmente uniformemente continuas, derivamos convergencias puntuales y uniformes.

Keywords and Phrases: error function based activation function, multivariate quasi-interpolation neural network approximation, Kantorovich-Shilkret type operator.

2010 AMS Mathematics Subject Classification: 41A17, 41A25, 41A30, 41A35.

1 Introduction

The author here performs multivariate error function based neural network approximation to continuous functions over $\mathbb{R}^N$, $N \in \mathbb{N}$, and then he extends his results to complex valued functions. The convergences here are with rates expressed via the multivariate modulus of continuity of the involved function and give by very tight Jackson type inequalities.

The author comes up with the "right" precisely defined flexible quasi-interpolation Baskakov-Shilkret type integral coefficient neural network operator associated to the error function.

Feed-forward neural network (FNNs) with one hidden layer with deal with, are expressed mathematicaly as

$$N_n(x) = \sum_{j=0}^{n} c_j \sigma(\langle a_j \cdot x \rangle + b_j), \quad x \in \mathbb{R}^s, \ s \in \mathbb{N},$$

where for $0 \le j \le n$, $b_j \in \mathbb{R}$ are the thresholds, $a_j \in \mathbb{R}^s$ are the connection weights, $c_j \in \mathbb{R}$ are the coefficients, $\langle a_j \cdot x \rangle$ is the inner product of a_j and x, and σ is the activation function of the network. In many fundamental neural network models the activation function is error function generated.

About neural networks in general you may read [4], [5], [6]. In recent years non-additive integrals, like the N. Shilkret one [7], have become fashionable and more useful in Economic theory, etc.

2 Background

Here we follow [7].

Let $\mathcal{F}$ be a σ-field of subsets of an arbitrary set Ω. An extended non-negative real valued function μ on $\mathcal{F}$ is called maxitive if $\mu(\varnothing) = 0$ and

$$\mu(\cup_{i \in I} E_i) = \sup_{i \in I} \mu(E_i), \tag{1}$$

where the set I is of cardinality at most countable. We also call μ a maxitive measure. Here f stands for a non-negative measurable function on Ω. In [7], Niel Shilkret developed his non-additive integral defined as follows:

$$(N^*) \int_D f d\mu := \sup_{y \in Y} \{y \cdot \mu(D \cap \{f \ge y\})\}, \tag{2}$$

where $Y = [0, m]$ or $Y = [0, m)$ with $0 < m \le \infty$, and $D \in \mathcal{F}$. Here we take $Y = [0, \infty)$.

It is easily proved that

$$(N^*) \int_D f d\mu = \sup_{y > 0} \{y \cdot \mu(D \cap \{f > y\})\}. \tag{3}$$

The Shilkret integral takes values in $[0, \infty]$.

The Shilkret integral ([7]) has the following properties:

$$(N^*) \int_{\Omega} \chi_E \, d\mu = \mu(E), \tag{4}$$

where χ_E is the indicator function on $E \in \mathcal{F}$,

$$(N*) \int_D cf d\mu = c (N^*) \int_D f d\mu, \quad c \geq 0, \tag{5}$$

$$(N^*) \int_D \sup_{n \in \mathbb{N}} f_n \, d\mu = \sup_{n \in \mathbb{N}} (N^*) \int_D f_n \, d\mu, \tag{6}$$

where f_n, $n \in \mathbb{N}$, is an increasing sequence of elementary (countably valued) functions converging uniformly to f. Furthermore we have

$$(N^*) \int_D f d\mu \geq 0, \tag{7}$$

$$f \geq g \text{ implies } (N^*) \int_D f d\mu \geq (N^*) \int_D g d\mu, \tag{8}$$

where $f, g : \Omega \to [0, \infty]$ are measurable.

Let $a \leq f(\omega) \leq b$ for almost every $\omega \in E$, then

$$a\mu(E) \leq (N^*) \int_E f d\mu \leq b\mu(E);$$

$$(N^*) \int_E 1 d\mu = \mu(E);$$

$f > 0$ almost everywhere and $(N^*) \int_E f d\mu = 0$ imply $\mu(E) = 0$;

$(N^*) \int_{\Omega} f d\mu = 0$ if and only $f = 0$ almost everywhere;

$(N^*) \int_{\Omega} f d\mu < \infty$ implies that

$$\overline{N}(f) := \{\omega \in \Omega | f(\omega) \neq 0\} \text{ has } \sigma\text{-finite measure}; \tag{9}$$

$$(N^*) \int_D (f + g) \, d\mu \leq (N^*) \int_D f d\mu + (N^*) \int_D g d\mu;$$

and

$$\left| (N^*) \int_D f d\mu - (N^*) \int_D g d\mu \right| \leq (N^*) \int_D |f - g| \, d\mu. \tag{10}$$

From now on in this article we assume that $\mu : \mathcal{F} \to [0, +\infty)$.

3 Main Results

We consider here the (Gauss) error special function ([1], [3])

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt, \quad x \in \mathbb{R}, \tag{11}$$

which is a sigmoidal type function and a strictly increasing function.

It has the properties

$$\operatorname{erf}(0) = 0, \ \operatorname{erf}(-x) = \operatorname{erf}(x), \ \operatorname{erf}(+\infty) = 1, \ \operatorname{erf}(-\infty) = -1,$$

and

$$(\operatorname{erf}(x))' = \frac{2}{\sqrt{\pi}} e^{-x^2}, \ x \in \mathbb{R},$$

$$\int \operatorname{erf}(x)\, dx = x\operatorname{erf}(x) + \frac{e^{-x^2}}{\sqrt{\pi}} + C,$$

where C is a constant.

The error function is related to the cumulative probability distribution function of the standard normal distribution

$$\Phi(x) = \frac{1}{2} + \frac{1}{2}\operatorname{erf}\left(\frac{x}{\sqrt{2}}\right).$$

We consider the activation function

$$\chi(x) = \frac{1}{4}\left(\operatorname{erf}(x+1) - \operatorname{erf}(x-1)\right), \ x \in \mathbb{R}, \tag{12}$$

and we notice that

$$\chi(-x) = \chi(x), \tag{13}$$

and even function.

Clearly $\chi(x) > 0$, all $x \in \mathbb{R}$.

We see that

$$\chi(0) = \frac{\operatorname{erf}(1)}{2} \simeq 0.4215. \tag{14}$$

Let $x > 0$, we have that

$$\chi'(x) < 0, \quad \text{for } x > 0. \tag{15}$$

That is χ is strictly decreasing on $[0, \infty)$ and is strictly increasing on $(-\infty, 0]$, and $\chi'(0) = 0$.

Clearly the x-axis is the horizontal asymptote on χ.

Conclusion, χ is a bell symmetric function with maximum $\chi(0) \simeq 0.4215$.

We further need

Theorem 3.1. *([2]) We have that*

$$\sum_{i=-\infty}^{\infty} \chi(x-i) = 1, \text{ all } x \in \mathbb{R}. \tag{16}$$

Theorem 3.2. *([2]) It holds*

$$\int_{-\infty}^{\infty} \chi(x)\, dx = 1. \tag{17}$$

So $\chi(x)$ is a density function on $\mathbb{R}$.

Theorem 3.3. *([2]) Let $0 < \alpha < 1$, and $n \in \mathbb{N}$ with $n^{1-\alpha} \geq 3$.*

It holds

$$\sum_{\substack{k=-\infty \\ : |nx-k| \geq n^{1-\alpha}}}^{\infty} \chi(nx-k) < \frac{1}{2\sqrt{\pi}\left(n^{1-\alpha}-2\right) e^{\left(n^{1-\alpha}-2\right)^2}}. \tag{18}$$

Remark 3.4. *We introduce*

$$Z(x_1, ..., x_N) := Z(x) := \prod_{i=1}^{N} \chi(x_i), \tag{19}$$

$x = (x_1, ..., x_N) \in \mathbb{R}^N$, $N \in \mathbb{N}$.

It has the properties:

(i)

$$Z(x) > 0, \ \forall\, x \in \mathbb{R}^N, \tag{20}$$

(ii)

$$\sum_{k=-\infty}^{\infty} Z(x-k) := \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \cdots \sum_{k_N=-\infty}^{\infty} Z(x_1-k_1, ..., x_N-k_N) = 1, \tag{21}$$

where $k := (k_1, ..., k_N) \in \mathbb{Z}^N$, $\forall\, x \in \mathbb{R}^N$,

hence

(iii)

$$\sum_{k=-\infty}^{\infty} Z(nx-k) = 1, \forall\, x \in \mathbb{R}^N, \ n \in \mathbb{N}, \tag{22}$$

and

(iv)

$$\int_{\mathbb{R}^N} Z(x)\, dx = 1, \tag{23}$$

that is Z is a multivariate density function.

Here $\|x\|_{\infty} := \max\{|x_1|, ..., |x_N|\}$, $x \in \mathbb{R}^N$, *also set* $\infty := (\infty, ..., \infty)$, $-\infty = (-\infty, ..., -\infty)$ *upon the multivariate context.*

It is also clear that (see (18))

(v)

$$\sum_{\begin{cases} k=-\infty \\ \left\| \frac{k}{n} - x \right\|_{\infty} > \frac{1}{n^{\beta}} \end{cases}}^{\infty} Z(nx-k) \leq \frac{1}{2\sqrt{\pi}\left(n^{1-\beta}-2\right) e^{\left(n^{1-\beta}-2\right)^2}}, \tag{24}$$

$0 < \beta < 1$, $n \in \mathbb{N} : n^{1-\beta} \geq 3$, $x \in \mathbb{R}^N$.

For $f \in C_B^{+}\left(\mathbb{R}^N\right)$ (continuous and bounded functions from $\mathbb{R}^N$ into $\mathbb{R}_+$), we define the first modulus of continuity

$$\omega_1(f,h) := \sup_{\substack{x,y \in \mathbb{R}^N \\ \|x-y\|_{\infty} \leq h}} |f(x) - f(y)|, \quad h > 0. \tag{25}$$

Given that $f \in C_U^{+}\left(\mathbb{R}^N\right)$ (uniformly continuous from $\mathbb{R}^N$ into $\mathbb{R}_+$), we have that

$$\lim_{h \to 0} \omega_1(f,h) = 0. \tag{26}$$

We make

Definition 3.5. *Let* $\mathcal{L}$ *be the Lebesgue* σ*-algebra on* $\mathbb{R}^N$, $N \in \mathbb{N}$, *and the maxitive measure* $\mu : \mathcal{L} \to [0, +\infty)$, *such that for any* $A \in \mathcal{L}$ *with* $A \neq \varnothing$, *we get* $\mu(A) > 0$.

For $f \in C_B^{+}\left(\mathbb{R}^N\right)$, *we define the multivariate Kantorovich-Shilkret type neural network operator for any* $x \in \mathbb{R}^N$:

$$T_n^{\mu}(f,x) = T_n^{\mu}(f, x_1, ..., x_N) :=$$

$$\sum_{k=-\infty}^{\infty} \left(\frac{(N^*) \int_{\left[0,\frac{1}{N}\right]^N} f\left(t + \frac{k}{n}\right) d\mu(t)}{\mu\left(\left[0,\frac{1}{n}\right]^N\right)} \right) Z(nx-k) =$$

$$\sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} ... \sum_{k_N=-\infty}^{\infty} \left(\frac{(N^*) \int_0^{\frac{1}{n}} ... \int_0^{\frac{1}{n}} f\left(t_1 + \frac{k_1}{n}, t_2 + \frac{k_2}{n}, ..., t_N + \frac{k_N}{n}\right) d\mu(t_1, ..., t_N)}{\mu\left(\left[0,\frac{1}{n}\right]^N\right)} \right) \tag{27}$$

$$\cdot \left(\prod_{i=1}^{N} Z(nx_i - k_i) \right),$$

where $x = (x_1, ..., x_N) \in \mathbb{R}^N$, $k = (k_1, ..., k_N)$, $t = (t_1, ..., t_N)$, $n \in \mathbb{N}$.

Clearly here $\mu\left(\left[0,\frac{1}{n}\right]^N\right) > 0$, $\forall\, n \in \mathbb{N}$.

Above we notice that

$$\left\| T_n^{\mu}(f) \right\|_{\infty} \leq \|f\|_{\infty}, \tag{28}$$

so that $T_n^{\mu}(f,x)$ *is well-defined.*

Remark 3.6. *Let* $t \in \left[0, \frac{1}{n}\right]^N$ *and* $x \in \mathbb{R}^N$, *then*

$$f\left(t+\frac{k}{n}\right) = f\left(t+\frac{k}{n}\right) - f(x) + f(x) \leq \left|f\left(t+\frac{k}{n}\right) - f(x)\right| + f(x), \tag{29}$$

hence

$$(N^*)\int_{\left[0,\frac{1}{n}\right]^N} f\left(t+\frac{k}{n}\right) d\mu(t) \leq$$

$$(N^*)\int_{\left[0,\frac{1}{n}\right]^N} \left|f\left(t+\frac{k}{n}\right) - f(x)\right| d\mu(t) + f(x)\,\mu\left(\left[0,\frac{1}{n}\right]^N\right). \tag{30}$$

That is

$$(N^*)\int_{\left[0,\frac{1}{n}\right]^N} f\left(t+\frac{k}{n}\right) d\mu(t) - f(x)\,\mu\left(\left[0,\frac{1}{n}\right]^N\right) \leq \tag{31}$$

$$(N^*)\int_{\left[0,\frac{1}{n}\right]^N} \left|f\left(t+\frac{k}{n}\right) - f(x)\right| d\mu(t).$$

Similarly we have

$$f(x) = f(x) - f\left(t+\frac{k}{n}\right) + f\left(t+\frac{k}{n}\right) \leq \left|f\left(t+\frac{k}{n}\right) - f(x)\right| + f\left(t+\frac{k}{n}\right),$$

hence

$$(N^*)\int_{\left[0,\frac{1}{n}\right]^N} f(x)\,d\mu(t) \leq (N^*)\int_{\left[0,\frac{1}{n}\right]^N} \left|f\left(t+\frac{k}{n}\right) - f(x)\right| d\mu(t)$$

$$+ (N^*)\int_{\left[0,\frac{1}{n}\right]^N} f\left(t+\frac{k}{n}\right) d\mu(t).$$

That is

$$f(x)\,\mu\left(\left[0,\frac{1}{n}\right]^N\right) - (N^*)\int_{\left[0,\frac{1}{n}\right]^N} f\left(t+\frac{k}{n}\right) d\mu(t) \leq \tag{32}$$

$$(N^*)\int_{\left[0,\frac{1}{n}\right]^N} \left|f\left(t+\frac{k}{n}\right) - f(x)\right| d\mu(t).$$

By (31) and (32) we derive

$$\left|(N^*)\int_{\left[0,\frac{1}{n}\right]^N} f\left(t+\frac{k}{n}\right) d\mu(t) - f(x)\,\mu\left(\left[0,\frac{1}{n}\right]^N\right)\right| \leq$$

$$(N^*)\int_{\left[0,\frac{1}{n}\right]^N} \left|f\left(t+\frac{k}{n}\right) - f(x)\right| d\mu(t). \tag{33}$$

In particular it holds

$$\left|\frac{(N^*)\int_{\left[0,\frac{1}{n}\right]^N} f\left(t+\frac{k}{n}\right) d\mu(t)}{\mu\left(\left[0,\frac{1}{n}\right]^N\right)} - f(x)\right| \leq$$

$$\frac{(N^*)\int_{\left[0,\frac{1}{n}\right]^N} \left|f\left(t+\frac{k}{n}\right) - f(x)\right| d\mu(t)}{\mu\left(\left[0,\frac{1}{n}\right]^N\right)}. \tag{34}$$

We present

Theorem 3.7. *Let* $f \in C_B^+\left(\mathbb{R}^N\right)$, $0<\beta<1$, $x \in \mathbb{R}^N$; $N, n \in \mathbb{N}$ *with* $n^{1-\beta} \geq 3$. *Then*

i)

$$\sup_{\mu}\left|T_n^{\mu}(f,x)-f(x)\right| \leq \omega_1\left(f, \frac{1}{n}+\frac{1}{n^{\beta}}\right)+\frac{\|f\|_{\infty}}{\sqrt{\pi}\left(n^{1-\beta}-2\right) e^{\left(n^{1-\beta}-2\right)^2}}=: \lambda_n, \tag{35}$$

ii)

$$\sup_{\mu}\left\|T_n^{\mu}(f)-f\right\|_{\infty} \leq \lambda_n. \tag{36}$$

Given that $f \in \left(C_U^+\left(\mathbb{R}^N\right) \cap C_B^+\left(\mathbb{R}^N\right)\right)$, *we obtain* $\lim_{n \to \infty} T_n^{\mu}(f)=f$, *uniformly.*

Proof. We observe that

$$\left|T_n^{\mu}(f,x)-f(x)\right|=$$

$$\left|\sum_{k=-\infty}^{\infty}\left(\frac{\left(N^*\right)\int_{\left[0,\frac{1}{n}\right]^N} f\left(t+\frac{k}{n}\right) d\mu(t)}{\mu\left(\left[0,\frac{1}{n}\right]^N\right)}\right) Z(nx-k)-\sum_{k=-\infty}^{\infty} f(x) Z(nx-k)\right|=$$

$$\left|\sum_{k=-\infty}^{\infty}\left(\left(\frac{\left(N^*\right)\int_{\left[0,\frac{1}{n}\right]^N} f\left(t+\frac{k}{n}\right) d\mu(t)}{\mu\left(\left[0,\frac{1}{n}\right]^N\right)}\right)-f(x)\right) Z(nx-k)\right| \leq \tag{37}$$

$$\sum_{k=-\infty}^{\infty}\left|\left(\frac{\left(N^*\right)\int_{\left[0,\frac{1}{n}\right]^N} f\left(t+\frac{k}{n}\right) d\mu(t)}{\mu\left(\left[0,\frac{1}{n}\right]^N\right)}\right)-f(x)\right| Z(nx-k) \overset{(34)}{\leq}$$

$$\sum_{k=-\infty}^{\infty}\left(\frac{\left(N^*\right)\int_{\left[0,\frac{1}{n}\right]^N}\left|f\left(t+\frac{k}{n}\right)-f(x)\right| d\mu(t)}{\mu\left(\left[0,\frac{1}{n}\right]^N\right)}\right) Z(nx-k)= \tag{38}$$

$$\sum_{\substack{k=-\infty \\ :\left\|\frac{k}{n}-x\right\|_{\infty} \leq \frac{1}{n^{\beta}}}}^{\infty}\left(\frac{\left(N^*\right)\int_{\left[0,\frac{1}{n}\right]^N}\left|f\left(t+\frac{k}{n}\right)-f(x)\right| d\mu(t)}{\mu\left(\left[0,\frac{1}{n}\right]^N\right)}\right) Z(nx-k)+$$

$$\sum_{\substack{k=-\infty \\ :\left\|\frac{k}{n}-x\right\|_{\infty} > \frac{1}{n^{\beta}}}}^{\infty}\left(\frac{\left(N^*\right)\int_{\left[0,\frac{1}{n}\right]^N}\left|f\left(t+\frac{k}{n}\right)-f(x)\right| d\mu(t)}{\mu\left(\left[0,\frac{1}{n}\right]^N\right)}\right) Z(nx-k) \leq$$

$$\sum_{\substack{k=-\infty \\ :\left\|\frac{k}{n}-x\right\|_{\infty} \leq \frac{1}{n^{\beta}}}}^{\infty}\left(\frac{\left(N^*\right)\int_{\left[0,\frac{1}{n}\right]^N} \omega_1\left(f,\|t\|_{\infty}+\left\|\frac{k}{n}-x\right\|_{\infty}\right) d\mu(t)}{\mu\left(\left[0,\frac{1}{n}\right]^N\right)}\right) Z(nx-k) \tag{39}$$

$$+2\|f\|_\infty \left(\sum_{\substack{k=-\infty \\ \left\{ : \left\| \frac{k}{n} - x \right\|_\infty > \frac{1}{n^\beta} \right.}}^{\infty} Z(nx-k) \right) \overset{(24)}{\leq}$$

$$\omega_1\left(f, \frac{1}{n} + \frac{1}{n^\beta}\right) + \frac{\|f\|_\infty}{\sqrt{\pi}\left(n^{1-\beta}-2\right) e^{\left(n^{1-\beta}-2\right)^2}}, \tag{40}$$

proving the claim. □

Additionally we give

Definition 3.8. *Denote by* $C_B^+\left(\mathbb{R}^N, \mathbb{C}\right) = \{f : \mathbb{R}^N \to \mathbb{C} | f = f_1 + if_2,$ *where* $f_1, f_2 \in C_B^+\left(\mathbb{R}^N\right)$, $N \in \mathbb{N}\}$. *We set for* $f \in C_B^+\left(\mathbb{R}^N, \mathbb{C}\right)$ *that*

$$T_n^\mu(f, x) := T_n^\mu(f_1, x) + iT_n^\mu(f_2, x), \tag{41}$$

$\forall\, n \in \mathbb{N}$, $x \in \mathbb{R}^N$, $i = \sqrt{-1}$.

Theorem 3.9. *Let* $f \in C_B^+\left(\mathbb{R}^N, \mathbb{C}\right)$, $f = f_1 + if_2$, $N \in \mathbb{N}$, $0 < \beta < 1$, $x \in \mathbb{R}^N$; $n \in \mathbb{N}$ *with* $n^{1-\beta} \geq 3$. *Then*

i)

$$\sup_\mu |T_n^\mu(f, x) - f(x)| \leq \left[\omega_1\left(f_1, \frac{1}{n} + \frac{1}{n^\beta}\right) + \omega_1\left(f_2, \frac{1}{n} + \frac{1}{n^\beta}\right)\right]$$
$$+ \frac{\left(\|f_1\|_\infty + \|f_2\|_\infty\right)}{\sqrt{\pi}\left(n^{1-\beta}-2\right) e^{\left(n^{1-\beta}-2\right)^2}} =: \psi_n, \tag{42}$$

and

ii)

$$\sup_\mu \|T_n^\mu(f) - f\| \leq \psi_n. \tag{43}$$

Proof.

$$|T_n^\mu(f, x) - f(x)| = |T_n^\mu(f_1, x) + iT_n^\mu(f_2, x) - f_1(x) - if_2(x)| =$$
$$|(T_n^\mu(f_1, x) - f_1(x)) + i(T_n^\mu(f_2, x) - f_2(x))| \leq$$
$$|T_n^\mu(f_1, x) - f_1(x)| + |T_n^\mu(f_2, x) - f_2(x)| \overset{(35)}{\leq} \tag{44}$$
$$\left(\omega_1\left(f_1, \frac{1}{n} + \frac{1}{n^\beta}\right) + \frac{\|f_1\|_\infty}{\sqrt{\pi}\left(n^{1-\beta}-2\right) e^{\left(n^{1-\beta}-2\right)^2}}\right) +$$
$$\left(\omega_1\left(f_2, \frac{1}{n} + \frac{1}{n^\beta}\right) + \frac{\|f_2\|_\infty}{\sqrt{\pi}\left(n^{1-\beta}-2\right) e^{\left(n^{1-\beta}-2\right)^2}}\right) =$$

$$\left[\omega_1\left(f_1, \frac{1}{n} + \frac{1}{n^\beta}\right) + \omega_1\left(f_2, \frac{1}{n} + \frac{1}{n^\beta}\right)\right] +$$
$$\frac{\left(\|f_1\|_\infty + \|f_2\|_\infty\right)}{\sqrt{\pi}\left(n^{1-\beta} - 2\right) e^{\left(n^{1-\beta}-2\right)^2}}, \tag{45}$$

proving the claim. □

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Mean curvature flow of certain kind of isoparametric foliations on non-compact symmetric spaces

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ABSTRACT

In this paper, we investigate the mean curvature flows starting from all leaves of the isoparametric foliation given by a certain kind of solvable group action on a symmetric space of non-compact type. We prove that the mean curvature flow starting from each non-minimal leaf of the foliation exists in infinite time, if the foliation admits no minimal leaf, then the flow asymptotes the self-similar flow starting from another leaf, and if the foliation admits a minimal leaf (in this case, it is shown that there exists the only one minimal leaf), then the flow converges to the minimal leaf of the foliation in C^∞-topology. These results give the geometric information between the leaves.

RESUMEN

En este artículo, investigamos el flujo por curvatura media comenzando desde cualquier hoja de una foliación isoparamétrica dada por la acción de un cierto grupo soluble en un espacio simétrico de tipo no-compacto. Demostramos que el flujo por curvatura media comenzando desde cualquier hoja no mínima de la foliación existe para tiempo infinito, si la foliación no admite hojas mínimas, entonces el flujo es asintótico al flujo autosemejante comenzando desde otra hoja; en cambio si el flujo admite una hoja mínima (en este caso, se muestra que la hoja mínima es única), entonces el flujo converge a dicha hoja mínima de la foliación en la topología C^∞. Estos resultados entregan información geométrica entre las hojas.

Keywords and Phrases: error function based activation function, multivariate quasi-interpolation neural network approximation, Kantorovich-Shilkret type operator.

2010 AMS Mathematics Subject Classification: 41A17, 41A25, 41A30, 41A35.

1 Introduction

In [6], we proved that the mean curvature flow starting from any non-minimal compact isoparametric (equivalently, equifocal) submanifold in a symmetric space of compact type collapses to one of its focal submanifolds in finite time. Here we note that parallel submanifolds and focal ones of the isoparametric submanifold give an isoparametric foliation consisting of compact leaves on the symmetric space, where an *isoparametric foliation* means a singular Riemannian foliation satisfying the following conditions:

(i) The mean curvature form is basic,

(ii) The regular leaves are submanifolds with section.

A singular Riemannian foliation satisfying only the first condition is called a *generalized isoparametric foliation*. Recently, M. M. Alexandrino and M. Radeschi [1] investigated the mean curvature flow starting from a regular leaf of a generalized isoparametric foliation consisting of compact leaves on a compact Riemannian manifold. In particular, they [1] generalized our result to the mean curvature flow starting from a regular leaf of the foliation in the case where the foliation is isoparametric and the ambient space curves non-negatively. On the other hand, we [7] proved that the mean curvature flow starting from a certain kind of non-minimal (not necessarily compact) isoparametric submanifold in a symmetric space of non-compact type (which curves non-positively) collapses to one of its focal submanifolds in finite time. Here we note that the isoparametric foliation associated with this isoparametric submanifold consists of curvature-adapted leaves. See the next paragraph about the definition of the curvature-adaptedness.

In this paper, we study the mean curvature flow starting from leaves of the isoparametiric foliation given by the action of a certain kind of solvable subgroup (see Examples 1 and 2) of the (full) isometry group of a symmetric space of non-compact type. Here we note that this isoparametric foliation consists of (not necessarily curvature-adapted) non-compact regular leaves. We shall explain the solvable group action which we treat in this paper. Let G/K be a symmetric space of non-compact type, $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ ($\mathfrak{k} := \mathrm{Lie}\, K$) be the Cartan decomposition associated with the symmetric pair (G, K), $\mathfrak{a}$ be the maximal abelian subspace of $\mathfrak{p}$, $\widetilde{\mathfrak{a}}$ be the Cartan subalgebra of $\mathfrak{g}$ containing $\mathfrak{a}$ and $\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}$ be the Iwasawa's decomposition. Let A, $\widetilde{A}$ and N be the connected Lie subgroups of G having $\mathfrak{a}$, $\widetilde{\mathfrak{a}}$ and $\mathfrak{n}$ as their Lie algebras, respectively. Let $\pi : G \to G/K$ be the natural projection.

Given metric. In this paper, we give G/K the G-invariant metric induced from the restriction $B|_{\mathfrak{p}\times\mathfrak{p}}$ of the Killing form B of $\mathfrak{g}$ to $\mathfrak{p} \times \mathfrak{p}$.

The symmetric space G/K is identified with the solvable group AN with a left-invariant metric through $\pi|_{AN}$. Fix a lexicographic ordering of $\mathfrak{a}$. Let $\mathfrak{g} = \mathfrak{g}_0 + \sum\limits_{\lambda\in\triangle} \mathfrak{g}_\lambda$, $\mathfrak{p} = \mathfrak{a} + \sum\limits_{\lambda\in\triangle_+} \mathfrak{p}_\lambda$ and

$\mathfrak{k} = \mathfrak{k}_0 + \sum_{\lambda \in \triangle_+} \mathfrak{k}_\lambda$ be the root space decompositions of $\mathfrak{g}$, $\mathfrak{p}$ and $\mathfrak{k}$ with respect to $\mathfrak{a}$, where we note that

$$\mathfrak{g}_\lambda = \{X \in \mathfrak{g} \,|\, \mathrm{ad}(a)X = \lambda(a)X \text{ for all } a \in \mathfrak{a}\} \quad (\lambda \in \triangle),$$
$$\mathfrak{p}_\lambda = \{X \in \mathfrak{p} \,|\, \mathrm{ad}(a)^2X = \lambda(a)^2X \text{ for all } a \in \mathfrak{a}\} \quad (\lambda \in \triangle_+),$$
$$\mathfrak{k}_\lambda = \{X \in \mathfrak{k} \,|\, \mathrm{ad}(a)^2X = \lambda(a)^2X \text{ for all } a \in \mathfrak{a}\} \quad (\lambda \in \triangle_+ \cup \{0\}).$$

Note that $\mathfrak{n} = \sum_{\lambda \in \triangle_+} \mathfrak{g}_\lambda$. Let $G = KAN$ be the Iwasawa decomposition of G. Now we shall give examples of a solvable group contained in AN whose action on $G/K(= AN)$ is (complex) hyperpolar. Since G/K is of non-compact type, π gives a diffeomorphism of AN onto G/K. Denote by $\langle\ ,\ \rangle$ the left-invariant metric of AN induced from the metric of G/K by $\pi|_{AN}$. Also, denote by $\langle\ ,\ \rangle^G$ the bi-invariant metric of G induced from the Killing form B. Note that $\langle\ ,\ \rangle \neq \iota^*\langle\ ,\ \rangle^G$, where ι is the inclusion map of AN into G. Denote by Exp the exponential map of the Riemannian manifold $AN(= G/K)$ at e and by $\exp_G$ the exponential map of the Lie group G. Let l be a r-dimensional subspace of $\mathfrak{a} + \mathfrak{n}$ and set $\mathfrak{s} := (\mathfrak{a} + \mathfrak{n}) \ominus l$, where $(\mathfrak{a} + \mathfrak{n}) \ominus l$ denotes the orthogonal complement of l in $\mathfrak{a} + \mathfrak{n}$ with respect to $\langle\ ,\ \rangle_e$ (e : is the identity element of G). According to the result in [5], if $\mathfrak{s}$ is a subalgebra of $\mathfrak{a} + \mathfrak{n}$ and $l_\mathfrak{p} := \mathrm{pr}_\mathfrak{p}(l)$ ($\mathrm{pr}_\mathfrak{p}$: the orthogonal projection of $\mathfrak{g}$ onto $\mathfrak{p}$) is abelian, then the S-action ($S := \exp_G(\mathfrak{s})$) gives an isoparametric foliation without singular leaf. We [5] gave examples of such a subalgebra $\mathfrak{s}$ of $\mathfrak{a} + \mathfrak{n}$.

Example 1. Let $\mathfrak{b}$ be a $r(\geq 1)$-dimensional subspace of $\mathfrak{a}$ and $\mathfrak{s}_\mathfrak{b} := (\mathfrak{a} + \mathfrak{n}) \ominus \mathfrak{b}$. It is clear that $\mathfrak{b}_\mathfrak{p}(= \mathfrak{b})$ is abelian and that $\mathfrak{s}_\mathfrak{b}$ is a subalgebra of $\mathfrak{a} + \mathfrak{n}$.

Example 2. Let $\{\lambda_1, \cdots, \lambda_k\}$ be a subset of a simple root system Π of $\triangle$ such that $H_{\lambda_1}, \cdots, H_{\lambda_k}$ are mutually orthogonal, $\mathfrak{b}$ be a subspace of $\mathfrak{a} \ominus \mathrm{Span}\{H_{\lambda_1}, \cdots, H_{\lambda_k}\}$ (where $\mathfrak{b}$ may be $\{0\}$) and l_i ($i = 1, \cdots, k$) be a one-dimensional subspace of $\mathbb{R}H_{\lambda_i} + \mathfrak{g}_{\lambda_i}$ with $l_i \neq \mathbb{R}H_{\lambda_i}$, where H_{λ_i} is the element of $\mathfrak{a}$ defined by $\langle H_{\lambda_i}, \cdot\rangle = \lambda_i(\cdot)$ and $\mathbb{R}H_{\lambda_i}$ is the subspace of $\mathfrak{a}$ spanned by H_{λ_i}. Set $l := \mathfrak{b} + \sum_{i=1}^{k} l_i$. Then, it is shown that $l_\mathfrak{p}$ is abelian and that $\mathfrak{s}_{\mathfrak{b}, l_1, \cdots, l_k} := (\mathfrak{a} + \mathfrak{n}) \ominus l$ is a subalgebra of $\mathfrak{a} + \mathfrak{n}$.

In Example 2, a unit vector of l_i is described as $\frac{1}{\cosh(\|\lambda_i\| t_i)} \xi^i - \frac{1}{\|\lambda_i\|} \tanh(\|\lambda_i\| t_i) H_{\lambda_i}$ for a unit vector ξ^i of $\mathfrak{g}_{\lambda_i}$ and some $t_i \in \mathbb{R}$, where $\|\lambda_i\| := \|H_{\lambda_i}\|$. Then we denote l_i by l_{ξ^i, t_i} if necessary and set $\xi^i_{t_i} := \frac{1}{\cosh(\|\lambda_i\| t_i)} \xi^i - \frac{1}{\|\lambda_i\|} \tanh(\|\lambda_i\| t_i) H_{\lambda_i}$. Set $S_\mathfrak{b} := \exp_G(\mathfrak{s}_\mathfrak{b})$ and $S_{\mathfrak{b},, l_1, \cdots, l_k} := \exp_G(\mathfrak{s}_{\mathfrak{b}, l_1, \cdots, l_k})$. Denote by $\mathfrak{F}_\mathfrak{b}$ and $\mathfrak{F}_{\mathfrak{b}, l_1, \cdots, l_k}$ the isoparametric foliations given by the $S_\mathfrak{b}$-action and the $S_{\mathfrak{b}, l_1, \cdots, l_k}$-one, respectively. A submanifold in a Riemannian manifold is said to be *curvature-adapted* if, for each normal vector v of the submanifold, the normal Jacobi operator $R(v) := R(\cdot, v)v$ preserves the tangent space of the submanifold invariantly and the restriction of $R(v)$ to the tangent space commutes with the shape operator A_v, where R is the curvature tensor of the ambient Riemannian manifold. According to the results in[5], the following facts hold for isoparametric foliations $\mathfrak{F}_{\mathfrak{b}}$ and $\mathfrak{F}_{\mathfrak{b},l_1,\cdots,l_k}$:

(i) All leaves of $\mathfrak{F}_{\mathfrak{b}}$ are curvature-adapted.

(ii) Let $\lambda_1,\cdots,\lambda_k\,(\in\triangle_+)$ be as in Example 2. If the root system $\triangle$ of G/K is non-reduced and $2\lambda_{i_0}\in\triangle_+$ for some $i_0\in\{1,\cdots,k\}$, then all leaves of $\mathfrak{F}_{\mathfrak{b},l_1,\cdots,l_k}$ are not curvature-adapted.

(iii) If $\mathfrak{b}\neq\{0\}$, then $\mathfrak{F}_{\mathfrak{b},l_1,\cdots,l_k}$ admits no minimal leaf. On the other hand, if $\mathfrak{b}=\{0\}$, then this action admits the only minimal leaf.

(iv) Let $l_1,\cdots,l_k$ be as in Example 2 and $\bar{l}_i$ $(i=1,\cdots,k)$ be the orthogonal projection of l_i onto $\mathfrak{g}_{\lambda_i}$. Then $\mathfrak{F}_{\mathfrak{b},\bar{l}_1,\cdots,\bar{l}_k}$ is congruent to $\mathfrak{F}_{\mathfrak{b},l_1,\cdots,l_k}$. In more detail, we have

$$L_{b\cdot\gamma_{\xi^1}(t_1)\cdot\cdots\cdot\gamma_{\xi^k}(t_k)}(S_{\mathfrak{b},l_1,\cdots,l_k}\cdot e)=S_{\mathfrak{b},\bar{l}_1,\cdots,\bar{l}_k}\cdot(b\cdot\gamma_{\xi^1}(t_1)\cdot\cdots\cdot\gamma_{\xi^k}(t_k)),$$

where γ_{ξ^i} $(i=1,\cdots,k)$ is the geodesic in $AN(=G/K)$ with $\gamma'_{\xi^i}(0)=\xi^i$, b is an element of $\exp(\mathfrak{b})$ and $L_{b\cdot\gamma_{\xi^1}(t_1)\cdot\cdots\cdot\gamma_{\xi^k}(t_k)}$ is the left translation by $b\cdot\gamma_{\xi^1}(t_1)\cdot\cdots\cdot\gamma_{\xi^k}(t_k)$. For example, in case of $k=1$ and $b=e$, the positional relation among the leaves of these foliations is as in Figure 1.

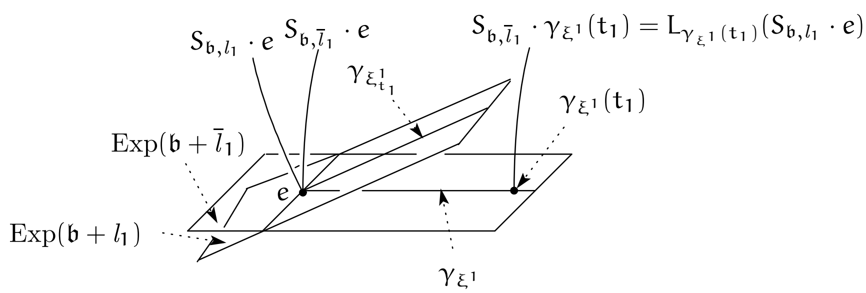


Figure 1.

According to the above facts (i) and (ii), the leaves of $\mathfrak{F}_{\mathfrak{b},l_1,\cdots,l_k}$ give examples of interesting isoparametric submanifolds in G/K.

In this paper, we shall prove the following facts for the mean curvature flows starting from the non-minimal leaves of $\mathfrak{F}_{\mathfrak{b},\bar{l}_1,\cdots,\bar{l}_k}$.

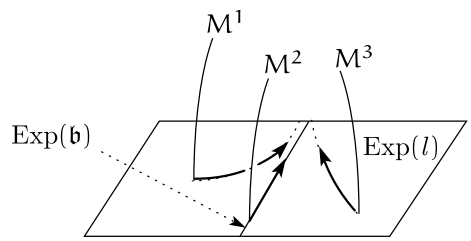
Theorem A. *Assume that* $\mathfrak{b}\neq\{0\}$. *Let* M *be any leaf of* $\mathfrak{F}_{\mathfrak{b},\bar{l}_1,\cdots,\bar{l}_k}$. *and* M_t $(0\le t<T)$ *be the mean curvature flow starting from* M. *Then the following statements* (i) $-$ (iii) *hold.*

(i) $T=\infty$ *holds.*

(ii) *If* M *passes through* $\exp(\mathfrak{b})$, *then the mean curvature flow* M_t *is self-similar.*

(iii) *If* M *does not pass through* $\exp(\mathfrak{b})$, *then the mean curvature flow* M_t *asymptotes the mean curvature flow starting from the leaf of* $\mathfrak{F}_{\mathfrak{b},\bar{l}_1,\cdots,\bar{l}_k}$ *passing through a point of* $\exp(\mathfrak{b})$.

Remark 1.1. The mean curvature flow starting from any leaf of $\mathfrak{F}_{\mathfrak{b}}$ is self-similar.



The mean curvature flows starting from leaves M^1 and M^3 of $\mathfrak{F}_{\mathfrak{b},\bar{l}_1,\cdots,\bar{l}_k}$ ($\mathfrak{b} \neq \{0\}$) asymptotes the mean curvature flow (which is self-similar) starting from a leaf M^2 of $\mathfrak{F}_{\mathfrak{b},\bar{l}_1,\cdots,\bar{l}_k}$.

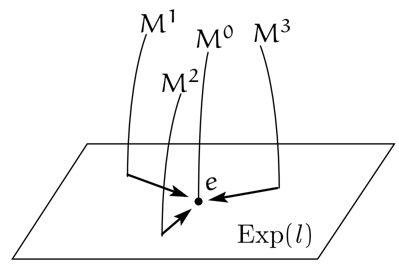
Figure 2.

Also, in case of $\mathfrak{b} = \{0\}$, we obtain the following fact.

Theorem B. *Let M be a leaf of $\mathfrak{F}_{\{0\},\bar{l}_1,\cdots,\bar{l}_k}$-action other than $S_{\{0\},\bar{l}_1,\cdots,\bar{l}_k} \cdot e$ and M_t $(0 \le t < T)$ be the mean curvature flow starting from M. Then the following statements* (i) $-$ (ii) *hold.*

(i) $T = \infty$ *holds.*

(ii) M_t *convergres to the only minimal leaf $S_{\{0\},\bar{l}_1,\cdots,\bar{l}_k} \cdot e$ (in C^∞-topology) as $t \to \infty$.*



The mean curvature flows starting from leaves M^1, M^2 and M^3 of $\mathfrak{F}_{\{0\},\bar{l}_1,\cdots,\bar{l}_k}$ converge to the only minimal leaf M^0 of $\mathfrak{F}_{\{0\},\bar{l}_1,\cdots,\bar{l}_k}$.

Figure 3.

The following question arises naturally.

Question. Let $\mathfrak{F}$ be an isoparametric foliation consisting of non-compact regular leaves on a non-positively curved Riemannian manifold. Assume that the leaves of $\mathfrak{F}$ are cohomogeneity compact (i.e., each leaf L is invariant under some subgroup action H_L of the isometry group of the ambient space and the quotient space L/H_L is compact). In what case, does the result similar to Theorem A or B hold for $\mathfrak{F}$?

2 Mean curvature flow.

In this section, we shall recall the notion of the mean curvature flow. Let f_t's ($t \in [0,T)$) be a one-parameter C^∞-family of immersions of a manifold M into a Riemannian manifold $\widetilde{M}$, where T is a positive constant or $T=\infty$. Define a map $F: M\times[0,T) \to \widetilde{M}$ by $F(x,t) = f_t(x)$ ($(x,t) \in M\times[0,T)$). Denote by π the natural projection of $M\times[0,T)$ onto M. For a vector bundle E over M, denote by π^*E the induced bundle of E by π. Also, denote by H_t and g_t the mean curvature vector field and the induced metric of f_t, respectively. Define a section g of $\pi^*(T^{(0,2)}M)$ by $g_{(x,t)} := (g_t)_x$ ($(x,t) \in M\times[0,T)$) and sections H of $F^*T\widetilde{M}$ by $H_{(x,t)} := (H_t)_x$ ($(x,t) \in M\times[0,T)$), where $T^{(0,2)}M$ is the tensor bundle of degree $(0,2)$ of M and $T\widetilde{M}$ is the tangent bundle of $\widetilde{M}$. The family f_t's ($0 \le t < T$) is called a *mean curvature flow* if it satisfies

$$F_*\left(\frac{\partial}{\partial t}\right) = H. \tag{1.1}$$

In particular, if f_t's are embeddings, then we call $M_t := f_t(M)$'s ($0 \in [0,T)$) rather than f_t's ($0 \le t < T$) a mean curvature flow. See [3], [4] and [2] and so on about the study of the mean curvature flow (treated as the evolution of an immersion).

3 The non-curvature-adaptedness of the leaves.

In [5], we proved the following statement:

($*$) *If the root system $\triangle$ of G/K is non-reduced and $2\lambda_{i_0} \in \triangle_+$ for some $i_0 \in \{1,\cdots,k\}$, then all leaves of $\mathfrak{F}_{\mathfrak{b},\bar{l}_1,\cdots,\bar{l}_k}$ are not curvature-adapted.*

(see the statement (ii) of Proposition 3.5 in [5]). However, there is a gap in the second-half part of the proof. In this section, we shall close the gap by recalculating the normal Jacobi operators of the leaves (see Proposition 3.5). We shall use the notations in Introduction. According to the fact (iv) stated in Introduction, we have

$$L_{\mathfrak{b}\cdot\gamma_{\xi^1}(t_1)\cdot\cdots\cdot\gamma_{\xi^k}(t_k)}(S_{\mathfrak{b},l_1,\cdots,l_k}\cdot e) = S_{\mathfrak{b},\bar{l}_1,\cdots,\bar{l}_k}\cdot(\mathfrak{b}\cdot\gamma_{\xi^1}(t_1)\cdot\cdots\cdot\gamma_{\xi^k}(t_k)).$$

Hence we suffice to show that the leaves $S_{\mathfrak{b},l_1,\cdots,l_k}\cdot e$'s are not curvature-adapted. As stated in Example 2, we set $\xi^i_{t_i} := \frac{1}{\cosh(||\lambda_i||t_i)}\xi^i - \frac{1}{||\lambda_i||}\tanh(||\lambda_i||t_i)H_{\lambda_i}$. For the shape operator of $S_{\mathfrak{b},l_1,\cdots,l_k}\cdot e$, we showed the following facts (see Lemma 3.2 of [5]).

Lemma 3.1[5]. *Let A be the shape tensor of $S_{\mathfrak{b},l_1,\cdots,l_k}\cdot e\,(\subset AN)$. Then, for A_{ξ_0} ($\xi_0 \in \mathfrak{b}$) and $A_{\xi^i_{t_i}}$ ($i=1,\cdots,k$), the following statements* (i) ~ (vii) *hold:*

(i) *For $X \in \mathfrak{a}\ominus(\mathfrak{b}+\sum_{i=1}^{k}\mathbb{R}H_{\lambda_i})$, we have $A_{\xi_0}X = A_{\xi^i_{t_i}}X = 0$ ($i=1,\cdots,k$).*

(ii) *For* $X \in \mathrm{Ker}(\mathrm{ad}(\xi^i)|_{\mathfrak{g}_{\lambda_i}}) \ominus \mathbb{R}\xi^i$, *we have* $A_{\xi_0}X = 0$ *and* $A_{\xi^i_{t_i}}X = -\|\lambda_i\|\tanh(\|\lambda_i\|t_i)X$.

(iii) *Assume that* $2\lambda_i \in \triangle_+$. *For* $X \in \mathfrak{g}_{2\lambda_i}$, *we have* $A_{\xi_0}([\theta\xi^i, X]) = 0$ *and*

$$A_{\xi^i_{t_i}}X = -2\|\lambda_i\|\tanh(\|\lambda_i\|t_i)X - \frac{1}{2\cosh(\|\lambda_i\|t_i)}[\theta\xi^i, X],$$
$$A_{\xi^i_{t_i}}([\theta\xi^i, X]) = -\frac{\|\lambda_i\|^2}{\cosh(\|\lambda_i\|t_i)}X - \|\lambda_i\|\tanh(\|\lambda_i\|t_i)[\theta\xi^i, X],$$

where θ *is the Cartan involution of* $\mathfrak{g}$ *with* $\mathrm{Fix}\,\theta = \mathfrak{k}$.

(iv) *For* $X \in (\mathbb{R}\xi^i + \mathbb{R}H_{\lambda_i}) \ominus l_i$, *we have* $A_{\xi_0}X = 0$ *and* $A_{\xi^i_{t_i}}X = -\|\lambda_i\|\tanh(\|\lambda_i\|t_i)X$.

(v) *For* $X \in (\mathfrak{g}_{\lambda_j} \ominus \mathbb{R}\xi^j) + ((\mathbb{R}\xi^j + \mathbb{R}H_{\lambda_j}) \ominus l_j) + \mathfrak{g}_{2\lambda_j}$ $(j \neq i)$, *we have* $A_{\xi_0}X = A_{\xi^i_{t_i}}X = 0$.

(vi) *For* $X \in \mathfrak{g}_\mu$ $(\mu \in \triangle_+ \setminus \{\lambda_1, \cdots, \lambda_k\})$, *we have* $A_{\xi_0}X = \mu(\xi_0)X$.

(vii) *Let* $k_i := \exp\left(\frac{\pi}{\sqrt{2}\|\lambda_i\|}(\xi^i + \theta\xi^i)\right)$, *where* $\exp$ *is the exponential map of* G. *Then* $\mathrm{Ad}(k_i) \circ A_{\xi^i_{t_i}} = -A_{\xi^i_{t_i}} \circ \mathrm{Ad}(k_i)$ *holds over* $\mathfrak{n} \ominus \sum_{i=1}^{k}(\mathfrak{g}_{\lambda_i} + \mathfrak{g}_{2\lambda_i})$, *where* Ad *is the adjoint representation of* G.

Remark 3.1. If $\lambda_i \in \triangle_+$, then we have $\|\lambda_i\| = \sqrt{2}$ from how to choose the metric of G/K (see Introduction).

According to (5.3) in Page 310 of [8], we have the following fact.

Lemma 3.2[8]. *Let* X *and* Y *be left-invariant vector fields on* AN *and* ∇ *be the Levi-Civita connection of the left-invariant metric* $\langle\ ,\ \rangle$ *of* AN. *Then we have*

$$\nabla_X Y = \frac{1}{2}\left(\ [X, Y] - \mathrm{ad}(X)^*(Y) - \mathrm{ad}(Y)^*(X)\ \right), \tag{3.2}$$

where $\mathrm{ad}(X)^*$ *(resp.* $\mathrm{ad}(Y)^*$*) is the adjoint operator of* $\mathrm{ad}(X)$ *(resp.* $\mathrm{ad}(Y)$*) with respect to* $\langle\ ,\ \rangle_e$ *and* $(\bullet)_{\mathfrak{a}+\mathfrak{n}}$ *is the the* $(\mathfrak{a}+\mathfrak{n})$*-component of* $(\bullet)$.

Let $\mathrm{pr}^1_{\mathfrak{a}+\mathfrak{n}}$ (resp. $\mathrm{pr}^2_{\mathfrak{a}+\mathfrak{n}}$) be the projection of $\mathfrak{g}$ onto $\mathfrak{a}+\mathfrak{n}$ with respect to the decomposition $\mathfrak{g} = \mathfrak{k} + (\mathfrak{a}+\mathfrak{n})$ (resp. $\mathfrak{g} = (\mathfrak{k}_0 + \sum_{\lambda\in\triangle_+}\mathfrak{p}_\lambda) + (\mathfrak{a}+\mathfrak{n})$). We [5] showed the following facts (see the proof of Lemma 3.2 in [5]).

Lemma 3.3[5]. (i) *For any* $H \in \mathfrak{a}$, *we have*

$$\mathrm{ad}(H)^* = \mathrm{ad}(H). \tag{3.3}$$

(ii) *For any* $X \in \mathfrak{g}_\lambda$*, we have*

$$\begin{aligned}\mathrm{ad}(X)^* &= -\mathrm{pr}_{\mathfrak{a}+\mathfrak{n}} \circ \mathrm{ad}(\theta X)\\ &= \begin{cases} 0 & \text{on } \mathfrak{a} \\ -\langle X, \cdot\rangle_e \otimes H_\lambda - \mathrm{pr}_{\mathfrak{n}} \circ \mathrm{pr}^1_{\mathfrak{a}+\mathfrak{n}} \circ \mathrm{ad}(X_{\mathfrak{k}}) & \\ +\mathrm{pr}_{\mathfrak{n}} \circ \mathrm{pr}^2_{\mathfrak{a}+\mathfrak{n}} \circ \mathrm{ad}(X_{\mathfrak{p}}) & \text{on } \mathfrak{n}, \end{cases}\end{aligned} \tag{3.4}$$

where $(\bullet)_{\mathfrak{k}}$ *(resp.* $(\cdot)_{\mathfrak{p}}$*) denotes the* $\mathfrak{k}$*-component (resp.* $\mathfrak{p}$*-component) of* $(\bullet)$*.*

According to (3.4), we have

$$\mathrm{ad}(X)^*(Y) = \begin{cases} 0 & (\lambda - \mu \in \triangle_+) \\ -\langle X, Y\rangle H_\lambda & (\lambda = \mu) \\ -[\theta X, Y] & (\mu - \lambda \in \triangle_+) \\ 0 & (\lambda - \mu \notin \triangle \cup \{0\}) \end{cases} \tag{3.5}$$

for any $X \in \mathfrak{g}_\lambda$ ($\lambda \in \triangle_+$) and any $Y \in \mathfrak{g}_\mu$ ($\mu \in \triangle_+$). For each $X \in \mathfrak{a} + \mathfrak{n}$, we denote by $\widetilde{X}$ the left-invariant vector field on AN with $(\widetilde{X})_e = X$. By using Lemma 3.2, (3.3), (3.4) and (3.5), we can derive the facts directly.

Lemma 3.4. *For any unit vector* X_λ, Y_λ *of* $\mathfrak{g}_\lambda$ *(*$\lambda \in \triangle_+$*) and* H_λ *(*$\lambda \in \triangle_+$*), we have*

$$\nabla_{\widetilde{H}_\lambda} \widetilde{H}_\mu = \nabla_{\widetilde{H}_\lambda} \widetilde{X}_\mu = 0, \quad \nabla_{\widetilde{X}_\lambda} \widetilde{H}_\mu = -\lambda(H_\mu)\widetilde{X}_\lambda \quad (\lambda,\ \mu \in \triangle_+)$$

and

$$\nabla_{\widetilde{X}_\lambda} \widetilde{Y}_\mu = \begin{cases} \frac{1}{2}\left([\widetilde{X}_\lambda, \widetilde{Y}_\mu] + \widetilde{\theta[Y_\mu, \theta X_\lambda]}\right) & (\lambda - \mu \in \triangle_+) \\ \frac{1}{2}[\widetilde{X}_\lambda, \widetilde{Y}_\mu] + \langle \widetilde{X}_\lambda, \widetilde{Y}_\mu\rangle \widetilde{H}_\lambda & (\lambda = \mu) \\ \frac{1}{2}\left([\widetilde{X}_\lambda, \widetilde{Y}_\mu] + \widetilde{\theta[X_\lambda, \theta Y_\mu]}\right) & (\mu - \lambda \in \triangle_+) \\ \frac{1}{2}[\widetilde{X}_\lambda, \widetilde{Y}_\mu] & (\lambda - \mu \notin \triangle \cup \{0\}) \end{cases}$$

From Lemma 3.4 and (3.5), we can derive the following facts for the normal Jacobi operators by somewhat long calculations.

Proposition 3.5. *Let* R *be the curvature tensor of* $AN(= G/K)$*. Then, for* $R(\xi_0)$ *(*$\xi^0 \in \mathfrak{b}$*) and* $R(\xi^i_{t_i})$ *(*$i = 1, \cdots, k$*), the following statements* (i) ~ (vi) *hold:*

(i) *For* $X \in \mathfrak{a} \ominus (\mathfrak{b} + \sum_{i=1}^{k} \mathbb{R}H_{\lambda_i})$*, we have* $R(\xi_0)(X) = R(\xi^i_{t_i})(X) = 0$ *(*$i = 1, \cdots, k$*).*

(ii) *For* $X \in \mathrm{Ker}(\mathrm{ad}(\xi^i)|_{\mathfrak{g}_{\lambda_i}}) \ominus \mathbb{R}\xi^i$, *we have* $R(\xi_0)(X) = 0$ *and* $R(\xi^i_{t_i})(X) = \frac{\|\lambda_i\|^2}{2}(1 - 3\tanh^2(\|\lambda_i\| t_i))X$.

(iii) *Assume that* $2\lambda_i \in \triangle_+$ *(hence* $\|\lambda_i\| = \sqrt{2}$*). For* $X \in \mathfrak{g}_{2\lambda_i}$, *we have* $R(\xi_0)(X) = R(\xi_0)([\theta\xi^i, X]) = 0$ *and*

$$\begin{aligned} R(\xi^i_{t_i})(X) &= -\|\lambda_i\|^2(1 + 3\tanh^2(\|\lambda_i\| t_i))X - \frac{3\|\lambda_i\|\tanh(\|\lambda_i\| t_i)}{2\cosh(\|\lambda_i\| t_i)}[\theta\xi^i, X] \\ R(\xi^i_{t_i})([\theta\xi^i, X]) &= -\frac{6\|\lambda_i\|\tanh(\|\lambda_i\| t_i)}{\cosh(\|\lambda_i\| t_i)}X + \frac{\sqrt{2}\|\lambda_i\|}{4}(1 - 3\tanh^2(\|\lambda_i\| t_i))[\theta\xi^i, X]. \end{aligned}$$

(iv) *For* $X \in (\mathbb{R}\xi^i + \mathbb{R}H_{\lambda_i}) \ominus \mathfrak{l}_i$, *we have* $R(\xi_0)(X) = 0$ *and* $R(\xi^i_{t_i})(X) = -\|\lambda_i\|^2 X$.

(v) *For* $X \in (\mathfrak{g}_{\lambda_j} \ominus \mathbb{R}\xi^j) + ((\mathbb{R}\xi^j + \mathbb{R}H_{\lambda_j}) \ominus \mathfrak{l}_j) + \mathfrak{g}_{2\lambda_j}$ *(*$j \neq i$*), we have* $R(\xi_0)(X) = R(\xi^i_{t_i})(X) = 0$.

(vi) *For* $X \in \mathfrak{g}_\mu$ *(*$\mu \in \triangle_+ \setminus \{\lambda_1, \cdots, \lambda_k\}$*), we have* $R(\xi_0)(X) = -\mu(\xi_0)^2 X$.

From Lemma 3.1 and Proposition 3.5, we can derive the following facts directly.

Proposition 3.6. *For* $[A_{\xi_0}, R(\xi_0)]$ *(*$\xi_0 \in \mathfrak{b}$*) and* $[A_{\xi^i_{t_i}}, R(\xi^i_{t_i})]$ *(*$i = 1, \cdots, k$*), the following statements* (i) ~ (vi) *hold:*

(i) *For* $X \in \mathfrak{a} \ominus (\mathfrak{b} + \sum_{i=1}^{k} \mathbb{R}H_{\lambda_i})$, *we have* $[A, R(\xi_0)](X) = [A_{\xi^i_{t_i}}, R(\xi^i_{t_i})](X) = 0$ *(*$i = 1, \cdots, k$*).*

(ii) *For* $X \in \mathrm{Ker}(\mathrm{ad}(\xi^i)|_{\mathfrak{g}_{\lambda_i}}) \ominus \mathbb{R}\xi^i$, *we have* $[A_{\xi_0}, R(\xi_0)](X) = [A_{\xi^i_{t_i}}, R(\xi^i_{t_i})](X) = 0$.

(iii) *Assume that* $2\lambda_i \in \triangle_+$ *(hence* $\|\lambda_i\| = \sqrt{2}$*). For* $X \in \mathfrak{g}_{2\lambda_i}$, *we have* $[A_{\xi_0}, R(\xi_0)](X) = [A_{\xi_0}, R(\xi_0)]([\theta\xi^i, X]) = 0$ *and*

$$\begin{aligned} [A_{\xi^i_{t_i}}, R(\xi^i_{t_i})](X) &= -\frac{3}{2\cosh^3(\sqrt{2}t_i)}[\theta\xi^i, X] \\ [A_{\xi^i_{t_i}}, R(\xi^i_{t_i})]([\theta\xi^i, X]) &= -\frac{6}{\cosh^3(\sqrt{2}t_i)}X. \end{aligned}$$

(iv) *For* $X \in (\mathbb{R}\xi^i + \mathbb{R}H_{\lambda_i}) \ominus \mathfrak{l}_i$, *we have* $[A_{\xi_0}, R(\xi_0)](X) = [A_{\xi^i_{t_i}}, R(\xi^i_{t_i})](X) = 0$.

(v) *For* $X \in (\mathfrak{g}_{\lambda_j} \ominus \mathbb{R}\xi^j) + ((\mathbb{R}\xi^j + \mathbb{R}H_{\lambda_j}) \ominus \mathfrak{l}_j) + \mathfrak{g}_{2\lambda_j}$ *(*$j \neq i$*), we have* $[A_{\xi_0}, R(\xi_0)](X) = [A_{\xi^i_{t_i}}, R(\xi^i_{t_i})](X) = 0$.

(vi) *For* $X \in \mathfrak{g}_\mu$ *(*$\mu \in \triangle_+ \setminus \{\lambda_1, \cdots, \lambda_k\}$*), we have* $[A_{\xi_0}, R(\xi_0)](X) = [A_{\xi^i_{t_i}}, R(\xi^i_{t_i})](X) = 0$.

From (iv) of Proposition 3.6, we can derive the statement $(*)$.

Also, we [5] showed the following fact in terms of Lemma 3.1.

Proposition 3.7[5]. *If* $\mathfrak{b} = \{0\}$*, then* $\mathfrak{F}_{\mathfrak{b},l_{\xi^1,t_1},\cdots,l_{\xi^k,t_k}}$ *admits the only minimal leaf.*

4 Proof of Theorem A

In this section, we shall prove Theorem A. We use the notations in Sections 1 and 3. Note that $\mathrm{Exp}|_{\mathfrak{a}} = \exp|_{\mathfrak{a}}$ and $\mathrm{Exp}|_{\mathfrak{n}} \neq \exp|_{\mathfrak{n}}$. Set $\Sigma := \mathrm{Exp}(T_e^{\perp} S_{\mathfrak{b},\bar{l}_1,\cdots,\bar{l}_k} \cdot e)(= \mathrm{Exp}(\mathfrak{b} + \mathbb{R}\{\xi^1,\cdots,\xi^k\}))$, which is the flat section of the $S_{\mathfrak{b},\bar{l}_1,\cdots,\bar{l}_k}$-action through e. Each leaf of $\mathfrak{F}_{\mathfrak{b},\bar{l}_1,\cdots,\bar{l}_k}$ meets Σ at the only one point. That is, Σ is regarded as the leaf space of this foliation. For $\xi_0 \in \mathfrak{b}$ and $t_i \in \mathbb{R}$ $(i = 1,\cdots,k)$, we set $x_{\xi_0,t_1,\cdots,t_k} := \mathrm{Exp}\xi_0 \cdot \gamma_{\xi^1(t_1)} \cdot \cdots \cdot \gamma_{\xi^k(t_k)}$. Also, denote by $\frac{D}{ds}(\bullet)$ the covariant derivative of vector fields $(\bullet)$ along curves in AN (with respect to the left-invariant metric). The following fact is well-known about the geodesics in rank one symmetric spaces of non-compact type but we shall give the proof.

Lemma 4.1. *The velocity vector* $\gamma'_{\xi^i}(s)$ $(i = 1,\cdots,k)$ *is described as*

$$\gamma'_{\xi^i}(s) = \frac{1}{\cosh(||\lambda_i||s)}(\widetilde{\xi^i})_{\gamma_{\xi^i}(s)} - \frac{\tanh(||\lambda_i||s)}{||\lambda_i||}(\widetilde{H_{\lambda_i}})_{\gamma_{\xi^i}(s)} \tag{4.1}$$

and $\gamma'_{\xi_0}(s)$ *is described as*

$$\gamma'_{\xi_0}(s) = (\widetilde{\xi_0})_{\gamma_{\xi_0}(s)} \tag{4.2}$$

Proof. Set $Y(s) := \frac{1}{\cosh(||\lambda_i||s)}(\widetilde{\xi^i})_{\gamma_{\xi^i}(s)} - \frac{\tanh(||\lambda_i||s)}{||\lambda_i||}(\widetilde{H_{\lambda_i}})_{\gamma_{\xi^i}(s)}$. It is clear that $Y(0) = \xi^i$. By using Lemma 3.4, we can show $\frac{D}{ds}Y = 0$. Hence we obtain $Y(s) = \gamma'_{\xi^i}(s)$. Also, it is clear that $(\widetilde{\xi_0})_{\gamma_{\xi_0}(0)} = \xi_0$. By using Lemma 3.4, we can show $\frac{D}{ds}(\widetilde{\xi_0})_{\gamma_{\xi_0}(s)} = 0$. Hence we obtain $(\widetilde{\xi_0})_{\gamma_{\xi_0}(s)} = \gamma'_{\xi_0}(s)$. q.e.d.

Next we shall show the following fact.

Lemma 4.2. *The point* $x_{\xi_0,t_1,\cdots,t_k}$ *belongs to* Σ.

Proof. It is clear that $\mathrm{Exp}(\xi_0)$ belongs to Σ. First we shall show that $\mathrm{Exp}(\xi_0) \cdot \gamma_{\xi^1(t_1)}$ belongs to Σ. Let γ_{ξ_0} be the geodesic in AN with $\gamma'_{\xi_0}(0) = \xi_0$. Since γ_{ξ^1} is a geodesic in AN and $L_{\mathrm{Exp}(\xi_0)}$ is an isometry of AN, $L_{\mathrm{Exp}(\xi_0)} \circ \gamma_{\xi^1}$ is a geodesic in AN. Hence we suffice to show that $(L_{\mathrm{Exp}(\xi_0)} \circ \gamma_{\xi^1})'(0) = (\widetilde{\xi^1})_{\mathrm{Exp}(\xi_0)}$ is tangent to Σ. Denote by $\widehat{\xi^1}$ the parallel vector field along γ_{ξ_0}. Take orthonormal bases $\{e_1^{\lambda},\cdots,e_{m_\lambda}^{\lambda}\}$ of $\mathfrak{g}_\lambda$ $(\lambda \in \triangle_+)$. Also, take an orthonormal base $\{e_1^0,\cdots,e_r^0\}$

of $\mathfrak{a}$. We describe $\widehat{\xi}^1$ as

$$\widehat{\xi}^1(s)=\sum_{i=1}^r a_i^0(s)(\widetilde{e_i^0})_{\gamma_{\xi_0}(s)}+\sum_{\lambda\in\triangle_+}\sum_{i=1}^{m_\lambda}a_i^\lambda(s)(\widetilde{e_i^\lambda})_{\gamma_{\xi_0}(s)}\quad(s\in\mathbb{R}),$$

where a_i^0 and a_i^λ are functions over $\mathbb{R}$. Fix $s_0\in\mathbb{R}$. By using Lemma 3.4, we can show

$$\begin{aligned}
\left.\frac{D}{ds}\right|_{s=s_0}\widehat{\xi}^1 &= \sum_{i=1}^r\left((a_i^0)'(s_0)(\widetilde{e_i^0})_{\gamma_{\xi_0}(s_0)}+(a_i^0)(s_0)\left.\frac{D}{ds}\right|_{s=s_0}((\widetilde{e_i^0})_{\gamma_{\xi_0}(s)})\right)\\
&\quad+\sum_{\lambda\in\triangle_+}\sum_{i=1}^{m_\lambda}\left((a_i^\lambda)'(s_0)(\widetilde{e_i^\lambda})_{\gamma_{\xi_0}(s_0)}+a_i^\lambda(s_0)\left.\frac{D}{ds}\right|_{s=s_0}((\widetilde{e_i^\lambda})_{\gamma_{\xi_0}(s)})\right)\\
&=\sum_{i=1}^r\left((a_i^0)'(s_0)(\widetilde{e_i^0})_{\gamma_{\xi_0}(s_0)}+(a_i^0)(s_0)\nabla_{\gamma'_{\xi_0}(s_0)}((\widetilde{e_i^0})_{\gamma_{\xi_0}(s_0)})\right)\\
&\quad+\sum_{\lambda\in\triangle_+}\sum_{i=1}^{m_\lambda}\left((a_i^\lambda)'(s_0)(\widetilde{e_i^\lambda})_{\gamma_{\xi_0}(s_0)}+a_i^\lambda(s_0)\nabla_{\gamma'_{\xi_0}(s_0)}((\widetilde{e_i^\lambda})_{\gamma_{\xi_0}(s_0)})\right)\\
&=\sum_{i=1}^r\left((a_i^0)'(s_0)(\widetilde{e_i^0})_{\gamma_{\xi_0}(s_0)}+(a_i^0)(s_0)(\nabla_{\widetilde{\xi_0}}\widetilde{e_i^0})_{\gamma_{\xi_0}(s_0)}\right)\\
&\quad+\sum_{\lambda\in\triangle_+}\sum_{i=1}^{m_\lambda}\left((a_i^\lambda)'(s_0)(\widetilde{e_i^\lambda})_{\gamma_{\xi_0}(s_0)}+a_i^\lambda(s_0)(\nabla_{\widetilde{\xi_0}}\widetilde{e_i^\lambda})_{\gamma_{\xi_0}(s_0)})\right)\\
&=\sum_{i=1}^r(a_i^0)'(s_0)(\widetilde{e_i^0})_{\gamma_{\xi_0}(s_0)}+\sum_{\lambda\in\triangle_+}\sum_{i=1}^{m_\lambda}(a_i^\lambda)'(s_0)(\widetilde{e_i^\lambda})_{\gamma_{\xi_0}(s_0)}=0,
\end{aligned}$$

that is, $(a_i^0)'(s_0)=(a_i^\lambda)'(s_0)=0$, where we use $\gamma'_{\xi_0}(s_0)=\widetilde{\xi_0}_{\gamma_{\xi_0}(s_0)}$. From the arbitrariness of s_0, we see that a_i^0 and a_i^λ are constant. Hence we obtain $\widehat{\xi}^1(s)=(\widetilde{\xi}^1)_{\gamma_{\xi_0}(s)}$. On the other hand, since ξ^1 is tangent to Σ and Σ is totally geodesic, $\widehat{\xi}^1(1)$ also is tangent to Σ. Hence we see that $(\widetilde{\xi}^1)_{\mathrm{Exp}(\xi_0)}$ is tangent to Σ. Therefore $\mathrm{Exp}(\xi_0)\cdot\gamma_{\xi^1(t_1)}$ belongs to Σ.

Next we shall show that $\mathrm{Exp}(\xi_0)\cdot\gamma_{\xi^1(t_1)}\cdot\gamma_{\xi^2(t_2)}$ belongs to Σ. Since γ_{ξ^2} is a geodesic in AN and $L_{\mathrm{Exp}(\xi_0)\cdot\gamma_{\xi^1}(t_1)}$ is an isometry of AN, $L_{\mathrm{Exp}(\xi_0)\cdot\gamma_{\xi^1}(t_1)}\circ\gamma_{\xi^2}$ is a geodesic in AN. Hence we suffice to show that $(L_{\mathrm{Exp}(\xi_0)\cdot\gamma_{\xi^1}(t_1)}\circ\gamma_{\xi^2})'(0)=(\widetilde{\xi}^2)_{\mathrm{Exp}(\xi_0)\cdot\gamma_{\xi^1}(t_1)}$ is tangent to Σ. Denote by $\widehat{\xi}^2$ the parallel vector field along $\overline{\gamma}_{\xi_1}:=L_{\mathrm{Exp}(\xi_0)}\circ\gamma_{\xi^1}$ with $\widehat{\xi}^2(0)=(\widetilde{\xi}^2)_{\mathrm{Exp}(\xi_0)}$. We describe $\widehat{\xi}^2$ as

$$\widehat{\xi}^2(s)=\sum_{i=1}^r b_i^0(s)(\widetilde{e_i^0})_{\overline{\gamma}_{\xi^1}(s)}+\sum_{\lambda\in\triangle_+}\sum_{i=1}^{m_\lambda}b_i^\lambda(s)(\widetilde{e_i^\lambda})_{\overline{\gamma}_{\xi^1}(s)}\quad(s\in\mathbb{R}),$$

where b_i^0 and b_i^λ are functions over $\mathbb{R}$. Fix $s_0 \in \mathbb{R}$. By using Lemma 3.4, we can show

$$
\begin{aligned}
\frac{D}{ds}\Big|_{s=s_0}\widehat{\xi}^2 &= \sum_{i=1}^{r}\left((b_i^0)'(s_0)(\widetilde{e_i^0})_{\overline{\gamma}_{\xi^1}(s_0)} + (b_i^0)(s_0)\,\frac{D}{ds}\Big|_{s=s_0}((\widetilde{e_i^0})_{\overline{\gamma}_{\xi^1}(s)})\right)\\
&\quad + \sum_{\lambda\in\triangle_+}\sum_{i=1}^{m_\lambda}\left((b_i^\lambda)'(s_0)(\widetilde{e_i^\lambda})_{\overline{\gamma}_{\xi^1}(s_0)} + b_i^\lambda(s_0)\,\frac{D}{ds}\Big|_{s=s_0}((\widetilde{e_i^\lambda})_{\overline{\gamma}_{\xi^1}(s)})\right)\\
&= \sum_{i=1}^{r}\left((b_i^0)'(s_0)(\widetilde{e_i^0})_{\overline{\gamma}_{\xi^1}(s_0)} + (b_i^0)(s_0)\nabla_{\overline{\gamma}'_{\xi^1}(s_0)}((\widetilde{e_i^0})_{\overline{\gamma}_{\xi^1}(s)})\right)\\
&\quad + \sum_{\lambda\in\triangle_+}\sum_{i=1}^{m_\lambda}\left((b_i^\lambda)'(s_0)(\widetilde{e_i^\lambda})_{\overline{\gamma}_{\xi^1}(s_0)} + b_i^\lambda(s_0)\nabla_{\overline{\gamma}'_{\xi^1}(s_0)}((\widetilde{e_i^\lambda})_{\overline{\gamma}_{\xi^1}(s)})\right) = 0.
\end{aligned}
\tag{4.3}
$$

Since $\gamma'_{\xi^1}(s_0) = \frac{1}{\cosh(||\lambda_1||s_0)}(\widetilde{\xi^1})_{\gamma_{\xi^1}(s_0)} - \frac{\tanh(||\lambda_1||s_0)}{||\lambda_1||}(\widetilde{H_{\lambda_1}})_{\gamma_{\xi^1}(s_0)}$ by Lemma 4.1, $\overline{\gamma}'_{\xi^1}(s_0)$ is described as

$$
\begin{aligned}
\overline{\gamma}'_{\xi^1}(s_0) &= (L_{\mathrm{Exp}(\xi_0)})_*(\gamma'_{\xi^1}(s_0))\\
&= \frac{1}{\cosh(||\lambda_1||s_0)}(\widetilde{\xi^1})_{\overline{\gamma}_{\xi^1}(s_0)} - \frac{\tanh(||\lambda_1||s_0)}{||\lambda_1||}(\widetilde{H}_{\lambda_1})_{\overline{\gamma}_{\xi^1}(s_0)}.
\end{aligned}
$$

Hence, by using Lemma 3.4, we have

$$
\begin{aligned}
\nabla_{\overline{\gamma}'_{\xi^1}(s_0)}((\widetilde{e_i^0})_{\overline{\gamma}_{\xi^1}} &= \frac{1}{\cosh(||\lambda_1||s_0)}(\nabla_{\widetilde{\xi^1}}\widetilde{e_i^0})_{\overline{\gamma}_{\xi^1}(s_0)} - \frac{\tanh(||\lambda_1||s_0)}{||\lambda_1||}(\nabla_{\widetilde{H_{\lambda_1}}}\widetilde{e_i^0})_{\overline{\gamma}_{\xi^1}(s_0)}\\
&= -\frac{\lambda_1(e_i^0)}{\cosh(||\lambda_1||s_0)}(\widetilde{\xi^1})_{\overline{\gamma}_{\xi^1}(s_0)}
\end{aligned}
\tag{4.4}
$$

and

$$
\begin{aligned}
\nabla_{\overline{\gamma}'_{\xi^1}(s_0)}((\widetilde{e_i^\lambda})_{\overline{\gamma}_{\xi^1}} &= \frac{1}{\cosh(||\lambda_1||s_0)}(\nabla_{\widetilde{\xi^1}}\widetilde{e_i^\lambda})_{\overline{\gamma}_{\xi^1}(s_0)} - \frac{\tanh(||\lambda_1||s_0)}{||\lambda_1||}(\nabla_{\widetilde{H_{\lambda_1}}}\widetilde{e_i^\lambda})_{\overline{\gamma}_{\xi^1}(s_0)}\\
&= \begin{cases}
\frac{1}{2\cosh(||\lambda_1||s_0)}\left([\widetilde{\xi^1},\widetilde{e_i^\lambda}] + \theta\widetilde{[e_i^\lambda,\theta\xi^1]}\right) & (\lambda_1-\lambda\in\triangle_+)\\
\frac{1}{2\cosh(||\lambda_1||s_0)}\left([\widetilde{\xi^1},\widetilde{e_i^\lambda}] + 2\langle\widetilde{\xi^1},\widetilde{e_i^\lambda}\rangle\widetilde{H}_{\lambda_1}\right) & (\lambda_1=\lambda)\\
\frac{1}{2\cosh(||\lambda_1||s_0)}\left([\widetilde{\xi^1},\widetilde{e_i^\lambda}] + \theta\widetilde{[\xi^1,\theta e_i^\lambda]}\right) & (\lambda-\lambda_1\in\triangle_+)\\
\frac{1}{2\cosh(||\lambda_1||s_0)}[\widetilde{\xi^1},\widetilde{e_i^\lambda}] & (\lambda_1-\lambda\notin\triangle\cup\{0\}).
\end{cases}
\end{aligned}
\tag{4.5}
$$

By substituting (4.4) and (4.5) into (4.3), we obtain

$$
\begin{aligned}
\frac{D}{ds}\Big|_{s=s_0}\widehat{\xi}^2 = & \sum_{i=1}^{r}\left((b_i^0)'(s_0)(\widetilde{e_i^0})_{\overline{\gamma}_{\xi^1}(s_0)} - \frac{\lambda_1(e_i^0)(b_i^0)(s_0)}{\cosh(\|\lambda_1\|s_0)}(\widetilde{\xi^1})_{\overline{\gamma}_{\xi^1}(s_0)}\right)\\
& + \sum_{\lambda\in\triangle_+}\sum_{i=1}^{m_\lambda}(b_i^\lambda)'(s_0)(\widetilde{e_i^\lambda})_{\overline{\gamma}_{\xi^1}(s_0)}\\
& + \sum_{\lambda_1-\lambda\in\triangle_+}\sum_{i=1}^{m_\lambda}\frac{b_i^\lambda(s_0)}{2\cosh(\|\lambda_1\|s_0)}\left([\widetilde{\xi^1},\widetilde{e_i^\lambda}]+\theta\widetilde{[e_i^\lambda,\theta\xi^1]}\right)\\
& + \sum_{\lambda-\lambda_1\in\triangle_+}\sum_{i=1}^{m_\lambda}\frac{b_i^\lambda(s_0)}{2\cosh(\|\lambda_1\|s_0)}\left([\widetilde{\xi^1},\widetilde{e_i^\lambda}]+\theta\widetilde{[\xi^1,\theta e_i^\lambda]}\right)\\
& + \sum_{\lambda-\lambda_1\notin\triangle\cup\{0\}}\sum_{i=1}^{m_\lambda}\frac{b_i^\lambda(s_0)}{2\cosh(\|\lambda_1\|s_0)}[\widetilde{\xi^1},\widetilde{e_i^\lambda}]\\
& + \sum_{i=1}^{m_{\lambda_1}}\frac{b_i^{\lambda_1}(s_0)}{2\cosh(\|\lambda_1\|s_0)}\left([\widetilde{\xi}^1,\widetilde{e_i^{\lambda_1}}]+2\langle\widetilde{\xi^1},\widetilde{e_i^{\lambda_1}}\rangle\widetilde{H}_{\lambda_1}\right)=0.
\end{aligned}
\tag{4.6}
$$

Without loss of generality, we may assume that $e_1^{\lambda_2}=\xi^2$. Hence we have $b_1^{\lambda_2}(0)=1$ and $b_i^\lambda(0)=0$ for any (λ,i) other than $(\lambda_2,1)$. From (4.6) and these relations, we obtain $b_1^{\lambda_2}\equiv 1$ and $b_i^\lambda\equiv 0$ for any (λ,i) other than $(\lambda_2,1)$, where we note that $\lambda_1-\lambda_2\notin\triangle\cup\{0\}$. Therefore we obtain $\widehat{\xi^2}=(\widetilde{\xi^2})_{\overline{\gamma}_{\xi^1}(s)}$. On the other hand, since $(\widehat{\xi^2})(0)$ is tangent to Σ and Σ is totally geodesic, $\widehat{\xi}^2(t_1)$ also is tangent to Σ. Hence we see that $(\widetilde{\xi^2})_{\mathrm{Exp}(\xi_0)\cdot\gamma_{\xi^1}(t_1)}$ is tangent to Σ. Therefore $\mathrm{Exp}(\xi_0)\cdot\gamma_{\xi^1(t_1)}\cdot\gamma_{\xi^2(t_2)}$ belongs to Σ. In the sequel, by repeating the same discussion, we can derive that $x_{\xi^0,t_1,\cdots,t_k}=\mathrm{Exp}(\xi_0)\cdot\gamma_{\xi^1(t_1)}\cdot\cdots\cdot\gamma_{\xi^k(t_k)}$ belongs to Σ.

It is clear that any point of Σ is described as $x_{\xi_0,t_1,\cdots,t_k}$ for some $\xi_0\in\mathfrak{b}$ and some $t_1,\cdots,t_k\in\mathbb{R}$. Fix an orthonormal base $\{e_1^0,\cdots,e_{m_0}^0\}$ of $\mathfrak{b}$, where $m_0:=\dim\mathfrak{b}$. Define vector fields E_i^0 $(i=1,\cdots,m_0)$ and E^j $(j=1,\cdots,k)$ along Σ by

$$
\begin{aligned}
&(E_i^0)_{x_{\xi_0,t_1,\cdots,t_k}}:=(L_{x_{\xi_0,t_1,\cdots,t_k}})_*(e_i^0)(=(\widetilde{e_i^0})_{x_{\xi_0,t_1,\cdots,t_k}})\\
\text{and}\qquad &(E^j)_{x_{\xi_0,t_1,\cdots,t_k}}:=(L_{x_{\xi_0,t_1,\cdots,t_k}})_*(\xi_{t_j}^j)(=(\widetilde{\xi_{t_j}^j})_{x_{\xi_0,t_1,\cdots,t_k}}).
\end{aligned}
$$

By imitating the discussions in the proofs of Lemmas 4.1 and 4.2, we can show the following fact for these vector fields.

Lemma 4.3. *The vector fields E_i^0 $(i=1,\cdots,m_0)$ and E^j $(j=1,\cdots,k)$ are tangent to Σ and they give a parallel orthonormal tangent frame field on Σ.*

Proof. Let $(\widehat{\xi^i})^j$ (resp. $(\widehat{\xi^i})^0$) be the parallel vector field along γ_{ξ^j} $(i\neq j)$ (resp. γ_{ξ_0}) with $(\widehat{\xi^i})_0^j=\xi^i$ (resp. $(\widehat{\xi^i})_0^0=\xi^i$) and $(\widehat{\xi_0})^j$ be the parallel vector field along γ_{ξ^j} with $(\widehat{\xi_0})_0^j=\xi_0$. According to Lemma 4.1, we have $(\gamma_{\xi^i})'(t)=(L_{\gamma_{\xi^i}(t)})_*(\xi_t^i)$ and $(\gamma_{\xi_0})'(t)=(L_{\gamma_{\xi_0}(t)})_*(\xi_0)$. Also, we can show $(\widehat{\xi^i})^j_{\gamma_{\xi^j}(t)} = (L_{\gamma_{\xi^j}(t)})_*(\xi^i)$ $(j \neq i)$, $(\widehat{\xi^i})^0_{\gamma_{\xi_0}(t)} = (L_{\gamma_{\xi_0}(t)})_*(\xi^i)$ and $(\widehat{\xi_0})^j_{\gamma_{\xi^j}(t)} = (L_{\gamma_{\xi^j}(t)})_*(\xi_0)$ by imitating the discussion in the proof of Lemma 4.2. On the basis of these facts, we can derive the statement of this lemma, where we note that Σ is flat.

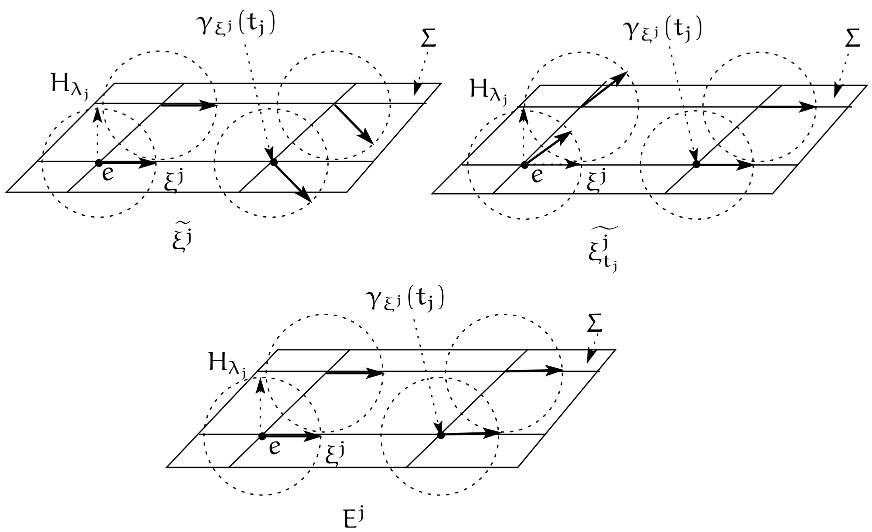


Figure 4.

By using these lemmas, we prove Theorem A.

Proof of Theorem A. In this proof, we use the notations as in Example 2. Set $M_{x_{\xi_0,t_1,\cdots,t_k}} := S_{\mathfrak{b},\bar{l}_1,\cdots,\bar{l}_k} \cdot x_{\xi_0,t_1,\cdots,t_k}$. Denote by $H^{x_{\xi_0,t_1,\cdots,t_k}}$ the mean curvature vector field of $M_{x_{\xi_0,t_1,\cdots,t_k}}$. Let $\{e^0_1,\cdots,e^0_{m_0}\}$ be an orthonormal base of $\mathfrak{b}$ and $(H_\lambda)_{\mathfrak{b}} = \sum_{i=1}^{m_0} H^i_\lambda e^0_i$ be the $\mathfrak{b}$-component of H_λ. According to the fact (iv) stated in Introduction, we have

$$M_{x_{\xi_0,t_1,\cdots,t_k}} = L_{x_{\xi_0,t_1,\cdots,t_k}}(S_{\mathfrak{b},l_{\xi^1,t_1},\cdots,l_{\xi^k,t_k}} \cdot e).$$

Denote by $\widehat{H}^{\xi_0,t_1,\cdots,t_k}$ the mean curvature vector field of $S_{\mathfrak{b},l_{\xi^1,t_1},\cdots,l_{\xi^k,t_k}} \cdot e$. According to Lemma 3.1, we have

$$(\widehat{H}^{\xi_0,t_1,\cdots,t_k})_e = \sum_{\lambda\in\triangle_+} m_\lambda (H_\lambda)_{\mathfrak{b}} - \sum_{i=1}^{k} \|\lambda_i\| \tanh(\|\lambda_i\| t_i)(m_{\lambda_i} + 2m_{2\lambda_i})\xi^i_{t_i}$$

and hence

$$\begin{aligned}(H^{x_{\xi_0,t_1,\cdots,t_k}})_{x_{\xi_0,t_1,\cdots,t_k}} &= \sum_{\lambda\in\triangle_+}\sum_{i=1}^{m_0} m_\lambda H^i_\lambda (E^0_i)_{x_{\xi_0,t_1,\cdots,t_k}} \\ &\quad - \sum_{i=1}^{k} \|\lambda_i\| \tanh(\|\lambda_i\| t_i)(m_{\lambda_i} + 2m_{2\lambda_i})(E^i)_{x_{\xi_0,t_1,\cdots,t_k}}.\end{aligned} \tag{4.7}$$

Define a tangent vector field Z over Σ by $Z_x := (H^x)_x$ $(x \in \Sigma)$. According to (4.7), we have

$$\begin{aligned} Z_{x_{\xi_0,t_1,\cdots,t_k}} = & \sum_{\lambda\in\triangle_+}\sum_{i=1}^{m_0} m_\lambda H^i_\lambda (E^0_i)_{x_{\xi_0,t_1,\cdots,t_k}} \\ & -\sum_{i=1}^{k} \|\lambda_i\| \tanh(\|\lambda_i\| t_i)(m_{\lambda_i}+2m_{2\lambda_i})(E^i)_{x_{\xi_0,t_1,\cdots,t_k}}. \end{aligned} \tag{4.8}$$

Define a coordinate $\phi = (u_1, \cdots, u_{m_0+k}) : \Sigma \to \mathbb{R}^{m_0+k}$ of Σ by

$$\phi(x_{\sum_{i=1}^{m_0} s_i e^0_i, t_1,\cdots,t_k}) := (s_1, \cdots, s_{m_0}, t_1, \cdots, t_k)$$

$(s_1, \cdots, s_{m_0}, t_1, \cdots, t_k \in \mathbb{R})$. We can show $\frac{\partial}{\partial u_i} = E^0_i$ $(i = 1, \cdots, m_0)$ and $\frac{\partial}{\partial u_{m_0+j}} = E^j$ $(j = 1, \cdots, k)$. Hence ϕ is a Euclidean coordinate of Σ. Under the identification of Σ and $\mathbb{R}^{m_0+k}$ by ϕ, we regard Z as a tangent vector field on $\mathbb{R}^{m_0+k}$. Then Z is described as

$$\begin{aligned} Z_{(u_1,\cdots,u_{m_0+k})} = (& \sum_{\lambda\in\triangle_+} m_\lambda H^1_\lambda, \cdots, \sum_{\lambda\in\triangle_+} m_\lambda H^{m_0}_\lambda, \\ & -\|\lambda_1\| \tanh(\|\lambda_1\| u_{m_0+1})(m_{\lambda_1}+2m_{2\lambda_1}), \\ & \cdots, -\|\lambda_k\| \tanh(\|\lambda_k\| u_{m_0+k})(m_{\lambda_k}+2m_{2\lambda_k})). \end{aligned} \tag{4.9}$$

Fix $(a_1, \cdots, a_{m_0}, t_1, \cdots, t_k) \in \mathbb{R}^{m_0+k}$. Let c be the integral curve of Z starting from $(a_1, \cdots, a_{m_0}, t_1, \cdots, t_k)$ and let $c = (c_1, \cdots, c_{m_0+k})$. We suffice to investigate c to investigate the mean curvature flow starting from $M_{x_{\sum_{i=1}^{m_0} a_i e^0_i, t_1,\cdots,t_k}}$. From $c'(t) = Z_{c(t)}$, we have $c'_i(t) = \sum_{\lambda\in\triangle_+} m_\lambda H^i_\lambda$ $(i = 1, \cdots, m_0)$ and $c'_{m_0+j}(t) = -(m_{\lambda_j}+2m_{2\lambda_j})\|\lambda_j\| \tanh(\|\lambda_j\| c_{m_0+j}(t))$ $(j = 1, \cdots, k)$. By solving $c'_i(t) = \sum_{\lambda\in\triangle_+} m_\lambda H^i_\lambda$ under the initial condition $c_i(0) = a_i$, we have

$$c_i(t) = a_i + t\sum_{\lambda\in\triangle_+} m_\lambda H^i_\lambda. \tag{4.10}$$

Also, by solving $c'_{m_0+j}(t) = -(m_{\lambda_j}+2m_{2\lambda_j})\|\lambda_j\| \tanh(\|\lambda_j\| c_{m_0+j}(t))$ under the initial condition $c_{m_0+j}(0) = t_j$, we have

$$c_{m_0+j}(t) = \frac{1}{\|\lambda_j\|}\mathrm{arcsinh}\left(e^{-\|\lambda_j\|^2(m_{\lambda_j}+2m_{2\lambda_j})t}\sinh(\|\lambda_j\| t_j)\right). \tag{4.11}$$

From (4.10) and (4.11), we can derive $T = \infty$, $\lim\limits_{t\to\infty}\sum_{i=1}^{m_0} c_i(t)^2 = \infty$ $(i = 1, \cdots, m_0)$ and $\lim\limits_{t\to\infty} c_{m_0+j}(t) = 0$ $(j = 1, \cdots, k)$. If $t_1 = \cdots = t_k = 0$, then we have $c_{m_0+j} \equiv 0$ $(j = 1, \cdots, m_0)$. Hence the mean curvature flow starting from $M_{x_{\xi_0,0,\cdots,0}}$ $(x_{\xi_0,0,\cdots,0} \in \mathrm{Exp}(\mathfrak{b}))$ consists of the leaves of $\mathfrak{F}_{\mathfrak{b},\bar{l}_1,\cdots,\bar{l}_k}$ through points of $\mathrm{Exp}(\mathfrak{b})$. Also, according to the fact (iv) stated in Introduction, the leaves of $\mathfrak{F}_{\mathfrak{b},\bar{l}_1,\cdots,\bar{l}_k}$ through points of $\mathrm{Exp}(\mathfrak{b})$ are congruent to $S_{\mathfrak{b},\bar{l}_1,\cdots,\bar{l}_k}\cdot e$. Therefore, the mean curvature flow starting from $M_{x_{\xi_0,0,\cdots,0}}$ is self-similar. From $\lim\limits_{t\to\infty}\sum_{i=1}^{m_0} c_i(t)^2 = \infty$ $(i = 1, \cdots, m_0)$ and $\lim\limits_{t\to\infty} c_{m_0+j}(t) = 0$ $(j = 1, \cdots, k)$, we see that the mean curvature flow starting from any leaf of $\mathfrak{F}_{\mathfrak{b},\bar{l}_1,\cdots,\bar{l}_k}$ asymptotes the mean curvature flow starting from the leaf of $\mathfrak{F}_{\mathfrak{b},\bar{l}_1,\cdots,\bar{l}_k}$ passing through a point of $\mathrm{Exp}(\mathfrak{b})$. q.e.d.

According to this proof, we obtain the following fact.

Corollary 4.1. (i) *The mean curvature flow starting from* $M_{x_{\xi_0},0,\cdots,0}$ *is self-similar.*

(ii) *The mean curvature flow starting from* $M_{x_{\xi_0},t_1,\cdots,t_k}$ *($(t_1,\cdots,t_k)\neq(0,\cdots,0)$) asymptotes the flow starting from* $M_{x_{\xi_0},0,\cdots,0}$. *In more detail, the distance between* $M_{x_{\xi_0},t_1,\cdots,t_k}$ *and* $M_{x_{\xi_0},0,\cdots,0}$ *is equal to*

$$\sqrt{\sum_{j=1}^{k}\frac{1}{\|\lambda_j\|^2}\mathrm{arcsinh}^2\left(e^{-\|\lambda_j\|^2(m_{\lambda_j}+2m_{2\lambda_j})t}\sinh(\|\lambda_j\|t_j)\right)},$$

which converges to zero as $t\to\infty$.

Next we prove Theorem B.

Proof of Theorem B. In case of $\mathfrak{b}=\{0\}$, the relation (4.9) is as follows:

$$\begin{aligned} Z_{(u_1,\cdots,u_k)} = (&-\|\lambda_1\|\tanh(\|\lambda_1\|u_{m_0+1})(m_{\lambda_1}+2m_{2\lambda_1}),\\ &\cdots,-\|\lambda_k\|\tanh(\|\lambda_k\|u_{m_0+k})(m_{\lambda_k}+2m_{2\lambda_k})). \end{aligned} \tag{4.12}$$

Hence, according to the dicussion in the proof of Theorem A, the mean curvature flow starting from any leaf of $\mathfrak{F}_{\mathfrak{b},\bar{l}_1,\cdots,\bar{l}_k}$ converges to the only minimal leaf $S_{\mathfrak{b},\bar{l},\cdots,\bar{l}_k}\cdot e$. Furthermore, the flow converges to the minimal leaf in C^∞-topology because the flow consists of $S_{\mathfrak{b},\bar{l}_1,\cdots,\bar{l}_k}$-orbits and the limit submanifold also is a $S_{\mathfrak{b},\bar{l}_1,\cdots,\bar{l}_k}$-orbit. q.e.d.

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Postulation of general unions of lines and +lines in positive characteristic

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ABSTRACT

A +line is a scheme $R \subset \mathbb{P}^r$ with a line as its reduction $L = R_{red}$ which is the union of L and a tangent vector $\nu \nsubseteq L$ with $\nu_{red} \in L$. Here we prove in arbitrary characteristic that for $r \geq 4$ a general union of lines and +lines has maximal rank. We use the case $r = 3$ proved by myself in a previous paper and adapt the characteristic zero proof of the case $r > 3$ given in the same paper.

RESUMEN

Una +línea es un esquema $R \subset \mathbb{P}^r$ con una línea como su reducción $L = R_{red}$ que es la unión de L y un vector tangente $\nu \nsubseteq L$, con $\nu_{red} \in L$. Acá demostramos que para $r \geq 4$ una unión general de líneas y +líneas tiene rango máximo en característica arbitraria. Usamos el caso $r = 3$ demostrado por el autor en un artículo anterior y adaptamos la demostración en característica cero del caso $r > 3$ dado en el mismo artículo anterior.

Keywords and Phrases: Hilbert function; decorated line; disjoint unions of lines.

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1 Introduction

The aim of this note is to extend to the positive characteristic case a results in [1]. This extension is sufficient to extend [2, 3] to the positive characteristic case.

A scheme $X \subset \mathbb{P}^r$ is said to have *maximal rank* if $h^0(\mathcal{I}_X(t)) \cdot h^1(\mathcal{I}_X(t)) = 0$ for all $t \in \mathbb{N}$. Fix a line $L \subset \mathbb{P}^r$, $r \geq 2$, and $P \in L$. A tangent vector of $\mathbb{P}^r$ with P as its support is a zero-dimensional scheme $Z \subset \mathbb{P}^r$ such that $\deg(Z) = 2$ and $Z_{red} = \{P\}$. The tangent vector Z is uniquely determined by P and the line $\langle Z \rangle$ spanned by Z. Conversely, for each line $D \subset \mathbb{P}^r$ with $P \in D$ there is a unique tangent vector ν with $\nu_{red} = P$ and $\langle \nu \rangle = D$. A +line $M \subset \mathbb{P}^r$ supported by L and with nilradical supported by P is the union $\nu \cup L$ of L and a tangent vector ν with P as its support and spanning a line $\langle \nu \rangle \neq L$. The set of all +lines of $\mathbb{P}^r$ supported by L and with a nilradical at P is an irreducible variety of dimension $r-1$ (the complement of L in the $(r-1)$-dimensional projective space of all lines of $\mathbb{P}^r$ containing P). Hence the set of all +lines of $\mathbb{P}^r$ supported by L is parametrized by an irreducible variety of dimension r. For any +line R and every integer $k > 0$ we have $h^0(\mathcal{O}_R(k)) = k+2$ and $h^1(\mathcal{O}_R(k)) = 0$.

For any integers $r \geq 3$, $t \geq 0$, $c \geq 0$ with $(t, c) \neq (0, 0)$ let $L(r, t, c)$ be the set of all schemes $X \subset \mathbb{P}^r$ which are the disjoint union of t lines and c +lines. Every element of $L(r, c, t)$ has the map $k \mapsto (k+1)t + (k+2)c$ as its Hilbert function.

Consider the following statement.

Theorem 1.1. *For all integers $r \geq 3$, $a \geq 0$ and $b \geq 0$, $(a, b) \neq (0, 0)$, a general union $X \subset \mathbb{P}^r$ of a lines and b +lines has maximal rank,*

This statement was proved in [1] when either $r = 3$ or $r \geq 4$ and the algebraically closed base field has characteristic zero. The aim of this note is to prove Theorem 1.1 in positive characteristic (using the case $r = 3$ proved in [1]). Hence we may assume $r \geq 4$. We also use numerical lemmas and elementary remarks contained in [1]. We only need to change all parts which quote [4, Lemma 1.4] or [6], the only characteristic zero tool used in [1]. We recall that the case $c = 0$ is due to R. Hartshorne and A. Hirschowitz ([7]).

2 Proof of Theorem 1.1

For all integers $r \geq 3$ and $k \geq 0$ let $H_{r,k}$ denote the following statement:

Assertion $H_{r,k}$, $r \geq 3$, $k \geq 0$: Fix $(t, c) \in \mathbb{N}^2 \setminus \{(0, 0)\}$ and take a general $X \in L(r, t, c)$. If $(k+1)t + (k+2)c \geq \binom{r+k}{k}$, then $h^0(\mathcal{I}_X(k)) = 0$. If $(k+1)t + (k+2)c \leq \binom{r+k}{k}$, then $h^1(\mathcal{I}_X(k)) = 0$.

For all integers $r \geq 3$ and $k \geq 0$ define the integers $m_{r,k}$ and $n_{r,k}$ by the relations

$$(k+1)m_{r,k} + n_{r,k} = \binom{r+k}{r}, \ 0 \leq n_{r,k} \leq k \tag{2.1}$$

From (2.1) for the pairs (r, k) and $(r, k-1)$ we get

$$m_{r,k-1} + (k+1)(m_{r,k} - m_{r,k-1}) + n_{r,k} - n_{r,k-1} = \binom{r+k-1}{r-1} \tag{2.2}$$

for all $k > 0$.

For all integers $r \geq 3$ and $k \geq 0$ set $u_{r,k} := \lceil \binom{r+k}{r}/(k+2) \rceil$ and $v_{r,k} := (k+2)u_{r,k} - \binom{r+k}{r}$. We have

$$(k+2)(u_{r,k} - v_{r,k}) + (k+1)v_{r,k} = \binom{r+k}{r}, \ 0 \leq v_{r,k} \leq k+1 \tag{2.3}$$

As in [1] we need the following assumption $B_{r,k}$:

Assumption $B_{r,k}$, $r \geq 4$, $k > 0$. Fix a hyperplane $H \subset \mathbb{P}^r$. There is $X \in L(r, m_{r,k} - n_{r,k}, n_{r,k})$ such that the support of the nilradical sheaf of X is contained in H and $h^0(\mathcal{I}_X(k)) = 0$.

For all $X \in L(r, m_{r,k} - n_{r,k}, n_{r,k})$ we have $h^0(\mathcal{O}_X(k)) = \binom{r+k}{r}$ and so $h^1(\mathcal{I}_X(k)) = h^0(\mathcal{I}_X(k))$.

Lemma 2.1. *We have* $m_{r,k} - m_{r,k-1} \geq n_{r,k-1} + n_{r,k}$ *if* $r \geq 4$ *and* $k \geq 2$.

Proof. Assume $m_{r,k} - m_{r,k-1} \leq n_{r,k-1} + n_{r,k} - 1$. From (2.1) we get

$$m_{r,k-1} + kn_{r,k-1} + (k+2)n_{r,k} - k - 1 \geq \binom{r+k-1}{r-1}$$

Since $n_{r,k-1} \leq k-1$ and $n_{r,k} \leq k$, we get $m_{r,k-1} \geq \binom{r+k-1}{r-1} - 2k^2 + 1$. Since $km_{r,k-1} \leq \binom{r+k-1}{r}$ and $k\binom{r+k-1}{r-1} - \binom{r+k-1}{r} = (r-1)\binom{r+k-1}{r}$, we get

$$2k^3 - k \geq (r-1)\binom{r+k-1}{r} \tag{2.4}$$

This inequality is false if $r = 4$ and $k \geq 2$, because it is equivalent to the inequality $k(2k^2 - 1) \geq (k+3)(k+2)(k+1)k/8$. Since the right hand side of (2.4) is an increasing function of r, we conclude for all $r \geq 5$ and $k \geq 2$. □

Lemma 2.2. *Fix an integer* $r \geq 4$ *and assume that Theorem 1.1 is true in* $\mathbb{P}^{r-1}$. *Then* $B_{r,k}$ *is true for all* $k > 0$.

Proof. Since the case $k = 1$ is true ([1, Remark 3]), we may assume $k \geq 2$ and use induction on k. By Lemma 2.1 we have $m_{r,k} - m_{r,k-1} \geq n_{r,k-1} + n_{r,k}$. Fix a solution $X \in L(r, m_{r,k-1} - n_{r,k-1}, n_{r,k-1})$ of $B_{r,k-1}$, say $X = A \sqcup B$ with $A \in L(r, m_{r,k-1} - n_{r,k-1}, 0)$, $B \in L(r, 0, n_{r,k-1})$ and the tangent vectors of B have support $S \subset H$. By semicontinuity we may assume that no irreducible component of X_{red} is contained in H, that no tangent vector associated to the nilradical of B is contained in H and that S is a general subset of H with cardinality $n_{r,k-1}$. Let $C_1 \subset H$ be a general union of $m_{r,k} - m_{r,k-1} - n_{r,k} - n_{r,k-1}$ lines. Let $C_2 \subset H$ be a general union of $n_{r,k-1}$ lines, each of them containing a different point of S. Let $E \subset H$ be a general union of $n_{r,k}$ +lines. Since S is general, $C_1 \cup C_2 \cup E$ is a general element of $L(r-1, m_{r,k}-m_{r,k-1}-n_{r,k}, r_{n,k})$. Since Theorem 1.1 is true in $\mathbb{P}^{r-1}$, by (2.2) we get $h^1(H, \mathcal{I}_{C_1\cup C_2\cup F}(k)) = 0$ and $h^0(H, \mathcal{I}_{C_1\cup C_2\cup E}(k)) = m_{r,k-1}-n_{r,k-1}$. Deforming A with $B\cup C_1\cup C_2\cup E$ fixed, we may assume $A\cap(B\cup C_1\cup C_2\cup E) = \emptyset$ and that $h^i(H, \mathcal{I}_{C_1\cup C_2\cup E\cup(A\cap H)}(k)) = 0$, $i = 0, 1$. Since $A\cap(B\cup C_1\cup C_2\cup E) = \emptyset$, $Y := A\cup B\cup C_1\cup C_2\cup E$ is a disjoint union of $n_{r,k}$ +lines with support in H (even contained in H), $m_{r,k}-2n_{r,k-1}-n_{r,k}$ lines and $n_{r,k-1}$ sundials in the sense of [5]. Hence Y is a flat limit of a family of elements $L(r, m_{r,k-1}-n_{r,k-1}, n_{r,k-1})$ whose nilpotent sheaf is contained in H ([7], [5]). By the semicontinuity theorem to prove $B_{r,k}$ it is sufficient to prove that $h^0(\mathcal{I}_Y(k)) = 0$. Since no tangent vector of B is contained in H, then $\mathrm{Res}_H(Y) = X$ and $Y\cap H = C_1\cup C_2\cup E\cup(A\cap H)$. Since $h^0(\mathcal{I}_X(k-1)) = 0$ and $h^0(H, \mathcal{I}_{C_1\cup C_2\cup E\cup(A\cap H)}(k)) = 0$, a residual exact sequence gives $h^0(\mathcal{I}_Y(k)) = 0$. □

Lemma 2.3. *Assume* $r\ge 4$ *and that Theorem 1.1 is true in* $H = \mathbb{P}^{r-1}$. *Fix an integer* $k\ge 2$ *and assume that* $H_{r,k-1}$ *is true. Fix integers* $a\ge 0$, $b\ge 0$, $e\ge 0$ *such that* $e\le 2\lfloor(k+2)/2\rfloor$, $(k+2)a+(k+1)b+4\lfloor(k+2)/2\rfloor \le \binom{r+k-1}{r-1}$. *Let* $X\subset H$ *be a general union of* a *+lines,* b *lines and* e *tangent vectors. Then* $h^1(H, \mathcal{I}_X(k)) = 0$.

Proof. It is sufficient to do the case $e = \lfloor(k+2)/2\rfloor$. Let $A\subset H$ be a general union of a lines and b 2-lines.

First assume that k is even. Let $L_1, L_2\subset H$ be general lines. Fix a general $S_i\subset L_i$ with $\sharp(S_i) = k/2$ and a general $P_i\in L_i$, $i = 1, 2$. Let $\nu_i\subset H$ be a general tangent vector of H with P_i as its support; in particular we assume $\nu_i\nsubseteq L_i$. Let $E_i\subset L_i$ be the union of the $k/2$ tangent vectors of L_i with $(E_i)_{red} = S_i$. Set $Y := A\cup E_1\cup\nu_1\cup E_2\cup\nu_2$. Let R_i the +lines with L_i as their supports and with ν_i as the tangent vectors associated to their nilpotent sheaf. We have $h^0(\mathcal{O}_{A\cup E_1\cup E_2\cup\nu_1\cup\nu_2}(k)) = h^0(\mathcal{O}_{A\cup R_1\cup R_2}(k))$, $h^1(\mathcal{O}_{A\cup E_1\cup E_2\cup\nu_1\cup\nu_2}(k)) = h^1(\mathcal{O}_{A\cup R_1\cup R_2}(k))$ and $h^0(\mathcal{I}_{A\cup E_1\cup E_2\cup\nu_1\cup\nu_2}(k)) = h^0(\mathcal{I}_{A\cup R_1\cup R_2}(k))$. Therefore we have $h^1(\mathcal{I}_{A\cup E_1\cup E_2\cup\nu_1\cup\nu_2}(k)) = h^1(\mathcal{I}_{A\cup R_1\cup R_2}(k))$. Since $(k+2)a+(k+1)b+2(k+2)\le\binom{r+k-1}{r-1}$ and Theorem 1.1 is true in $\mathbb{P}^{r-1}$, we have $h^1(\mathcal{I}_{A\cup R_1\cup R_2}(k)) = 0$. Hence $h^1(\mathcal{I}_{A\cup E_1\cup E_2\cup\nu_1\cup\nu_2}(k)) = 0$. The semicontinuity theorem gives $h^1(H, \mathcal{I}_X(k)) = 0$.

Now assume that k is even. Let $F_i\subset L_i$ be any disjoint union of $(k+1)/2$ tangent vectors. We have $h^0(\mathcal{O}_{A\cup F_1\cup F_2}(k)) = h^0(\mathcal{O}_{A\cup L_1\cup L_2}(k))$, $h^1(\mathcal{O}_{A\cup F_1\cup F_2}(k)) = h^1(\mathcal{O}_{A\cup L_1\cup L_2}(k))$ and $h^0(\mathcal{I}_{A\cup F_1\cup F_2}(k)) = h^0(\mathcal{I}_{A\cup L_1\cup L_2}(k))$. Therefore we obtain $h^1(\mathcal{I}_{A\cup F_1\cup F_2}(k)) = h^1(\mathcal{I}_{A\cup L_1\cup L_2}(k))$. Since $(k+2)a+(k+1)b+2(k+1)\le\binom{r+k-1}{r-1}$ and Theorem 1.1 is true in $\mathbb{P}^{r-1}$, we have $h^1(\mathcal{I}_{A\cup L_1\cup L_2}(k)) = 0$. Therefore $h^1(\mathcal{I}_{A\cup F_1\cup F_2}(k)) = 0$. The semicontinuity theorem gives $h^1(H, \mathcal{I}_X(k)) = 0$. □

Proof of Theorem 1.1: By [1] we may assume $r\ge 4$. By induction on r we may also assume that Theorem 1.1 is true in $\mathbb{P}^{r-1}$. By [1, Remark 3] it is sufficient to prove $H_{r,k}$ for all integers $k\ge 1$. $H_{r,1}$ is true ([1, Lemma 3]). Hence we may assume $k\ge 2$ and that $H_{r,k-1}$ is true. By [1, Remark 4] it is sufficient to prove $H_{r,k}$ for the pairs (t, c) such that either $t = 0$ and $\binom{r+k}{r}-k-1\le c(k+2)\le\binom{r+k}{r}$ or $t(k+1)+(k+2)c = \binom{r+k}{r}$ and $c>0$; in the former case either

$\nu_{r,k} = 0$ and $c = u_{r,k}$ or $\nu_{r,k} > 0$ and $c = u_{r,k} - 1$; in the latter case we have $t + c \geq u_{r,k}$. If $c < n_{r,k-1}$, then we use step (b) of the proof of Theorem 1 in [1], because we gave a characteristic free proof of $B_{r,k}$ (Lemma 2.2). The case $c \geq n_{r,k-1}$ and $t \geq m_{r,k-1} - n_{r,k-1}$ was proved as step (a1) without using the characteristic zero assumption. Hence we may assume $c \geq n_{r,k-1}$ and $t < m_{r,k-1} - n_{r,k-1}$, i.e. the case of step (a2) of the proof in [1].

(i) Assume $t = 0$ and hence either $\nu_{r,k} = 0$ and $c = u_{r,k}$ or $\nu_{r,k} > 0$ and $c = u_{r,k} - 1$. Fix a general $U \in L(r, 0, \nu_{r,k-1}, u_{r,k-1} - \nu_{r,k-1})$, say $U = A \sqcup B$ with A the union of the $\nu_{r,k-1}$ lines. By $H_{r,k-1}$ we have $h^i(\mathcal{I}_U(k-1)) = 0$, $i = 0, 1$. It is easy to check using (2.3) that $u_{r,k} > u_{r,k-1}$. Hence $c \geq u_{r,k-1}$. Let $E \subset H$ be a general union of $c - u_{r,k-1}$ +lines. We may assume $E \cap (H \cap U)) = \emptyset$. Let $G \subset H$ be a general union of $\nu_{r,k-1}$ tangent vectors of H with the only restriction that $G_{red} = A \cap H$. For general A (and hence a general $A \cap H$)) the scheme $E \cup G$ is a general union inside H of $u_{r,k} - u_{k-1}$ +lines and $\nu_{r,k-1}$ tangent vectors. We have $\nu_{r,k-1} \leq k$. Using (2.3) for the integer $k - 1$ is easy to check that if $\nu_{r,k-1} > 0$, then $u_{r,k-1} - \nu_{r,k-1} \geq 2(k+2) - 2\nu_{r,k-1}$. Hence Lemma 2.3 gives $h^1(H, \mathcal{I}_{E \cup G}(k)) = 0$. Since $B \cap H$ is a general union of

(ii) Assume $t > 0$, $c > 0$, $t(k+1) + (k+2)c = \binom{r+k}{r}$ and $t < m_{r,k-1} - n_{r,k-1}$. First assume $t \leq 2\lfloor (k+2)/2 \rfloor$. In this case we may use the proof given in [1] (step (a2)) quoting Lemma 2.3 instead of [4, Lemma 1.4] for the postulation of the t tangent vectors, because $m_{r,k-1} - t \geq 2k + 2$ in this case. Therefore we may assume $t \geq k + 1$. Since $t < m_{r,k-1} - n_{r,k-1}$, we have $k \geq 3$ and $kt < \binom{r+k-1}{r}$. Set $d := \lfloor (\binom{r+k-1}{r} - kt)/(k+1) \rfloor$ and $z := (k+1)d + kt - \binom{r+k}{r}$. We have $0 \leq z \leq k + 1$. Fix a general $W \in L(r, t, d)$. Since $H_{r,x}$ holds for $x = k - 1, k - 2$, we have $h^0(\mathcal{I}_W(k-2)) = 0$ and $h^1(\mathcal{I}_W(k)) = 0$ and $h^0(\mathcal{I}_W(k)) = z$. Since S is general in H and $\sharp(S) = z$, we get $h^i(\mathcal{I}_{W \cup S}(k-1)) = 0$, $i = 0, 1$. Since $kt + (k+1)t + z = \binom{r+k-1}{r}$ and $t(k+1) + (k+2)c = \binom{r+k}{r}$, we get

$$t + d + (k+2)(c - d - z) + (k+1)z = \binom{r+k-1}{r-1} \tag{2.5}$$

Claim 1: We have $c \geq d + z$.

Proof of Claim 1: Assume $c \leq d + z - 1$. From (2.5) we get $t + d + (k+1)z - (k+1) \geq \binom{r+k-1}{r-1}$ and hence $k(t + d) + (k+1)kz - (k+1)k \geq k\binom{r+k-1}{r-1}$. Since $kt + (k+1)d + z = \binom{r+k-1}{r}$ and $z \leq k$, we get $(k+1)k^2 - k(k+1) - k \geq k\binom{r+k-1}{r-1} - \binom{r+k-1}{r}$, i.e. $k^3 - 2k \geq (r-1)\binom{r+k-1}{r}$. Call $\phi(r, k)$ the difference between the right hand side and the left hand side of this inequality. We have $\phi(r, k) = (r-1)\binom{r+k-1}{r} - k^3 + 2k$, which is positive if $r \geq 4$ and $k \geq 2$.

Let $M \subset H$ be a general union of $c - d - z$ +lines of H. Let $N \subset H$ be z general lines of H, each of them containing a different point of Z. Since S is general, $M \cup N$ has the Hilbert function of a general element of $L(r - 1, z, c - d - z)$ and hence it has maximal rank. By (2.5) we have $h^1(H, \mathcal{I}_{M \cup N}(k)) = 0$ and $h^0(\mathcal{I}_{M \cup N}(k)) = t + d$. Let $Z \subset \mathbb{P}^r$ be a general union of z +lines of $\mathbb{P}^r$ with N as their support. We have $G \cap H = N$ and $\mathrm{Res}_H(Z) = S$. Since $W \cup M \cup Z \in L(r, t, c)$, it is sufficient to prove that $h^i(\mathcal{I}_{W \cup M \cup Z}(k)) = 0$, $i = 0, 1$. Since $\mathrm{Res}_H(W \cup M \cup Z) = W \cup S$, we have $h^i(\mathcal{I}_{\mathrm{Res}_H(W \cup M \cup Z)}(k-1)) = 0$. Since $W \cap H$ is a general union of $d + c$ points of H and $(W \cup M \cup Z) = (W \cap H) \cup M \cup N$ as schemes, (2.5) gives $h^i(H, \mathcal{I}_{H \cap (W \cup M \cup Z)}(k)) = 0$. Apply the

Castelnuovo's lemma. □

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Yamabe Solitons with potential vector field as torse forming

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ABSTRACT

The Riemannian manifolds whose metric is Yamabe soliton with potential vector field as torse forming admitting Riemannian connection, semisymmetric metric connection and projective semisymmetric connection have been studied. An example is constructed to verify the theorem concerning Riemannian connection.

RESUMEN

Se estudian las variedades Riemannianas cuya métrica es un solitón de Yamabe con vector de potencial que forma un virol (superficie desarrollable) con respecto a conexiones Riemanniana, semisimétrica métrica y proyectiva semisimétrica. Se construye un ejemplo explícito para verificar las hipótesis del teorema en el caso de la conexión Riemanniana.

Keywords and Phrases: Yamabe soliton, torse forming vector field, torqued vector field, semisymmetric metric connection, projective semisymmetric connection.

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1 Introduction

The curvature tensor, Ricci tensor and scalar curvature of a Riemannian manifold M of dimension n equipped with Riemannian metric g with respect to Levi-Civita connection ∇ are denoted by R, S and r respectively. Hamilton ([5], [6]) introduced the notion of Yamabe flow, which is an evolution equation for metrics on M as follows:

$$\frac{\partial}{\partial t} g = -rg.$$

When $n = 2$, the Yamabe flow is equivalent to the Ricci flow. However, for $n > 2$, they do not agree.

A Yamabe soliton on M is, a special solution of the Yamabe flow, a triplet (g, V, σ) such that

$$\frac{1}{2}\pounds_V g = (r - \sigma)g, \tag{1.1}$$

where $\pounds_V$ is the Lie derivative in the direction of $V \in \chi(M)$ and σ is a constant. The nature of such soliton depends on the behaviour of σ. The Yamabe soliton is said to be shrinking, steady and expanding according as $\sigma < 0$, $= 0$ and > 0 respectively. If $\sigma \in C^\infty(M)$ then the metric satisfying (1.1) is called almost Yamabe soliton [1]. For $n = 2$ such soliton is equivalent with Ricci soliton, but for $n > 2$, they do not. Yamabe solitons have been studied by several authors such as [5], [6], [9], [10] and references there in.

As a generalization of concircular, concurrent and parallel vector field, Yano [14] introduced the torse-forming vector field. A nowhere vanishing vector field τ is said to be a torse-forming on M if

$$\nabla_X \tau = fX + \gamma(X)\tau, \tag{1.2}$$

where $f \in C^\infty(M)$ and γ is an 1-form.

If the 1-form γ in (1.2) vanishes identically, then τ is concircular [13]. Concircular vector fields also known as geodesis vector fields since integral curves of such vector fields are geodesis. Recently, Chen [2] studied Ricci solitons with concircular vector field. If $f = 1$ and $\gamma = 0$ then τ is concurrent [16]. The vector field τ is recurrent if it satisfies (1.2) with $f = 0$. Also if $f = \gamma = 0$, the vector field τ in (1.2) is parallel vector field.

As a consequence of torse forming vector field, recently Chen [3] introduced a new vector field, called torqued vector field. If the vector field τ satisfies (1.2) with $\gamma(\tau) = 0$ then τ is called torqued vector field. Here, f is known as the torqued function and the 1-form is the torqued form of τ.

In this paper we have studied Yamabe solitons, whose potential vector field is torse forming, on Riemannian manifolds with respect to Riemannian connection (RC), semisymmetric metric connection (SSMC) and projective semisymmetric connection (PSSC) and prove the following:

Theorem 1.1. *Let (g, τ, σ) be a Yamabe soliton on M with respect to RC ∇. Then the following holds:*

τ	*condition of existence*	*conditions of shrinking, steady and expanding*
torse-forming	$r - f - \frac{1}{n}\gamma(\tau)=$*constant*	$r - f - \frac{1}{n}\gamma(\tau) \lesseqqgtr 0$
concircular	$r - f=$ *constant*	$r - f \lesseqqgtr 0$
concurrent	$r=$*constant*	$r \lesseqqgtr 1$
recurrent	$r - \frac{1}{n}\gamma(\tau)=$ *constant*	$r - \frac{1}{n}\gamma(\tau) \lesseqqgtr 0$
parallel	$r =$ *constant*	$r \lesseqqgtr 0$
torqued	$r - f =$*constant*	$r - f \lesseqqgtr 0$

Theorem 1.2. *Let (g, τ, σ) be a Yamabe soliton on M with respect to SSMC $\bar{\nabla}$. Then the following holds:*

τ	*condition of existence*	*conditions of shrinking, steady and expanding*
torse-forming	$r - f - 2(n-1)a$ $-\frac{1}{n}\{(n-1)\pi(\tau) + \gamma(\tau)\}=$*constant*	$r - f - 2(n-1)a$ $-\frac{1}{n}\{(n-1)\pi(\tau) + \gamma(\tau)\} \lesseqqgtr 0$
concircular	$r - f - (n-1)\{2a + \frac{1}{n}\pi(\tau)\}=$*constant*	$r - f - (n-1)\{2a + \frac{1}{n}\pi(\tau)\} \lesseqqgtr 0$
concurrent	$r - 1 - (n-1)\{2a + \frac{1}{n}\pi(\tau)\} = \mathsf{constant}$	$r - 1 - (n-1)\{2a + \frac{1}{n}\pi(\tau)\} \lesseqqgtr 0$
recurrent	$r - 2(n-1)a$ $-\frac{1}{n}\{(n-1)\pi(\tau) + \gamma(\tau)\}=$ *constant*	$r - 2(n-1)a$ $-\frac{1}{n}\{(n-1)\pi(\tau) + \gamma(\tau)\} \lesseqqgtr 0$
parallel	$r - (n-1)\{2a + \frac{1}{n}\pi(\tau)\} = \mathsf{constant}$	$r - (n-1)\{2a + \frac{1}{n}\pi(\tau)\} \lesseqqgtr 0$
torqued	$r - f - (n-1)\{2a + \frac{1}{n}\pi(\tau)\}$ $=$*constant*	$r - f - (n-1)\{2a + \frac{1}{n}\pi(\tau)\} \lesseqqgtr 0$

Theorem 1.3. *Let (g, τ, σ) be a Yamabe soliton on M with respect to PSSC $\widetilde{\nabla}$. Then the following holds:*

τ	*condition of existence*	*conditions of shrinking, steady and expanding*
torse-forming	$r - f + Tr\cdot\beta - (n-1)Tr\cdot\alpha$ $-\frac{1}{n}\{(n-1)\pi(\tau)+\gamma(\tau)\}=$*constant*	$r - f + Tr\cdot\beta - (n-1)Tr\cdot\alpha$ $-\frac{1}{n}\{(n-1)\pi(\tau)+\gamma(\tau)\} \lesseqqgtr 0$
concircular	$r - f + Tr\cdot\beta - (n-1)\{Tr\cdot\alpha$ $+\frac{1}{n}\pi(\tau)\} =$*constant*	$r - f + Tr\cdot\beta - (n-1)\{Tr\cdot\alpha$ $+\frac{1}{n}\pi(\tau)\} \lesseqqgtr 0$
concurrent	$r + Tr\cdot\beta - (n-1)\{Tr\cdot\alpha$ $+\frac{1}{n}\pi(\tau)\} =$*constant*	$r - 1 + Tr\cdot\beta$ $-(n-1)\{Tr\cdot\alpha + \frac{1}{n}\pi(\tau)\} \lesseqqgtr 0$
recurrent	$r + Tr\cdot\beta - (n-1)Tr\cdot\alpha$ $-\frac{1}{n}\{(n-1)\pi(\tau)+\gamma(\tau)\}=$ *constant*	$r + Tr\cdot\beta - (n-1)Tr\cdot\alpha$ $-\frac{1}{n}\{(n-1)\pi(\tau)+\gamma(\tau)\} \lesseqqgtr 0$
parallel	$r + Tr\cdot\beta - (n-1)\{Tr\cdot\alpha + \frac{1}{n}\pi(\tau)\}$ $=$*constant*	$r + Tr\cdot\beta - (n-1)\{Tr\cdot\alpha +$ $\frac{1}{n}\pi(\tau)\} \lesseqqgtr 0$
torqued	$r - f + Tr\cdot\beta - (n-1)\{Tr\cdot\alpha +$ $\frac{1}{n}\pi(\tau)\}=$*constant*	$r - f + Tr\cdot\beta - (n-1)\{Tr\cdot\alpha +$ $\frac{1}{n}\pi(\tau)\} \lesseqqgtr 0$

Section 2 consists with preliminaries. The proof of our theorems are given in section 3. In section 4, we have constructed an example to verify Theorem 1.1.

Remark. The conditions of existence of Theorem 1.1, Theorem 1.2 and Theorem 1.3 are only necessary. Finding sufficient conditions for the existence of solitons is a much deeper problem and this is not addressed in the present manuscript.

2 Preliminaries

The relation between the semisymmetric metric connection (SSMC) $\bar{\nabla}$ and ∇ of M is given by ([4], [7], [15])

$$\bar{\nabla}_X Y = \nabla_X Y + \pi(Y)X - g(X, Y)\rho, \tag{2.1}$$

where $\pi(X) = g(X, \rho)$ for all $X \in \chi(M)$. If $\bar{R}$ (resp. $\bar{S}$ and $\bar{r}$) are the curvature tensor (respectively Ricci tensor and scalar curvature) of M with respect to SSMC, then [4]

$$\bar{R}(X, Y)Z = R(X, Y)Z - P(Y, Z)X + P(X, Z)Y - g(Y, Z)LX + g(X, Z)LY, \tag{2.2}$$

$$\bar{S}(Y, Z) = S(Y, Z) - (n-2)P(Y, Z) - a g(Y, Z), \tag{2.3}$$

$$\bar{r} = r - 2(n-1)a, \tag{2.4}$$

where P is a tensor field of type $(0,2)$ given by

$$P(X,Y) = g(LX,Y) = (\nabla_X \pi)(Y) - \pi(X)\pi(Y) + \frac{1}{2}\pi(\rho)g(X,Y)$$

and $a = Tr.P$ for any $X, Y \in \chi(M)$. The relation between projective semisymmetric connection $\widetilde{\nabla}$ and ∇ is [17]

$$\widetilde{\nabla}_X Y = \nabla_X Y + \psi(Y)X + \psi(X)Y + \phi(Y)X - \phi(X)Y, \tag{2.5}$$

where the 1-forms ϕ and ψ are given by $\phi(X) = \frac{1}{2}\pi(X)$ and $\psi(X) = \frac{n-1}{2(n+1)}\pi(X)$. If $\widetilde{R}$ (resp. $\widetilde{r}$ and $\widetilde{S}$) are the curvature tensor, Ricci tensor and scalar curvature of M with respect to $\widetilde{\nabla}$, then ([12], [17])

$$\widetilde{R}(X,Y)Z = R(X,Y)Z + \beta(X,Y)Z + \alpha(X,Z)Y - \alpha(Y,Z)X, \tag{2.6}$$

$$\widetilde{S}(Y,Z) = S(Y,Z) + \beta(Y,Z) - (n-1)\alpha(Y,Z), \tag{2.7}$$

$$\widetilde{r} = r + Tr.\beta - (n-1)Tr.\alpha, \tag{2.8}$$

for all $X, Y, Z \in \chi(M)$, where

$$\beta(X,Y) = \frac{1}{2}[(\nabla_Y \pi)(X) - (\nabla_X \pi)(Y)],$$

$$\alpha(X,Y) = \frac{n-1}{2(n+1)}(\nabla_X \pi)(Y) + \frac{1}{2}(\nabla_Y \pi)(X) - \frac{n^2}{(n+1)^2}\pi(X)\pi(Y).$$

3 Proof of the Theorems

Proof of the Theorem 1.1. Let (g, τ, σ) be a Yamabe soliton on M. Then from (1.1) we get

$$\frac{1}{2}(\pounds_\tau g)(X,Y) = (r - \sigma)g(X,Y). \tag{3.1}$$

Now from (1.2) we have

$$\begin{aligned}(\pounds_\tau g)(X,Y) &= g(\nabla_X \tau, Y) + g(X, \nabla_Y \tau) \\ &= 2fg(X,Y) + \gamma(X)g(\tau,Y) + \gamma(Y)g(\tau,X)\end{aligned} \tag{3.2}$$

for all $X, Y \in \chi(M)$. In view of (3.2), (3.1) yields

$$(r - \sigma - f)g(X,Y) = \frac{1}{2}\{\gamma(X)g(\tau,Y) + \gamma(Y)g(\tau,X)\}. \tag{3.3}$$

Taking contraction of (3.3) over X and Y we get

$$n(r - \sigma - f) = \gamma(\tau). \tag{3.4}$$

This leads to the following:

Proposition 3.1. *Let (g, τ, σ) be a Yamabe soliton on M with respect to RC ∇. If τ is torse-forming then this soliton is shrinking, steady and expanding according as*

$$r - f - \frac{1}{n}\gamma(\tau) \lesseqqgtr 0,$$

provided as $r - f - \frac{1}{n}\gamma(\tau)$ is constant.
From Proposition 3.1, we obtain Theorem 1.1.
Proof of the Theorem 1.2. We now consider (g, τ, σ) is a Yamabe soliton on M with respect to semisymmetric metric connection. Then we have

$$\frac{1}{2}(\bar{\mathcal{L}}_\tau g)(X, Y) = (\bar{r} - \sigma)g(X, Y), \tag{3.5}$$

where $\bar{\mathcal{L}}_\tau$ is the Lie derivative along τ of $\bar{\nabla}$. From (2.1) we get

$$\begin{aligned} (\bar{\mathcal{L}}_\tau g)(X, Y) &= g(\bar{\nabla}_X \tau, Y) + g(X, \bar{\nabla}_Y \tau) \\ &= g(\nabla_X \tau + \pi(\tau)X - g(X, \tau)\rho, Y) \\ &+ g(X, \nabla_Y \tau + \pi(\tau)Y - g(Y, \tau)\rho) \\ &= (\mathcal{L}_\tau g)(X, Y) + 2\pi(\tau)g(X, Y) \\ &- [g(X, \tau)\pi(Y) + g(Y, \tau)\pi(X)]. \end{aligned} \tag{3.6}$$

Using (2.4) and (3.6) in (3.5), we get

$$\begin{aligned} \frac{1}{2}(\mathcal{L}_\tau g)(X, Y) &= (r - \sigma)g(X, Y) - \{2(n-1)a + \pi(\tau)\}g(X, Y) \\ &+ \frac{1}{2}[g(X, \tau)\pi(Y) + g(Y, \tau)\pi(X)]. \end{aligned} \tag{3.7}$$

In view of (3.2), (3.7) yields

$$\begin{aligned} &\{r - \sigma - f - 2(n-1)a - \pi(\tau)\}g(X, Y) \\ &+ \frac{1}{2}[\{\pi(Y) - \gamma(Y)\}g(\tau, X) + \{\pi(X) - \gamma(X)\}g(\tau, Y)] = 0. \end{aligned} \tag{3.8}$$

Contracting (3.8) over X and Y, we get

$$n\{r - \sigma - f - 2(n-1)a\} - (n-1)\pi(\tau) - \gamma(\tau) = 0. \tag{3.9}$$

This leads to the following:

Proposition 3.2. *Let (g, τ, σ) be a Yamabe soliton on M with respect to SSMC $\bar{\nabla}$. If τ is torse-forming then this soliton is shrinking, steady and expanding according as*

$$r - f - 2(n-1)a - \frac{1}{n}\{(n-1)\pi(\tau) + \gamma(\tau)\} \lesseqqgtr 0,$$

provided $r - f - 2(n-1)a - \frac{1}{n}\{(n-1)\pi(\tau) + \gamma(\tau)\}$ is constant.
From Proposition 3.2, we obtain Theorem 1.2.
Proof of the Theorem 1.3. We now consider (g, τ, σ) is a Yamabe soliton on M with respect to $\widetilde{\widetilde{\nabla}}$. Then we have

$$\frac{1}{2}(\widetilde{\mathcal{L}}_\tau g)(X, Y) = (\widetilde{r} - \sigma)g(X, Y), \tag{3.10}$$

where $\widetilde{\mathcal{L}}_\tau$ is the Lie derivative along τ of $\widetilde{\widetilde{\nabla}}$. From (2.5) we get

$$\begin{aligned}(\widetilde{\mathcal{L}}_\tau g)(X, Y) &= g(\widetilde{\widetilde{\nabla}}_X \tau, Y) + g(X, \widetilde{\widetilde{\nabla}}_Y \tau) \\ &= (\mathcal{L}_\tau)(X, Y) + \frac{1}{n+1}\{2n\pi(\tau)g(X, Y) \\ &- \pi(X)g(\tau, Y) - \pi(Y)g(X, \tau)\}.\end{aligned} \tag{3.11}$$

Using (2.8) and (3.11) in (3.10), we get

$$\begin{aligned}\frac{1}{2}(\mathcal{L}_\tau g)(X, Y) &= (r - \sigma)g(X, Y) \\ &+ [\mathrm{Tr}\cdot\beta - (n-1)\mathrm{Tr}\cdot\alpha]g(X, Y) \\ &- \frac{1}{2(n+1)}\{2n\pi(\tau)g(X, Y) \\ &- \pi(X)g(Y, \tau) - \pi(Y)g(X, \tau)\}.\end{aligned} \tag{3.12}$$

In view of (3.2), (3.12) yields

$$\begin{aligned}&\{r - \sigma - f + \mathrm{Tr}\cdot\beta - (n-1)\mathrm{Tr}\cdot\alpha - \frac{n}{n+1}\pi(\tau)\}g(X, Y) \\ &+\frac{1}{2}[\{\frac{\pi(Y)}{n+1} - \gamma(Y)\}g(\tau, X) + \{\frac{\pi(X)}{n+1} - \gamma(X)\}g(\tau, Y)] = 0.\end{aligned} \tag{3.13}$$

Contracting (3.13) over X and Y, we get

$$n\{r - \sigma - f + \mathrm{Tr}\cdot\beta - (n-1)\mathrm{Tr}\cdot\alpha\} - (n-1)\pi(\tau) - \gamma(\tau) = 0. \tag{3.14}$$

This leads to the following:

Proposition 3.3. *Let (g, τ, σ) be a Yamabe soliton on M with respect to $\widetilde{\widetilde{\nabla}}$. If τ is torse-forming then this soliton is shrinking, steady and expanding according as*

$$r - f + \mathrm{Tr}\cdot\beta - (n-1)\mathrm{Tr}\cdot\alpha - \frac{1}{n}\{(n-1)\pi(\tau) + \gamma(\tau)\} \lesseqgtr 0,$$

provided $r - f + \mathrm{Tr}\cdot\beta - (n-1)\mathrm{Tr}\cdot\alpha - \frac{1}{n}\{(n-1)\pi(\tau) + \gamma(\tau)\}$ is constant.
From Proposition 3.3, we obtain Theorem 1.3.

4 Example

Here we construct an example to verify Theorem 1.1.

Example: Let us consider a 3-dimensional manifold $M = \{(x, y, z) \in \mathbb{R}^3 : z \neq 0\}$. Let $\{e_1, e_2, e_3\}$ be a linearly independent global frame on M given by

$$e_1 = z^2 \frac{\partial}{\partial x},\ e_2 = z^2 \frac{\partial}{\partial y},\ e_3 = \frac{\partial}{\partial z}.$$

Let g be the Riemannian metric defined by $g(e_i, e_j) = \begin{cases} 1, & \text{i=j} \\ 0, & \text{i} \neq \text{j} \end{cases}$.

These vector field and such metric is used in ([8], [11]). Using Koszul formula, we have [11]

$$\begin{aligned}
&\nabla_{e_1} e_1 = \frac{2}{z} e_3, \quad \nabla_{e_1} e_2 = 0, \quad \nabla_{e_1} e_3 = -\frac{2}{z} e_1, \\
&\nabla_{e_2} e_1 = 0, \quad \nabla_{e_2} e_2 = \frac{2}{z} e_3, \quad \nabla_{e_2} e_3 = -\frac{2}{z} e_2, \\
&\nabla_{e_3} e_1 = 0, \quad \nabla_{e_3} e_2 = 0, \quad \nabla_{e_3} e_3 = 0.
\end{aligned}$$

The scalar curvature of this manifold is also computed in [11] and it is $r = -\frac{32}{z^2}$.
Since $\{e_1, e_2, e_3\}$ forms a basis, any vector field $X, Y, U \in \chi(M)$ can be written as
$X = a_1 e_1 + b_1 e_2 + c_1 e_3, \quad Y = a_2 e_1 + b_2 e_2 + c_2 e_3, \quad U = a_3 e_1 + b_3 e_2 + c_3 e_3,$
where $a_i, b_i, c_i \in \mathbb{R}^+$ for $i = 1, 2, 3$ such that

$$\frac{a_1 a_2 + b_1 b_2}{c_1} + c_1 \left(\frac{b_2}{b_1} - \frac{a_2}{a_1} - 1\right) \neq 0.$$

If we choose the 1-form γ by $\gamma(W) = g(W, \frac{2}{z} e_3)$ for any $W \in \chi(M)$ and considering $f \in C^\infty(M)$ as

$$f = \frac{2}{z} \left\{ \frac{a_1 a_2 + b_1 b_2}{c_1} + c_1 \left(\frac{b_2}{b_1} - \frac{a_2}{a_1} - 1 \right) \right\}.$$

Then the relation

$$\nabla_X Y = fX + \gamma(X) Y \tag{4.1}$$

holds. Consequently Y is a torse-forming vector field. Now from (4.1) we get

$$\begin{aligned}
(\pounds_Y g)(X, U) &= g(\nabla_X Y, U) + g(X, \nabla_U Y) \\
&= 2fg(X, U) + \gamma(X) g(Y, U) + \gamma(U) g(Y, X).
\end{aligned} \tag{4.2}$$

Also we can calculate

$$\begin{cases} g(X, U) = a_1 a_3 + b_1 b_3 + c_1 c_3, \\ g(Y, U) = a_2 a_3 + b_2 b_3 + c_2 c_3, \\ g(Y, X) = a_1 a_2 + b_1 b_2 + c_1 c_2, \end{cases} \tag{4.3}$$

$$\gamma(X) = \frac{2c_1}{z}, \gamma(Y) = \frac{2c_2}{z}, \gamma(U) = \frac{2c_3}{z}. \tag{4.4}$$

In view of (4.3) and (4.4), (4.2) yields

$$\begin{aligned}\frac{1}{2}(\pounds_Y g)(X,U) &= \frac{1}{z}\Big[\Big\{\frac{2(a_1a_2+b_1b_2)}{c_1}+2c_1\Big(\frac{b_2}{b_1}-\frac{a_2}{a_1}-1\Big)\Big\}(a_1a_3+b_1b_3+c_1c_3)\\ &+ c_1(a_2a_3+b_2b_3+c_2c_3)+c_3(a_1a_2+b_1b_2+c_1c_2)\Big]. \qquad (4.5)\end{aligned}$$

Also

$$(r-\sigma)g(X,U) = \Big(-\frac{32}{z^2}-\sigma\Big)(a_1a_3+b_1b_3+c_1c_3). \qquad (4.6)$$

Assuming that $a_1a_3+b_1b_3+c_1c_3 \neq 0$ and

$$3c_1(a_2a_3+b_2b_3+c_2c_3)+3c_3(a_1a_2+b_1b_2+c_1c_2)-2c_2(a_1a_3+b_1b_3+c_1c_3)=0,$$

we get (g, Y, σ) is an Yamabe soliton, i.e $\frac{1}{2}(\pounds_Y g)(X,U) = (r-\sigma)g(X,U)$ holds, provided

$$\begin{aligned}\sigma &= -\frac{32}{z^2}-\frac{2}{z}\Big\{\frac{(a_1a_2+b_1b_2)}{c_1}+c_1\Big(\frac{b_2}{b_1}-\frac{a_2}{a_1}-1\Big)\Big\}\\ &- \frac{c_1(a_2a_3+b_2b_3+c_2c_3)+c_3(a_1a_2+b_1b_2+c_1c_2)}{(a_1a_3+b_1b_3+c_1c_3)z}\\ &= r-f-\frac{1}{3}\gamma(Y)\\ &= \text{constant}.\end{aligned}$$

Thus the condition of existence of Yamabe soliton (g, Y, σ) on a 3-dimensional Riemannian manifold with potential vector field Y as torse forming in Theorem 1.1 is verified.

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Study of global asymptotic stability in nonlinear neutral dynamic equations on time scales

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ABSTRACT

This paper is mainly concerned the global asymptotic stability of the zero solution of a class of nonlinear neutral dynamic equations in C^1_{rd}. By converting the nonlinear neutral dynamic equation into an equivalent integral equation, our main results are obtained via the Banach contraction mapping principle. The results obtained here extend the work of Yazgan, Tunc and Atan [17].

RESUMEN

Este artículo está mayormente interesado en la estabilidad global asintótica de la solución cero de una clase de ecuaciones nolineales neutrales dinámicas en C^1_{rd}. Transformando la ecuación nolineal neutral dinámica en una ecuación integral equivalente, nuestros resultados principales son obtenidos a través del principio de la aplicación contractiva de Banach. Los resultados obtenidos aquí son una extensión del trabajo de Yazgan, Tunc y Atan [17].

Keywords and Phrases: Fixed points, neutral dynamic equations, asymptotic stability, time scales.

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The concept of time scales analysis is a fairly new idea. In 1988, it was introduced by the German mathematician Stefan Hilger in his Ph.D. thesis [13]. It combines the traditional areas of continuous and discrete analysis into one theory. After the publication of two textbooks in this area by Bohner and Peterson [7] and [8], more and more researchers were getting involved in this fast-growing field of mathematics. The study of dynamic equations brings together the traditional research areas of differential and difference equations. It allows one to handle these two research areas at the same time, hence shedding light on the reasons for their seeming discrepancies. In fact, many new results for the continuous and discrete cases have been obtained by studying the more general time scales case (see [1, 3, 4, 6, 14] and the references therein).

There is no doubt that the Lyapunov method have been used successfully to investigate stability properties of wide variety of ordinary, functional and partial equations. Nevertheless, the application of this method to problem of stability in differential equations with delay has encountered serious difficulties if the delay is unbounded or if the equation has unbounded term. It has been noticed that some of theses difficulties vanish by using the fixed point technic. Other advantages of fixed point theory over Lyapunov's method is that the conditions of the former are average while those of the latter are pointwise (see [2, 5, 9, 10, 11, 12, 15, 17] and references therein).

In paper, we consider the following neutral nonlinear dynamic equations with variable delays given by

$$x^{\triangle}(t) = -a(t)x^{\sigma}(t) + b(t)g(x(t)) + c(t)f\left(x^{\widetilde{\triangle}}(t-\tau_1(t))\right) + q(t,x(t),x(t-\tau_2(t))), \quad (0.1)$$

with the initial condition

$$x(t) = \varphi(t), \ t \in [d_{t_0}, t_0] \cap \mathbb{T},$$

where

$$d_{t_0} = \inf_{t\in[t_0,\infty)\cap\mathbb{T}} \{t-\tau_1(t), t-\tau_2(t)\},$$

for each $t_0 \in [0,\infty) \cap \mathbb{T}$ and $\mathbb{T}$ is an unbounded above and below time scale and such that $t_0 \in \mathbb{T}$.

Our results are obtained with no need of further assumptions on the delta-differentiable of the neutral coefficient c and the twice delta-differentiable of τ_i with $\tau_i^{\triangle}(t) \neq 1$ for $t \in [0,\infty) \cap \mathbb{T}$, so that for a given initial function $\varphi \in \Phi_{t_0}$ a mapping P for (0.1) is constructed in such a way to map a, carefully chosen, closed convex nonempty subset D of a Banach space X into itself on which P is a contraction mapping possessing a fixed point. This procedure will enable us to establish and prove by means of the contraction mapping theorem ([16], p. 2) the global asymptotic stability in C^1_{rd} for the zero solution of (0.1) with a less restrictive conditions. In the special case $\mathbb{T} = \mathbb{R}$, Yazgan, Tunc and Atan in [17] show that the zero solution of (0.1) is globally asymptotically stable in C^1_{rd} by using the contraction mapping theorem. Then, the results obtained here extend the work of Yazgan, Tunc and Atan [17].

1 Preliminaries

In this section, we consider some advanced topics in the theory of dynamic equations on a time scales. Again, we remind that for a review of this topic we direct the reader to the monographs of Bohner and Peterson [7] and [8].

A time scale $\mathbb{T}$ is a closed nonempty subset of $\mathbb{R}$. For $t \in \mathbb{T}$ the forward jump operator σ, and the backward jump operator ρ, respectively, are defined as $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$ and $\rho(t) = \sup\{t \in \mathbb{T} : s < t\}$. These operators allow elements in the time scale to be classified as follows. We say t is right scattered if $\sigma(t) > t$ and right dense if $\sigma(t) = t$. We say t is left scattered if $\rho(t) < t$ and left dense if $\rho(t) = t$. The graininess function $\mu : \mathbb{T} \to [0, \infty)$, is defined by $\mu(t) = \sigma(t) - t$ and gives the distance between an element and its successor. We set $\inf \emptyset = \sup \mathbb{T}$ and $\sup \emptyset = \inf \mathbb{T}$. If $\mathbb{T}$ has a left scattered maximum M, we define $\mathbb{T}^k = \mathbb{T} \diagdown \{M\}$. Otherwise, we define $\mathbb{T}^k = \mathbb{T}$. If $\mathbb{T}$ has a right scattered minimum m, we define $\mathbb{T}_k = \mathbb{T} \diagdown \{m\}$. Otherwise, we define $\mathbb{T}_k = \mathbb{T}$.

Let $t \in \mathbb{T}^k$ and let $f : \mathbb{T} \to \mathbb{R}$. The delta derivative of $f(t)$, denoted $f^{\triangle}(t)$, is defined to be the number (when it exists), with the property that, for each $\epsilon > 0$, there is a neighborhood U of t such that

$$\left| f(\sigma(t)) - f(s) - f^{\triangle}(t)[\sigma(t) - s] \right| \leq \epsilon |\sigma(t) - s|,$$

for all $s \in U$. If $\mathbb{T} = \mathbb{R}$ then $f^{\triangle}(t) = f'(t)$ is the usual derivative. If $\mathbb{T} = \mathbb{Z}$ then $f^{\triangle}(t) = \triangle f(t) = f(t+1) - f(t)$ is the forward difference of f at t.

A function f is right dense continuous (rd-continuous), $f \in C_{rd} = C_{rd}(\mathbb{T}, \mathbb{R})$, if it is continuous at every right dense point $t \in \mathbb{T}$ and its left-hand limits exist at each left dense point $t \in \mathbb{T}$. The function $f : \mathbb{T} \to \mathbb{R}$ is differentiable on $\mathbb{T}^k$ provided $f^{\triangle}(t)$ exists for all $t \in \mathbb{T}^k$. $f \in C^1_{rd} = C^1_{rd}(\mathbb{T}, \mathbb{R})$ if $f^{\triangle} \in C_{rd}(\mathbb{T}, \mathbb{R})$.

We are now ready to state some properties of the delta-derivative of f. Note $f^{\sigma}(t) = f(\sigma(t))$.

Theorem 1.1 ([7, Theorem 1.20]). *Assume* $f, g : \mathbb{T} \to \mathbb{R}$ *are differentiable at* $t \in \mathbb{T}^k$ *and let* α *be a scalar.*

(i) $(f+g)^{\triangle}(t) = g^{\triangle}(t) + f^{\triangle}(t)$.

(ii) $(\alpha f)^{\triangle}(t) = \alpha f^{\triangle}(t)$.

(iii) *The product rules*

$$\begin{aligned}(fg)^{\triangle}(t) &= f^{\triangle}(t) g(t) + f^{\sigma}(t) g^{\triangle}(t), \\ (fg)^{\triangle}(t) &= f(t) g^{\triangle}(t) + f^{\triangle}(t) g^{\sigma}(t).\end{aligned}$$

(iv) *If* $g(t) g^{\sigma}(t) \neq 0$ *then*

$$\left(\frac{f}{g}\right)^{\triangle}(t) = \frac{f^{\triangle}(t) g(t) - f(t) g^{\triangle}(t)}{g(t) g^{\sigma}(t)}.$$

The next theorem is the chain rule on time scales ([7, Theorem 1.93]).

Theorem 1.2 (Chain Rule)**.** *Assume* $\nu : \mathbb{T} \to \mathbb{R}$ *is strictly increasing and* $\widetilde{\mathbb{T}} := \nu(\mathbb{T})$ *is a time scale. Let* $\omega : \widetilde{\mathbb{T}} \to \mathbb{R}$. *If* $\nu^{\triangle}(t)$ *and* $\omega^{\widetilde{\triangle}}(\nu(t))$ *exist for* $t \in \mathbb{T}^k$, *then* $(\omega \circ \nu)^{\triangle} = \left(\omega^{\widetilde{\triangle}} \circ \nu\right) \nu^{\triangle}$.

In the sequel we will need to differentiate and integrate functions of the form $f(t - \tau(t)) = f(\nu(t))$ where, $\nu(t) := t - \tau(t)$. Our next theorem is the substitution rule ([7, Theorem 1.98]).

Theorem 1.3 (Substitution)**.** *Assume* $\nu : \mathbb{T} \to \mathbb{R}$ *is strictly increasing and* $\widetilde{\mathbb{T}} := \nu(T)$ *is a time scale. If* $f : \mathbb{T} \to \mathbb{R}$ *is rd-continuous function and* ν *is differentiable with rd-continuous derivative, then for* $a, b \in T$,

$$\int_a^b f(t)\, \nu^{\triangle}(t) \triangle t = \int_{\nu(a)}^{\nu(b)} \left(f \circ \nu^{-1}\right)(s)\, \widetilde{\triangle} s.$$

A function $p : \mathbb{T} \to \mathbb{R}$ is said to be regressive provided $1 + \mu(t)\, p(t) \neq 0$ for all $t \in \mathbb{T}^k$. The set of all regressive rd-continuous function $f : \mathbb{T} \to \mathbb{R}$ is denoted by $\mathcal{R}$. The set of all positively regressive functions $\mathcal{R}^+$, is given by $\mathcal{R}^+ = \{f \in \mathcal{R} : 1 + \mu(t)\, f(t) > 0 \text{ for all } t \in \mathbb{T}\}$.

Let $p \in \mathcal{R}$ and $\mu(t) \neq 0$ for all $t \in \mathbb{T}$. The exponential function on $\mathbb{T}$ is defined by

$$e_p(t, s) = \exp\left(\int_s^t \frac{1}{\mu(z)} \log(1 + \mu(z)\, p(z))\, \Delta z\right).$$

It is well known that if $p \in \mathcal{R}^+$, then $e_p(t, s) > 0$ for all $t \in \mathbb{T}$. Also, the exponential function $y(t) = e_p(t, s)$ is the solution to the initial value problem $y^{\triangle} = p(t)\, y$, $y(s) = 1$. Other properties of the exponential function are given by the following lemma.

Lemma 1.4 ([7, Theorem 2.36])**.** *Let* $p, q \in \mathcal{R}$. *Then*

(i) $e_0(t, s) = 1$ *and* $e_p(t, t) = 1$,

(ii) $e_p(\sigma(t), s) = (1 + \mu(t)\, p(t))\, e_p(t, s)$,

(iii) $\frac{1}{e_p(t,s)} = e_{\ominus p}(t, s)$, *where* $\ominus p(t) = -\frac{p(t)}{1+\mu(t)p(t)}$,

(iv) $e_p(t, s) = \frac{1}{e_p(s,t)} = e_{\ominus p}(s, t)$,

(v) $e_p(t, s)\, e_p(s, r) = e_p(t, r)$,

(vi) $e_p^{\triangle}(., s) = p e_p(., s)$ *and* $\left(\frac{1}{e_p(.,s)}\right)^{\triangle} = -\frac{p(t)}{e_p^{\sigma}(.,s)}$.

Lemma 1.5 ([1])**.** *If* $p \in \mathcal{R}^+$, *then*

$$0 < e_p(t, s) \leq \exp\left(\int_s^t p(u) \triangle u\right), \ \forall t \in \mathbb{T}.$$

2 Global asymptotic stability

In this section, we shall study the global asymptotic stability in C^1_{rd} of the zero solution to (0.1). We introduce the following hypotheses.

(H_1) $a, b, c \in C_{rd}([0,\infty)\cap\mathbb{T},\mathbb{R})$, $g, f \in C(\mathbb{R},\mathbb{R})$, $q \in C_{rd}([0,\infty)\cap\mathbb{T}\times\mathbb{R}\times\mathbb{R},\mathbb{R})$, $\tau_i \in C_{rd}([0,\infty)\cap\mathbb{T},(0,\infty)\cap\mathbb{T})$ and $(id-\tau_i)([0,\infty)\cap\mathbb{T})$ is closed with $t-\tau_i(t)\to\infty$ as $t\to\infty$, $i=1,2$.

(H_2) For $t\in[0,\infty)\cap\mathbb{T}$, $g(0)=f(0)=q(t,0,0)=0$, and there exist $L_g, L_f>0$, $L_1, L_2\in C_{rd}([0,\infty)\cap\mathbb{T},(0,\infty))$ such that

$$|g(x_1)-g(x_2)|\le L_g|x_1-x_2|,$$

$$|f(x_1)-f(x_2)|\le L_f|x_1-x_2|,$$

$$|q(t,x_1,y_1)-q(t,x_2,y_2)|\le L_1(t)|x_1-x_2|+L_2(t)|y_1-y_2|,$$

for any $x_i, y_i\in\mathbb{R}$, $i=1,2$.

(H_3) $a\in\mathcal{R}^+$ is bounded on $[0,\infty)\cap\mathbb{T}$ and $\lim_{t\to\infty}\inf\int_0^t \frac{1}{\mu(s)}\log(1+\mu(s)a(s))\Delta s>-\infty$.

(H_4) There exists $\alpha\in(0,1)$ such that for $t\in[0,\infty)\cap\mathbb{T}$,

$$\int_0^t e_{\ominus a}(t,u)\left[L_g|b(u)|+L_f|c(u)|+L_1(u)+L_2(u)\right]\Delta u\le\alpha,$$

and

$$\begin{aligned}&|a(t)|\int_0^{\sigma(t)} e_{\ominus a}(\sigma(t),u)\left[L_g|b(u)|+L_f|c(u)|+L_1(u)+L_2(u)\right]\Delta u\\&+L_g|b(t)|+L_f|c(t)|+L_1(t)+L_2(t)\ \le\ \alpha.\end{aligned}$$

For each $t_0\in[0,\infty)\cap\mathbb{T}$ denote $C_{rd}^1(t_0)=C_{rd}^1([d_{t_0},t_0]\cap\mathbb{T},\mathbb{R})$ with the norm defined by

$$|x|_{t_0}=\max_{t\in[d_{t_0},t_0]\cap\mathbb{T}}\left\{|x(t)|,\left|x^{\triangle}(t)\right|\right\}$$

for $x\in C_{rd}^1(t_0)$. In addition, denote

$$\begin{aligned}\Phi_{t_0}=\{\varphi\in C_{rd}^1(t_0):\varphi^{\triangle}(t_0)&=-a(t_0)\varphi^{\sigma}(t_0)+b(t_0)g(\varphi(t_0))\\&+c(t_0)f\left(\varphi^{\widetilde{\triangle}}(t_0-\tau_1(t_0))\right)+q(t_0,\varphi(t_0),\varphi(t_0-\tau_2(t_0)))\}.\end{aligned}$$

For each $t_0\in[0,\infty)\cap\mathbb{T}$, we always assume that the initial function for (0.1) is of the type $\varphi\in\Phi_{t_0}$. For convenience of stating our main result, we shall give the following definitions.

Definition 2.1. *For each* $(t_0,\varphi)\in[0,\infty)\cap\mathbb{T}\times\Phi_{t_0}$, x *is said to be a solution of (0.1) through* (t_0,φ) *if* $x\in C_{rd}^1([d_{t_0},\infty)\cap\mathbb{T})$ *satisfies (0.1) on* $[t_0,\infty)\cap\mathbb{T}$ *and* $x(t)=\varphi(t)$ *for* $t\in[d_{t_0},t_0]\cap\mathbb{T}$. *We denote such a solution by* $x(t)=x(t,t_0,\varphi)$.

Definition 2.2. (i) *The zero solution of (0.1) is said to be stable in* C_{rd}^1 *if, for any* $t_0\in[0,\infty)\cap\mathbb{T}$, $\varepsilon>0$ *there is a* $\delta=\delta(\varepsilon,t_0)$ *such that* $\varphi\in\Phi_{t_0}$ *and* $|\varphi|_{t_0}<\delta$ *implies*

$$\max_{s\in[d_{t_0},t]\cap\mathbb{T}}\left\{|x(s)|,\left|x^{\triangle}(s)\right|\right\}<\varepsilon,$$

for $t \in [t_0, \infty) \cap \mathbb{T}$.

(ii) *The zero solution of (0.1) is said to be globally asymptotically stable in* C^1_{rd} *if it is stable in* C^1_{rd}, *and for any* $t_0 \in [0, \infty) \cap \mathbb{T}$, $\varphi \in \Phi_{t_0}$ *implies*

$$\lim_{t\to\infty} x(t, t_0, \varphi) = \lim_{t\to\infty} x^{\triangle}(t, t_0, \varphi) = 0.$$

In view of the definition of solution of (0.1), it is clear that the conditions imposed on the initial functions are very natural. From the above assumptions, it is easy to see that for each $(t_0, \varphi) \in [0, \infty) \cap \mathbb{T} \times \Phi_{t_0}$, there exists a unique solution $x(t) = x(t, t_0, \varphi)$ of (0.1) defined on $[d_{t_0}, \infty) \cap \mathbb{T}$. By (H_2), (0.1) has the zero solution.

Theorem 2.3. *Assume that* $(H_1) - (H_4)$ *hold. Then the zero solution of (0.1) is globally asymptotically stable in* C^1_{rd} *if and only if*

$$\int_0^t \frac{1}{\mu(s)} \log(1 + \mu(s) a(s)) \Delta s \to \infty \text{ as } t \to \infty. \tag{2.1}$$

Proof. (i) Suppose that (2.1) holds. For any $t_0 \in [0, \infty) \cap \mathbb{T}$, let

$$X = \left\{ x \in C^1_{rd}([d_{t_0}, \infty) \cap \mathbb{T}) : \lim_{t\to\infty} x(t) = \lim_{t\to\infty} x^{\triangle}(t) = 0 \right\},$$

with the norm defined by

$$\|x\|_{t_0} = \max_{t \in [d_{t_0}, \infty) \cap \mathbb{T}} \left\{ |x(t)|, \left|x^{\triangle}(t)\right| \right\},$$

for $x \in X$. Since X is a closed vectorial subspace of $C^1_{rd}([d_{t_0}, \infty) \cap \mathbb{T})$ and $C^1_{rd}\left([d_{t_0}, \infty) \cap \mathbb{T}, \|.\|_{t_0}\right)$ is a Banach space, then $\left(X, \|.\|_{t_0}\right)$ is also a Banach space. For any $\varphi \in \Phi_{t_0}$, let

$$D = \{x \in X : x(t) = \varphi(t) \text{ for } t \in [d_{t_0}, t_0] \cap \mathbb{T}\}.$$

It is easy to see that D is a nonempty, closed subset of X.

Multiply both sides of (0.1) by $e_a(t, t_0)$ and then integrate from t_0 to t to obtain

$$\begin{aligned}
&\int_{t_0}^t [x(u) e_a(u, t_0)]^{\Delta} \Delta u \\
&= \int_{t_0}^t e_a(u, t_0) \left[b(u) g(x(u)) + c(u) f\left(x^{\tilde{\triangle}}(u - \tau_1(u)) \right) \right. \\
&\left. + q(u, x(u), x(u - \tau_2(u))) \right] \Delta u.
\end{aligned}$$

As a consequence, we arrive at

$$\begin{aligned}
&x(t) e_a(t, t_0) - x(t_0) \\
&= \int_{t_0}^t e_a(u, t_0) \left[b(u) g(x(u)) + c(u) f\left(x^{\tilde{\triangle}}(u - \tau_1(u)) \right) \right. \\
&\left. + q(u, x(u), x(u - \tau_2(u))) \right] \Delta u.
\end{aligned}$$

By dividing both sides of the above equation by $e_a(t,t_0)$, we obtain

$$x(t) = \varphi(t_0)\, e_{\ominus a}(t,t_0) + \int_{t_0}^{t} e_{\ominus a}(t,u)\Big[b(u)\, g(x(u)) + c(u)\, f\Big(x^{\widetilde{\triangle}}(u-\tau_1(u))\Big) \\ +q(u, x(u), x(u-\tau_2(u)))\Big]\Delta u. \tag{2.2}$$

Use (2.2) to define the operator $P : D \to C_{rd}([d_{t_0},\infty)\cap\mathbb{T})$ by $(Px)(t) = \varphi(t)$ for $t \in [d_{t_0}, t_0] \cap \mathbb{T}$ and

$$(Px)(t) = \varphi(t_0)\, e_{\ominus a}(t,t_0) + \int_{t_0}^{t} e_{\ominus a}(t,u)\Big[b(u)\, g(x(u)) + c(u)\, f\Big(x^{\widetilde{\triangle}}(u-\tau_1(u))\Big) \\ +q(u, x(u), x(u-\tau_2(u)))\Big]\Delta u, \tag{2.3}$$

for $t \in [t_0,\infty)\cap\mathbb{T}$.

Firstly, we prove $Px \in D$ for any $x \in D$. From (2.3), for $t > t_0$,

$$\begin{aligned}
(Px)^{\triangle}(t) &= -\varphi(t_0)\, a(t)\, e_{\ominus a}(\sigma(t), t_0) \\
&\quad + e_{\ominus a}(\sigma(t),t)\Big[b(t)\, g(x(t)) + c(t)\, f\Big(x^{\widetilde{\triangle}}(t-\tau_1(t))\Big) + q(t, x(t), x(t-\tau_2(t)))\Big] \\
&\quad - a(t)\int_{t_0}^{t} e_{\ominus a}(\sigma(t),u)\Big[b(u)\, g(x(u)) + c(u)\, f\Big(x^{\widetilde{\triangle}}(u-\tau_1(u))\Big) \\
&\quad +q(u, x(u), x(u-\tau_2(u)))\Big]\Delta u \\
&= -a(t)\, \varphi(t_0)\, e_{\ominus a}(\sigma(t), t_0) \\
&\quad - a(t)\int_{t_0}^{\sigma(t)} e_{\ominus a}(\sigma(t),u)\Big[b(u)\, g(x(u)) + c(u)\, f\Big(x^{\widetilde{\triangle}}(u-\tau_1(u))\Big) \\
&\quad +q(u, x(u), x(u-\tau_2(u)))\Big]\Delta u \\
&\quad + b(t)\, g(x(t)) + c(t)\, f\Big(x^{\widetilde{\triangle}}(t-\tau_1(t))\Big) + q(t, x(t), x(t-\tau_2(t))) \\
&= -a(t)\,(Px)^{\sigma}(t) + b(t)\, g(x(t)) + c(t)\, f\Big(x^{\widetilde{\triangle}}(t-\tau_1(t))\Big) + q(t, x(t), x(t-\tau_2(t))).
\end{aligned} \tag{2.4}$$

By the definition of Φ_{t_0}, (2.4) yields on a time scale

$$\begin{aligned}
(Px)^{\triangle}_{+}(t_0) &= -a(t_0)\,\varphi^{\sigma}(t_0) + b(t_0)\, g(\varphi(t_0)) + c(t_0)\, f\Big(\varphi^{\widetilde{\triangle}}(t_0-\tau_1(t_0))\Big) \\
&\quad + q(t_0, \varphi(t_0), \varphi(t_0-\tau_2(t_0))) \\
&= \varphi^{\triangle}_{-}(t_0).
\end{aligned}$$

Hence, $Px \in C^1_{rd}([d_{t_0},\infty)\cap\mathbb{T})$ for $x \in D$.

For $x \in D$, $\lim_{t\to\infty} x(t) = \lim_{t\to\infty} x^{\triangle}(t) = 0$. Note that $\lim_{t\to\infty} t - \tau_i(t) = \infty$, $i = 1, 2$. Therefore, for any $\varepsilon > 0$, there exists $T > 0$ such that for $t \geq T$,

$$\max\left\{|x(t)|, |x(t-\tau_2(t))|, \left|x^{\widetilde{\triangle}}(t-\tau_1(t))\right|\right\} < \varepsilon. \tag{2.5}$$

It follows from (2.3), (2.5) and (H_2) and (H_4) that for $t > T$ and $x \in D$,

$$
\begin{aligned}
&\left|(Px)(t)\right| \\
&\leq |\varphi(t_0)|\, e_{\ominus a}(t,t_0) + \int_{t_0}^{T} e_{\ominus a}(t,u) \Big| b(u)\, g(x(u)) + c(u)\, f\left(x^{\widetilde{\triangle}}(u-\tau_1(u))\right) \\
&+q(u, x(u), x(u-\tau_2(u)))\big|\, \Delta u \\
&+\int_{T}^{t} e_{\ominus a}(t,u) \Big| b(u)\left(g(x(u)) - g(0)\right) + c(u)\left(f\left(x^{\widetilde{\triangle}}(u-\tau_1(u))\right) - f(0)\right) \\
&+q(u, x(u), x(u-\tau_2(u))) - q(u,0,0)\big|\, \Delta u \\
&\leq e_{\ominus a}(t,t_0)\left[|\varphi(t_0)| + \int_{t_0}^{T} e_a(u,t_0) \Big| b(u)\, g(x(u)) + c(u)\, f\left(x^{\widetilde{\triangle}}(u-\tau_1(u))\right)\right. \\
&\left.+q(u, x(u), x(u-\tau_2(u)))\big|\, \Delta u\right] \\
&+\int_{T}^{t} e_{\ominus a}(t,u)\left[L_g |b(u)|\, |x(u)| + L_f |c(u)| \left|x^{\widetilde{\triangle}}(u-\tau_1(u))\right|\right. \\
&\left.+L_1(u)\, |x(u)| + L_2(u)\, |x(u-\tau_2(u))|\right] \Delta u \\
&\leq e_{\ominus a}(t,t_0)\left[|\varphi(t_0)| + \int_{t_0}^{T} e_a(u,t_0) \Big| b(u)\, g(x(u)) + c(u)\, f\left(x^{\widetilde{\triangle}}(u-\tau_1(u))\right)\right. \\
&\left.+q(u, x(u), x(u-\tau_2(u)))\big|\, \Delta u\right] \\
&+\varepsilon \int_{T}^{t} e_{\ominus a}(t,u)\left[L_g |b(u)| + L_f |c(u)| + L_1(u) + L_2(u)\right] \Delta u \\
&\leq e_{\ominus a}(t,t_0)\left[|\varphi(t_0)| + \int_{t_0}^{T} e_a(u,t_0) \Big| b(u)\, g(x(u)) + c(u)\, f\left(x^{\widetilde{\triangle}}(u-\tau_1(u))\right)\right. \\
&\left.+q(u, x(u), x(u-\tau_2(u)))\big|\, \Delta u\right] + \alpha\varepsilon.
\end{aligned}
$$

From (2.1), there exists $T_1 > T$ such that for $t > T_1$,

$$
\begin{aligned}
&e_{\ominus a}(t,t_0)\left[|\varphi(t_0)| + \int_{t_0}^{T} e_a(u,t_0) \Big| b(u)\, g(x(u)) + c(u)\, f\left(x^{\widetilde{\triangle}}(u-\tau_1(u))\right)\right. \\
&\left.+q(u, x(u), x(u-\tau_2(u)))\big|\, \Delta u\right] \;<\; \varepsilon.
\end{aligned}
$$

Hence, $\lim_{t\to\infty}(Px)(t) = 0$ for $x \in D$. In addition, it follows from (2.4) and (H_2) that

$$
\begin{aligned}
&\left|(Px)^{\triangle}(t)\right| \\
&\leq \left|a(t)\,(Px)^{\sigma}(t)\right| + |b(t)\left(g(x(t)) - g(0)\right)| + \left|c(t)\left(f\left(x^{\widetilde{\triangle}}(t-\tau_1(t))\right) - f(0)\right)\right| \\
&+ |q(t, x(t), x(t-\tau_2(t))) - q(t,0,0)| \\
&\leq \left|a(t)\,(Px)^{\sigma}(t)\right| + L_g |b(t)|\, |x(t)| + L_f |c(t)| \left|x^{\widetilde{\triangle}}(t-\tau_1(t))\right| \\
&+ L_1(t)\, |x(t)| + L_2(t)\, |x(t-\tau_2(t))|.
\end{aligned}
$$

This, together with (H_3) and (H_4), yields $\lim_{t\to\infty}(Px)^{\triangle}(t) = 0$ for $x \in D$. Therefore, $Px \in D$ for $x \in D$, i.e. $P : D \to D$.

Secondly, we show that $P : D \to D$ is a contraction mapping. For any $x, y \in D$, it follows from (2.3), (H_2) and (H_4) that for $t \in [t_0, \infty) \cap \mathbb{T}$,

$$\begin{aligned}
&|(Px)(t) - (Py)(t)| \\
&\leq \int_{t_0}^{t} e_{\ominus a}(t,u)\left[L_g|b(u)||x(u) - y(u)| + L_f|c(u)|\left|x^{\widetilde{\triangle}}(u-\tau_1(u)) - y^{\widetilde{\triangle}}(u-\tau_1(u))\right|\right. \\
&+ |q(u, x(u), x(u-\tau_2(u))) - q(u, y(u), y(u-\tau_2(u)))|]\,\Delta u \\
&\leq \|x-y\|_{t_0}\int_{t_0}^{t} e_{\ominus a}(t,u)\left[L_g|b(u)| + L_f|c(u)| + L_1(u) + L_2(u)\right]\Delta u \\
&\leq \alpha\|x-y\|_{t_0}. \qquad (2.6)
\end{aligned}$$

In addition, it follows from (2.4), (2.6), (H_2) and (H_4) that for $t \in [t_0, \infty) \cap \mathbb{T}$,

$$\begin{aligned}
&\left|(Px)^{\triangle}(t) - (Py)^{\triangle}(t)\right| \\
&\leq |a(t)|\left|(Px)^{\sigma}(t) - (Py)^{\sigma}(t)\right| + L_g|b(t)||x(t) - y(t)| + L_f|c(t)|\left|x^{\widetilde{\triangle}}(t-\tau_1(t)) - y^{\widetilde{\triangle}}(t-\tau_1(t))\right| \\
&+ |q(t, x(t), x(t-\tau_2(t))) - q(t, y(t), y(t-\tau_2(t)))| \\
&\leq \|x-y\|_{t_0}\left[|a(t)|\int_{t_0}^{\sigma(t)} e_{\ominus a}(\sigma(t),u)\left[L_g|b(u)| + L_f|c(u)| + L_1(u) + L_2(u)\right]\Delta u\right. \\
&\left. + L_g|b(t)| + L_f|c(t)| + L_1(t) + L_2(t)\right] \\
&\leq \alpha\|x-y\|_{t_0}. \qquad (2.7)
\end{aligned}$$

From (2.6) and (2.7), $P : D \to D$ is a contraction mapping. By the contraction mapping principle, P has a unique fixed point x in D, which is a unique solution of (0.1) through (t_0, φ) and satisfies

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} x^{\triangle}(t) = 0. \qquad (2.8)$$

Finally, we show that the zero solution of (0.1) is stable in C^1_{rd}. Let

$$K = \sup_{t\in[t_0,\infty)\cap\mathbb{T}}\{e_{\ominus a}(t,t_0)\} \text{ and } A = \sup_{t\in[t_0,\infty)\cap\mathbb{T}}\{|a(t)|\}.$$

From (2.1) and (H_3), $K, A \in (0, \infty)$. For any $\varepsilon > 0$, let $\delta > 0$ such that

$$\delta < \varepsilon\min\left\{1, \frac{1-\alpha}{K}, \frac{1-\alpha}{KA}\right\}.$$

If $x(t) = x(t, t_0, \varphi)$ is a solution of (0.1) with $|\varphi|_{t_0} < \delta$, then $x(t) = (Px)(t)$ on $[t_0, \infty) \cap \mathbb{T}$. We claim that $\|x\|_{t_0} < \varepsilon$. Otherwise, there exists $t_1 > t_0$ such that

$$\max\left\{|x(t_1)|, \left|x^{\triangle}(t_1)\right|\right\} = \varepsilon,$$

and

$$\max\left\{|x(t)|,\left|x^{\triangle}(t)\right|\right\}<\varepsilon,$$

for $t\in[d_{t_0},t_1)\cap\mathbb{T}$. If $|x(t_1)|=\varepsilon$, then it follows from (2.3), (H_2) and (H_4) that

$$\begin{aligned}|x(t_1)| &\le |\varphi(t_0)|\,e_{\ominus a}(t_1,t_0)\\ &\quad+\int_{t_0}^{t_1} e_{\ominus a}(t_1,u)\left|b(u)\,g(x(u))+c(u)\,f\left(x^{\tilde{\triangle}}(u-\tau_1(u))\right)+q(u,x(u),x(u-\tau_2(u)))\right|\Delta u\\ &\le K\delta+\varepsilon\int_{t_0}^{t_1} e_{\ominus a}(t_1,u)\left[L_g|b(u)|+L_f|c(u)|+L_1(u)+L_2(u)\right]\Delta u\\ &\le K\delta+\alpha\varepsilon\\ &<\varepsilon.\end{aligned}$$

This is a contradiction. If $\left|x^{\triangle}(t_1)\right|=\varepsilon$, then it follows from (2.4) and (H_2) and (H_4) that

$$\begin{aligned}&\left|x^{\triangle}(t_1)\right|\\ &\le|\varphi(t_0)\,a(t_1)|\,e_{\ominus a}(\sigma(t_1),t_0)+|b(t_1)|\,|g(x(t_1))|\\ &+|c(t_1)|\left|f\left(x^{\tilde{\triangle}}(t_1-\tau_1(t_1))\right)\right|+|q(t_1,x(t_1),x(t_1-\tau_2(t_1)))|\\ &+|a(t_1)|\int_{t_0}^{\sigma(t_1)} e_{\ominus a}(\sigma(t_1),u)\left|b(u)\,g(x(u))+c(u)\,f\left(x^{\tilde{\triangle}}(u-\tau_1(u))\right)+q(u,x(u),x(u-\tau_2(u)))\right|\Delta u\\ &\le KA\delta+\varepsilon\left\{|a(t_1)|\int_{t_0}^{\sigma(t_1)} e_{\ominus a}(\sigma(t_1),u)\left[L_g|b(u)|+L_f|c(u)|+L_1(u)+L_2(u)\right]\Delta u\right.\\ &\left.+L_g|b(t_1)|+L_f|c(t_1)|+L_1(t_1)+L_2(t_1)\right\}\\ &\le KA\delta+\alpha\varepsilon\\ &<\varepsilon.\end{aligned}$$

This is also a contradiction. Hence, the zero solution of (0.1) is stable in C^1_{rd}. This, together with (2.8), implies that the zero solution of (0.1) is globally asymptotically stable in C^1_{rd}.

(ii) Assume that the zero solution of (0.1) is globally asymptotically stable in C^1_{rd}. Now we prove that (2.1) holds. Otherwise, set

$$l=\liminf_{t\mapsto\infty}\int_0^t\frac{1}{\mu(s)}\log(1+\mu(s)\,a(s))\,\Delta s,\ K_0=\sup_{t\in[0,\infty)\cap\mathbb{T}}\{e_{\ominus a}(t,0)\},\ A_0=\sup_{t\in[0,\infty)\cap\mathbb{T}}\{|a(t)|\},$$

thus it follows from (H_3) that $l\in(-\infty,\infty)$, $K_0\in(0,\infty)$, $A_0\in[0,\infty)$. Hence, there exists an increasing sequence $\{t_n\}\subset[0,\infty)\cap\mathbb{T}$ such that $\lim_{t\to\infty}t_n=\infty$ and

$$\lim_{n\mapsto\infty}\int_0^{t_n}\frac{1}{\mu(s)}\log(1+\mu(s)\,a(s))\,\Delta s=l. \tag{2.9}$$

Denote

$$I_n=\int_0^{t_n} e_a(u,0)\left[L_g|b(u)|+L_f|c(u)|+L_1(u)+L_2(u)\right]\Delta u,\quad n=1,2,....$$

From (H_4), it follows that

$$I_n = e_a(t_n, 0) \int_0^{t_n} e_{\ominus a}(t_n, u) [L_g |b(u)| + L_f |c(u)| + L_1(u) + L_2(u)] \Delta u \leq \alpha e_a(t_n, 0).$$

This, together with (2.9), implies that the sequence $\{I_n\}$ is bounded. Further, there exists a convergent subsequence. For brevity of notation, we may assume that $\{I_n\}$ is convergent. Therefore, there exists a positive integer m such that for any integer $n > m$,

$$\int_{t_m}^{t_n} e_a(u, 0) [L_g |b(u)| + L_f |c(u)| + L_1(u) + L_2(u)] \Delta u < \frac{1-\alpha}{8B(e^{-l}+1)}, \tag{2.10}$$

and

$$e_{\ominus a}(t_n, t_m) > \frac{1}{2}, \quad e_{\ominus a}(t_n, 0) < e^{-l} + 1, \quad e_a(t_m, 0) < e^l + 1, \tag{2.11}$$

where $B = \max\left\{K_0\left(e^l+1\right), K_0 A_0\left(e^l+1\right), 1\right\}$.

For any $\delta > 0$, consider the solution $x(t) = x(t, t_m, \varphi)$ of (0.1) with $|\varphi|_{t_m} < \delta$ and $|\varphi(t_m)| > \delta/2$. It follows from (2.3), (2.4), (2.11), (H_2) and (H_4) that for $t \in [t_m, \infty) \cap \mathbb{T}$,

$$\begin{aligned}
&|x(t)| \\
&\leq |\varphi(t_m)| e_{\ominus a}(t, t_m) \\
&+ \int_{t_m}^{t} e_{\ominus a}(t, u) \left| b(u) g(x(u)) + c(u) f\left(x^{\widetilde{\triangle}}(u - \tau_1(u))\right) + q(u, x(u), x(u - \tau_2(u)))\right| \Delta u \\
&\leq |\varphi(t_m)| e_{\ominus a}(t, 0) e_a(t_m, 0) + \|x\|_{t_m} \int_{t_m}^{t} e_{\ominus a}(t, u) [L_g |b(u)| + L_f |c(u)| + L_1(u) + L_2(u)] \Delta u \\
&\leq K_0\left(e^l+1\right)\delta + \|x\|_{t_m} \int_0^t e_{\ominus a}(t, u) [L_g |b(u)| + L_f |c(u)| + L_1(u) + L_2(u)] \Delta u \\
&\leq B\delta + \alpha \|x\|_{t_m},
\end{aligned}$$

and

$$\begin{aligned}
&\left|x^{\triangle}(t)\right| \\
&\leq |\varphi(t_m)| |a(t)| e_{\ominus a}(\sigma(t), t_m) + |b(t)| |g(x(t))| \\
&+ |c(t)| \left| f\left(x^{\widetilde{\triangle}}(t - \tau_1(t))\right)\right| + |q(t, x(t), x(t - \tau_2(t)))| \\
&+ |a(t)| \int_{t_m}^{\sigma(t)} e_{\ominus a}(\sigma(t), u) \left| b(u) g(x(u)) + c(u) f\left(x^{\widetilde{\triangle}}(u - \tau_1(u))\right) + q(u, x(u), x(u - \tau_2(u)))\right| \Delta u \\
&\leq K_0 A_0 \left(e^l + 1\right)\delta \\
&+ \|x\|_{t_m} \left\{ |a(t)| \int_{t_m}^{\sigma(t)} e_{\ominus a}(\sigma(t), u) [L_g |b(u)| + L_f |c(u)| + L_1(u) + L_2(u)] \Delta u \right. \\
&\left. + L_g |b(t)| + L_f |c(t)| + L_1(t) + L_2(t) \right\} \\
&\leq B\delta + \alpha \|x\|_{t_m}.
\end{aligned}$$

Hence, $\|x\|_{t_m} \leq B\delta + \alpha \|x\|_{t_m}$, i.e.

$$\|x\|_{t_m} \leq \frac{B}{1-\alpha}\delta. \tag{2.12}$$

It follows from (2.3),(2.10)-(2.12) and (H_2) that for any $n > m$,

$$\begin{aligned}
&|x(t_n)| \\
&\geq |\varphi(t_m)|\, e_{\ominus a}(t_n, t_m) \\
&- e_{\ominus a}(t_n, 0) \int_{t_m}^{t_n} e_a(u, 0) \left| b(u)\, g(x(u)) + c(u)\, f\left(x^{\tilde{\Delta}}(u - \tau_1(u))\right) + q(u, x(u), x(u - \tau_2(u))) \right| \Delta u \\
&\geq |\varphi(t_m)|\, e_{\ominus a}(t_n, t_m) - \|x\|_{t_m} e_{\ominus a}(t_n, 0) \int_{t_m}^{t_n} e_a(u, 0) \left[L_g |b(u)| + L_f |c(u)| + L_1(u) + L_2(u)\right] \Delta u \\
&> \frac{1}{4}\delta - \frac{B}{1-\alpha}\delta \left(e^{-l} + 1\right) \frac{1-\alpha}{8B\left(e^{-l}+1\right)} = \frac{1}{8}\delta.
\end{aligned}$$

This contradicts the fact that $\lim_{n\to\infty} t_n = \infty$ and the zero solution of (0.1) is globally asymptotically stable in C^1_{rd}. The proof is complete. □

Example 2.4. *Let $\mathbb{T} = \mathbb{R}$. Consider the following neutral differential equation*

$$x'(t) = -a(t)\, x(t) + b(t)\, g(x(t)) + c(t)\, f(x'(t - \tau_1(t))) + q(t, x(t), x(t - \tau_2(t))), \tag{2.13}$$

where

$$\begin{aligned}
\tau_1(t) &= t/2 + 1,\ \ \tau_2(t) = t/3 + 2,\ \ a(t) = \frac{1}{t+1},\ \ b(t) = \frac{1}{15(t+1)}, \\
c(t) &= \frac{1}{12(t+1)},\ \ g(x) = 1 - \cos(x),\ \ f(x) = \sin(x), \\
q(t, x, y) &= \frac{1}{16(t+1)} \sin(x + y).
\end{aligned}$$

Obviously $a, b, c \in C([0,\infty), \mathbb{R})$, $g, f \in C(\mathbb{R}, \mathbb{R})$, $q \in C([0,\infty) \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $\tau_1, \tau_2 \in C([0,\infty), (0,\infty))$ *with* $t - \tau_i(t) \to \infty$ *as* $t \to \infty$, $i = 1, 2$. *A simple calculation shows that*

$$g(0) = f(0) = q(t, 0, 0) = 0,\ \int_0^\infty a(s)\, ds = \infty,$$

$$L_1(t) = L_2(t) = \frac{1}{16(t+1)},\ \ L_g = 1,\ \ L_f = 1,$$

$$\int_0^t e^{-\int_u^t a(s) ds} \left[L_g |b(u)| + L_f |c(u)| + L_1(u) + L_2(u)\right] du \leq \frac{11}{40},$$

and

$$\begin{aligned}
a(t) \int_0^t e^{-\int_u^t a(s) ds} &\left[L_g |b(u)| + L_f |c(u)| + L_1(u) + L_2(u)\right] du \\
+ L_g |b(t)| + L_f |c(t)| + L_1(t) + L_2(t) &\leq \frac{22}{40}.
\end{aligned}$$

It is easy to see that all the conditions of Theorem 2.3 hold for $\alpha = \frac{22}{40} < 1$. Thus, Theorem 2.3 implies that the zero solution of (2.13) is globally asymptotically stable in C^1.

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Ball comparison between Jarratt's and other fourth order method for solving equations

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ABSTRACT

The convergence order of iterative methods is determined using high order derivatives and Taylor series, and without providing computable error bounds, uniqueness of the solution results or information on how to choose the initial point. We address all these problems by using hypotheses only on the first derivative. Moreover, to achieve all these we present our technique using a comparison between the convergence radii of Jarratt's fourth order method and another method of the same convergence order.

RESUMEN

El orden de convergencia de métodos iterativos es determinado usando derivadas de orden alto y series de Taylor, y sin poder entregar cotas de error calculables, resultados de unicidad de soluciones o información de cómo elegir el punto inicial. Tratamos estos problemas usando hipótesis sólo en la primera derivada. Más aún, para responder todos los anteriores, presentamos una técnica que usa una comparación entre el radio de convergencia del método de cuarto orden de Jarratt y otro método con el mismo orden de convergencia.

Keywords and Phrases: Jarratt method; Banach space; Ball convergence.

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1 Introduction

Let $\mathcal{B}_1$ and $\mathcal{B}_2$ stand for Banach spaces, with $\Omega \subseteq \mathcal{B}_1$ being nonempty, open and convex. Consider an equation

$$F(x) = 0, \tag{1.1}$$

where $F : \Omega \longrightarrow \mathcal{B}_2$ is a differentiable in the of Fréchet-sense. The task of finding a solution p of equation (1.1) is very difficult in general. It is even harder to find a solution p in closed form, since this can be achieved in some special cases. That explains why most authors develop iterative methods, to generate a sequence approximating p under some initial conditions.

Notice that, solution methods for equation (1.1) is an important area of research, since a plethora of problems from diverse disciplines such that Mathematics, Optimization, Mathematical Programming, Chemistry, Biology, Physics, Economics, Statistics, Engineering and other disciplines can be modeled into an equation of the form (1.1) using mathematical modeling [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16]. The most popular method is without a doubt Newton's method (NM)

$$x_{n+1} = x_n - F'(x_n)^{-1}F(x_n),\ x_0 \in \Omega,\ \text{and all } n = 0, 1, 2, \ldots. \tag{1.2}$$

NM converges quadratically to p for x_0 sufficiently close to p [10]. To increase the convergence order numerous methods have been proposed [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16]. The order of these methods is almost exclusively been obtained using Taylor series, and hypotheses on high order derivatives. No computable error bounds or uniqueness results are given, and the choice of the initial point is a shot in the dark.

Iterative methods are usually studied based on: semi-local and local convergence. The semi-local convergence matter is, based on the information around an initial point, to give conditions ensuring the convergence of the iterative procedure; while the local one is, based on the information around a solution, to find estimates of the radii of convergence balls [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16].

A radius of convergence about p determines a ball such that if an initial point is selected from that ball convergence of the method to p is guaranteed. To deal with all these problems we have selected two popular fourth order methods. In particular, we compare the radii of convergence of fourth order Jarratt's iterative method defined [9, 12] for $n = 0, 1, 2, \ldots,$ as

$$\begin{aligned} y_n &= x_n - \frac{2}{3}F'(x_n)^{-1}F(x_n) \\ x_{n+1} &= x_n - \frac{1}{2}[(3F'(y_n) - F'(x_n))^{-1}(3F'(y_n) + F'(x_n))] \\ &\quad \times F'(x_n)^{-1}F(x_n), \end{aligned} \tag{1.3}$$

to the fourth order Sharma's method [13] defined for $n = 0, 1, 2, \ldots,$ as

$$\begin{aligned} y_n &= x_n - \frac{2}{3}F'(x_n)^{-1}F(x_n) \\ x_{n+1} &= x_n - \frac{1}{2}[-I + \frac{9}{4}F'(y_n)^{-1}F'(x_n) + \frac{3}{4}F'(x_n)^{-1}F'(y_n)] \\ &\quad \times F'(x_n)^{-1}F(x_n). \end{aligned} \tag{1.4}$$

Earlier convergence analysis of these methods, in the special case when $\mathcal{B}_1 = \mathcal{B}_2 = \mathbb{R}^k$ used, assumptions of the Fréchet derivatives of F of order up to five [9, 12, 13]. But these assumptions limit the applicability of methods (1.3) and (1.4).

Let as an example, $\mathcal{B}_1 = \mathcal{B}_2 = \mathbb{R}$, $\Omega = [-\frac{1}{2}, \frac{3}{2}]$. Define F on Ω as

$$F(x) = x^3 \log x^2 + x^5 - x^4$$

Then, we have $p = 1$, and

$$F'(x) = 3x^2 \log x^2 + 5x^4 - 4x^3 + 2x^2,$$

$$F''(x) = 6x \log x^2 + 20x^3 - 12x^2 + 10x,$$

$$F'''(x) = 6 \log x^2 + 60x^2 = 24x + 22.$$

Clearly, $F'''(x)$ is not bounded on Ω. So, methods (1.3) and (1.4) cannot be applied to solve the above example, if we use the analysis in the earlier studies. In this study, our analysis uses only the assumptions on the first Fréchet derivative of F.

Moreover, we provide computable upper estimates on $\|x_n - p\|$, a radius of convergence as well as uniqueness results based on generalized Lipschitz conditions. Hence, we extend the applicability of these methods. Our technique can be used to extend the applicability of other high order methods along the same lines.

The rest of the study is organized as follows. In Section 2 , the local convergence analysis is given and numerical examples are given in the last Section 4.

2 Local convergence

It is convenient for the local convergence analysis of method (1.3) and method (1.4) to introduce some fucntions and parameters. First for method (1.3): Let $\omega_0 : S \longrightarrow S$ be a continuous and increasing function with $w_0(0) = 0$, where $S = [0, \infty)$. Suppose that equation

$$\omega_0(t) = 1 \tag{2.1}$$

has at least one positive solution. Denote by ρ_0 the smallest such solution. Set $S_0 = [0, \rho_0)$. Let also $\omega : S_0 \longrightarrow S$ and $\omega_1 : S_0 \longrightarrow S$ be continuous and increasing functions with $\omega(0) = 0$. Define functions φ_1 and $\bar{\varphi}_1$ on the interval S_0 by

$$\varphi_1(t) = \frac{\int_0^1 \omega((1-\theta)t)d\theta + \frac{1}{3}\int_0^1 \omega_1(\theta t)d\theta}{1-\omega_0(t)}$$

and

$$\bar{\varphi}_1(t) = \varphi_1(t) - 1.$$

Suppose that

$$\omega_0(0) < 3. \tag{2.2}$$

Then, we get by (2.2) that $\bar{\varphi}_1(0) < 0$ and $\bar{\varphi}_1(t) \longrightarrow \infty$ as $t \longrightarrow \rho_0^-$. The intermediate value theorem guarantees that equation $\bar{\varphi}_1(t) = 0$ has at least one solution in $(0, \rho_0)$. Denote by R_1 the smallest such solution. Suppose that equation

$$\omega_0(\varphi_1(t)t) = 1 \tag{2.3}$$

has at least one positive solution. Denote by ρ_1 the smallest such solution. Set $\rho = \min\{\rho_0, \rho_1\}$ and $S_1 = [0, \rho)$. Define functions φ_2 and $\bar{\varphi}_2$ on S_1 by

$$\begin{aligned}\varphi_2(t) \quad = \quad & \frac{\int_0^1 \omega((1-\theta)t)d\theta}{1-\omega_0(t)} + \frac{3}{8}[\frac{(\omega_0(\varphi_1(t)t)+\omega_0(t))^2}{(1-\omega_0(t))(1-\omega_0(\varphi_1(t)t))} \\ & +2\frac{w_0(\varphi_1(t)t)+\omega_0(t)}{1-\omega_0(\varphi_1(t)t)}]\frac{\int_0^1 \omega_1(\theta t)d\theta}{1-\omega_0(t)}\end{aligned}$$

and

$$\bar{\varphi}_2(t) = \varphi_2(t) - 1.$$

We get that $\bar{\varphi}_2(0) = -1$ and $\bar{\varphi}_2(t) \longrightarrow \infty$ as $t \longrightarrow \rho^-$. Denote by R_2 the smallest such solution of equation $\bar{\varphi}_2(t) = 0$. Moreover, define a radius of convergence R by

$$R = \min\{R_1, R_2\}. \tag{2.4}$$

It follows that for each $t \in [0, R)$

$$0 \le \omega_0(t) < 1 \tag{2.5}$$

$$0 \le \omega_0(\varphi_1(t)t) < 1 \tag{2.6}$$

$$0 \le \varphi_1(t) < 1 \tag{2.7}$$

and

$$0 \le \varphi_2(t) < 1. \tag{2.8}$$

Let us introduce conditions (A):

(a1) $F : \Omega \longrightarrow \mathcal{B}_2$ is continuously differentiable in the sense of Fréchet and there exists $p \in \Omega$ such that $F(p) = 0$ and $F'(p)^{-1} \in \mathcal{L}(\mathcal{B}_2, \mathcal{B}_1)$.

(a2) There exists function $\omega_0 : S \longrightarrow S$ continuous and increasing with $\omega_0(0) = 0$ and for each $x \in \Omega$

$$\|F'(p)^{-1}(F'(x) - F'(p))\| \leq \omega_0(\|x - p\|)$$

and (2.2) holds. Set $\Omega_0 = \Omega \cap U(p, \rho_0)$, where ρ_0 is given in (2.1).

(a3) There exist functions $\omega : S_0 \longrightarrow S, \omega_1 : S_0 \longrightarrow S$ continuous and increasing with $\omega(0) = 0$ such that for each $x, y \in \Omega_0$

$$\|F'(p)^{-1}(F'(y) - F'(x))\| \leq \omega(\|y - x\|)$$

and

$$\|F'(p)^{-1}F'(x)\| \leq \omega_1(\|x - p\|).$$

(a4) $\bar{U}(p, R) \subset \Omega, \rho_0, \rho_1$ exist and are given by (2.1) and (2.3), respectively.

(a5) There exists $R^* \geq R$ such that

$$\int_0^1 \omega_0(\theta R^*) d\theta < 1.$$

Set $\Omega_1 = \Omega \cap \bar{U}(p, R^*)$.

Next, the local convergence analysis is given for method (1.3) based on the conditions (A) and the preceding notation.

Theorem 2.1. *Suppose that the conditions (A) hold. Then, sequence* $\{x_n\}$ *generated by (1.3), starting at* $x_0 \in U(p, R) - \{p\}$ *is well defined, remains in* $U(p, R)$ *for each* $n = 0, 1, 2, 3, \ldots$ *and converges to* p*. Moreover, the following error bounds hold*

$$\|y_n - p\| \leq \varphi_1(\|x - p\|)\|x - p\| \leq \|x - p\| < R \tag{2.9}$$

and

$$\|x_{n+1} - p\| \leq \varphi_2(\|x - p\|)\|x - p\| \leq \|x - p\|, \tag{2.10}$$

where functions φ_1 *and* φ_2 *are given previously and* R *is defined in (2.4). Furthermore, the limit point* p *is the only solution of equation* $F(x) = 0$ *in the set* Ω_1*, which is defined in (a5).*

Proof. Mathematical induction is utilized to show (2.9) and (2.10). Let $x \in U(p, R) - \{p\}$. Then, by (a1), (a2), (2.1), (2.4) and (2.5), we obtain in turn that

$$\|F'(p)^{-1}(F'(x) - F'(p))\| \leq \omega_0(\|x - p\|) \leq \omega_0(R) < 1. \tag{2.11}$$

In view of (2.11) and the Banach lemma on invertible operators [7, 8, 10], $F'(x)^{-1} \in \mathcal{L}(\mathcal{B}_2, \mathcal{B}_1)$ and

$$\|F'(x)^{-1}F'(p)\| \leq \frac{1}{1 - \omega(\|x - p\|)}. \tag{2.12}$$

The point y_0 is well defined by the first substep of method (1.3) and (2.12) for $x = x_0$. We can write by (a1)

$$F(x) = F(x) - F(p) = \int_0^1 F'(p + \theta(x - p))(x - p)d\theta. \tag{2.13}$$

Then, by the second hypothesis in (a3), we get by (2.13) that

$$\|F'(p)^{-1}F'(p)\| \leq \int_0^1 \omega_1(\theta\|x - p\|)d\theta\|x - p\|. \tag{2.14}$$

Using the first substep of method (1.3) for $n = 0$, (a3), (2.4), (2.7), (2.12) (for $x = x_0$) and (2.14), we have in turn from

$$y_0 - p = x_0 - p - F'(x_0)^{-1}F(x_0) + \frac{1}{3}F'(x_0)^{-1}F(x_0)$$

that

$$\begin{aligned}
\|y_0 - p\| &\leq \|F'(x_0)^{-1}F'(p)\|\|\int_0^1 F'(p)^{-1}(F'(p + \theta(x_0 - p)) - F'(x_0))d\theta(x - p)\| \\
&\quad \frac{1}{3}\|F'(x_0)^{-1}F'(p)\|\|F'(p)^{-1}F(x_0)\| \\
&\leq \frac{[\int_0^1 \omega((1 - \theta)\|x_0 - p\|)d\theta + \frac{1}{3}\int_0^1 \omega_1(\theta\|x_0 - p\|)d\theta]}{1 - \omega_0(\|x_0 - p\|)} \\
&\quad \times\|x_0 - p\| \\
&= \varphi_1(\|x_0 - p\|)\|x_0 - p\| \leq \|x_0 - p\| < R,
\end{aligned} \tag{2.15}$$

which implies that (2.9) holds for $n = 0$ and $y_0 \in U(p, R)$. Moreover, $F'(y_0)^{-1} \in \mathcal{L}(\mathcal{B}_2, \mathcal{B}_1)$, so x_1 is well defined by the second substep of method (1.3) for $n = 0$ and (2.6). Furthermore, by (2.4), (2.8), (2.12) (for $x = y_0$), (2.14) (for $x = y_0$) and the estimate

$$\begin{aligned}
x_1 - p &= x_0 - p - F'(x_0)^{-1}F(x_0) \\
&\quad -\frac{1}{2}[-3I + \frac{9}{4}F'(y_0)^{-1}F'(x_0) \\
&\quad +\frac{3}{4}F'(x_0)^{-1}F'(y_0)]F'(x_0)^{-1}F(x_0) \\
&= x_0 - p - F'(x_0)^{-1}F(x_0) \\
&\quad -\frac{3}{2}[-I + \frac{3}{4}F'(y_0)^{-1}F'(x_0) \\
&\quad +\frac{1}{4}F'(x_0)^{-1}F'(y_0)]F'(x_0)^{-1}F(x_0) \\
&= x_0 - p - F'(x_0)^{-1}F(x_0) \\
&\quad -\frac{3}{8}[F'(x_0)^{-1}(F'(y_0) - F'(x_0))F'(y_0)^{-1}(F'(y_0) - F'(x_0)) \\
&\quad -2F'(y_0)^{-1}(F'(y_0) - F'(x_0)]F'(x_0)^{-1}F(x_0),
\end{aligned} \tag{2.16}$$

we have in turn that

$$\begin{aligned}
\|x_1 - p\| &\leq \|x_0 - p - F'(x_0)^{-1}F(x_0)\| \\
&+\frac{3}{8}[\|F'(x_0)^{-1}F'(p)\|(\|F'(p)^{-1}(F'(y_0) - F'(x_0))\| \\
&+\|F'(p)^{-1}(F'(x_0) - F'(p))\|)^2 \\
&\|F'(y_0)^{-1}F'(p)\| \\
&+2\|F'(y_0)^{-1}F'(p)\|(\|F'(p)^{-1}(F'(y_0) - F'(x_0)\| \\
&+\|F'(p)^{-1}(F'(x_0) - F'(p))\|)] \\
&\|F'(x_0)^{-1}F'(p)\|\|F'(p)^{-1}F(x_0)\| \\
&\leq \left\{\frac{\int_0^1 \omega((1-\theta)\|x_0 - p\|)d\theta}{1 - \omega_0(\|x_0 - p\|)}\right. \\
&\frac{3}{8}\left[\frac{(\omega_0(\|y_0 - p\|) + \omega_0(\|x_0 - p\|))^2}{(1 - \omega_0(\|x_0 - p\|))(1 - \omega_0(\|y_0 - p\|))}\right. \\
&\left.+2\ \frac{\omega_0(\|x_0 - p\|) + \omega_0(\|y_0 - p\|)}{1 - \omega_0(\|y_0 - p\|)}\right] \\
&\left.\frac{\int_0^1 \omega_1(\theta\|x_0 - p\|)d\theta}{1 - \omega_0(\|x_0 - p\|)}\right\}\|x_0 - p\| \\
&\leq \varphi_2(\|x_0 - p\|)\|x_0 - p\| \leq \|x_0 - p\|, \qquad (2.17)
\end{aligned}$$

so (2.10) holds for $n = 0$ and $x_1 \in U(p, R)$, where we also used the following estimates in the derivativation of (2.16):

$$\begin{aligned}
&-I + \frac{3}{4}F'(y_0)^{-1}F'(x_0) + \frac{1}{4}F'(x_0)^{-1}F'(y_0) \qquad (2.18) \\
= &-\frac{3}{4}I + \frac{3}{4}F'(y_n)^{-1}F'(x_n) - \frac{1}{4}I + \frac{1}{4}F'(x_0)^{-1}F'(y_0) \\
= &\ \frac{3}{4}[F'(y_0)^{-1}F'(x_0) - I] + \frac{1}{4}[F'(x_0)^{-1}F'(y_0) - I] \\
= &\ \frac{3}{4}F'(y_0)^{-1}(F'(x_0) - F'(y_0)) + \frac{1}{4}F'(x_0)^{-1}(F'(y_0) - F'(x_0)) \\
= &\ \frac{1}{4}F'(x_0)^{-1}(F'(y_0) - F'(x_0)) - \frac{1}{4}F'(y_0)^{-1}(F'(y_0) - F'(x_0)) \\
&-\frac{2}{4}F'(y_0)^{-1}(F'(y_0) - F'(x_0)) \\
= &\ \frac{1}{4}(F'(x_0)^{-1} - F'(y_0)^{-1})(F'(y_0) - F'(x_0)) \\
&-\frac{1}{2}F'(y_0)^{-1}(F'(y_0) - F'(x_0)) \\
= &\ \frac{1}{4}F'(x_0)^{-1}(F'(y_0) - F'(x_0))F'(y_0)^{-1}(F'(y_0) - F'(x_0)) \\
&-\frac{1}{2}F'(y_0)^{-1}(F'(y_0) - F'(x_0)). \qquad (2.19)
\end{aligned}$$

The induction for (2.9) and (2.10) is completed, if x_m, y_m, x_{m+1} replace x_0, y_0, x_1 in the preceding estimations. Then, from the estimate

$$\|x_{m+1} - p\| \leq r\|x_m - p\| < R,\ r = \varphi_2(\|x_0 - p\|) \in [0, 1) \tag{2.20}$$

we conclude that $\lim_{m\longrightarrow\infty} x_m = p$ and $x_{m+1} \in U(p, R)$. Finally, let $G = \int_0^1 F'(p_1 + \theta(p - p_1))d\theta$ for $p_1 \in \Omega_1$ with $F(p_1) = 0$. Then, by (a2), we get that

$$\begin{aligned}\|F'(p)^{-1}(G - F'(p))\| &\leq \int_0^1 \omega_0(\theta\|p - p_1\|)d\theta \\ &\leq \int_0^1 \omega_0(\theta R^*)d\theta < 1 \end{aligned}\tag{2.21}$$

leading to $G^{-1} \in \mathcal{L}(\mathcal{B}_2, \mathcal{B}_1)$. Then, from the identity

$$0 = F(p) - F(p_1) = G(p - p_1),$$

we deduce that $p_1 = p$.

Next, we study the local convergence analysis of method (1.4) in an analogous way. Let $\omega_0, \omega, \omega_1, \rho_0, \varphi_1$ and $\bar{\varphi}_1$ are previously. Suppose that equation

$$q(t) = 1 \tag{2.22}$$

where $q(t) = \frac{1}{2}(3\omega_0(\varphi_1(t)t) + \omega_0(t))$ has at least one positive solution. Denote by ρ_1 the smallest such solution. Set $D_1 = [0, \rho)$ where $\rho = \min\{\rho_0, \rho_1\}$. Define functions φ_3 and $\bar{\varphi}_3$ on D_1 by

$$\begin{aligned}\varphi_3(t) &= \frac{\int_0^1 \omega((1-\theta)t)d\theta}{1 - \omega_0(t)} \\ &+ \frac{3}{2}\frac{(\omega_0(t) + \omega_0(\varphi_1(t)t))\int_0^1 \omega_1(\theta t)d\theta}{(1 - q(t))(1 - \omega_0(t))}\end{aligned}$$

and

$$\bar{\varphi}_3 = \varphi_3 - 1.$$

We get $\bar{\varphi}_3(t) = -1$ and $\bar{\varphi}_3(t) \longrightarrow \infty$ as $t \longrightarrow \rho^-$. Denote by R_3 the smallest solution of equation $\bar{\varphi}_3 = 0$ in $(0, \rho)$. Define a radius of convergence R by

$$R = \min\{R_1, R_3\}. \tag{2.23}$$

Consider the conditions (A) again but with R given in (2.23) and ρ_1 given in (2.22). Call these conditions (A)'. Then, for each $t \in [0, R)$, we have

$$0 \leq \omega_0(t) < 1 \tag{2.24}$$

$$0 \leq q(t) < 1 \tag{2.25}$$

$$0 \leq \varphi_1(t) < 1 \tag{2.26}$$

and

$$0 \leq \varphi_3(t) < 1. \tag{2.27}$$

Theorem 2.2. *Suppose that the conditions (A) hold. Then, sequence* $\{x_n\}$ *defined by (1.4), starting at* $x_0 \in U(p, R) - \{p\}$ *is well defined, remains in* $U(p, R)$ *for each* $n = 0, 1, 2, 3, \ldots$ *and converges to* p*. Moreover, the following error bounds hold*

$$\|y_n - p\| \leq \varphi_1(\|x - p\|)\|x - p\| \leq \|x - p\| < R \tag{2.28}$$

and

$$\|x_{n+1} - p\| \leq \varphi_3(\|x - p\|)\|x - p\| \leq \|x - p\|, \tag{2.29}$$

where functions φ_1 *and* φ_3 *are given previously and* R *is defined in (2.23). Furthermore, the limit point* p *is the only solution of equation* $F(x) = 0$ *in the set* Ω_1*, which is defined previously.*

Proof. It follows as in Theorem 2.1 but notice

$$\begin{aligned}
& \|(2F'(p))^{-1}(3F'(y_0) - F'(x_0) - 3F'(p) + F'(p))\| \\
\leq\ & \frac{1}{2}(3\|F'(p)^{-1}(F'(y_0) - F'(p))\| \\
& + \|F'(p)^{-1}(F'(x_0) - F'(p))\|) \\
\leq\ & \frac{1}{2}(3\omega_0(\|y_0 - p\|) + \omega_0(\|x_0 - p\|)) \\
\leq\ & \frac{1}{2}(3\omega_0(\varphi_1(\|x_0 - p\|)\|x_0 - p\|) + \omega_0(\|x_0 - p\|) \\
=\ & q(\|x_0 - p\|) < 1
\end{aligned} \tag{2.30}$$

and

$$\begin{aligned}
x_1 - p &= x_0 - p - \frac{1}{2}[(3F'(y_0) - F'(x_0))^{-1}(3F'(y_0) - F'(x_0)) \\
& \quad + 2F'(x_0)]F'(x_0)^{-1}F(x_0) \\
&= x_0 - p - F'(x_0)^{-1}F(x_0) \\
& \quad - \frac{1}{2}[-I + 2(3F'(y_0) - F'(x_0))^{-1}]F'(x_0)^{-1}F(x_0) \\
&= x_0 - p - F'(x_0)^{-1}F(x_0) - \frac{3}{2}(3F'(y_0) - F'(x_0))^{-1} \\
& \quad \times (F'(x_0) - F'(y_0))F'(x_0)^{-1}F(x_0),
\end{aligned} \tag{2.31}$$

where for the derivation of (2.31), we also used the estimate

$$\begin{aligned}
& -I + 2(3F'(y_0) - F'(x_0))^{-1}F'(x_0) \\
=\ & (3F'(y_0) - F'(x_0))^{-1}[-(3F'(y_0) - F'(x_0)) + 2F'(x_0)] \\
=\ & 3(3F'(y_0) - F'(x_0))^{-1}[F'(x_0) - F'(y_0)],
\end{aligned}$$

so we get by (2.31)

$$
\begin{aligned}
\|x_1 - p\| &\leq \|x_0 - p - F'(x_0)^{-1}F(x_0)\| \\
&+\frac{3}{2}\|(3F'(y_0) - F'(x_0))^{-1}F'(p)\| \\
&\times[\|F'(p)^{-1}(F'(y_0) - F'(p))\| + \|F'(p)^{-1}(F'(x_0) - F'(p))\|] \\
&\times\|F'(x_0)^{-1}F'(p)\|\|F'(p)^{-1}F(p)\| \\
&\leq \Bigg[\frac{\int_0^1 \omega_0((1-\theta)\|x_0 - p\|)d\theta}{1 - \omega_0(\|x_0 - p\|)} \\
&+\frac{3}{2}\,\frac{(\omega_0(\|x_0 - p\|) + \omega_0(\|y_0 - p\|))\int_0^1 \omega_1(\theta\|x_0 - p\|)d\theta}{(1 - q(\|x_0 - p\|))(1 - \omega_0(\|x_0 - p\|))}\Bigg]\|x_0 - p\| \\
&\leq \varphi_3(\|x_0 - p\|)\|x_0 - p\| \leq \|x_0 - p\| < R,
\end{aligned} \tag{2.32}
$$

which shows (2.29) for $n = 0$ and $x_1 \in U(p, R)$. The rest of the proof as identical to the one in Theorem 2.1 is omitted.

□

Remark 2.3. *(a) Let* $\omega_0(t) = L_0 t, \omega(t) = Lt$. *Then, the radius* $r_A = \frac{2}{2L_0 + L}$ *was obtained by Argyros in [4] as the convergence radius for Newton's method under condition (2.12)-(2.14). Notice that the convergence radius for Newton's method given independently by Rheinboldt [14] and Traub [16] is given by*

$$\rho = \frac{2}{3L} < r_A,$$

where $\omega_1(t) = L_1 t$ *replaces* $\omega(t)$, *and* L_1 *is the Lipschitz constant on* Ω. *Notice that* $\Omega_0 \subseteq \Omega$, *so* $L_0 \leq L_1$ *and* $L \leq L_1$. *As an example, let us consider the function* $f(x) = e^x - 1$. *Then* $p = 0$. *Set* $D = U(0, 1)$. *Then, we have that* $L_0 = e - 1 < L = e^{\frac{1}{e-1}} < L_1 = e$, *so* $\rho = 0.24252961 < r_A = 0.3827$.

Moreover, the new error bounds [4, 5, 6, 7, 8] are:

$$\|x_{n+1} - p\| \leq \frac{L}{1 - L_0\|x_n - p\|}\|x_n - p\|^2,$$

whereas the old ones [10, 14, 16]

$$\|x_{n+1} - p\| \leq \frac{L}{1 - L\|x_n - p\|}\|x_n - p\|^2.$$

Clearly, the new error bounds are more precise, if $L_0 < L$. *Then, the radius of convergence of method (1.3) or method (1.4) cannot be larger than* r_A.

(b) The local results can be used for projection methods such as Arnoldi's method, the generalized minimum residual method(GMREM), the generalized conjugate method(GCM) for combined Newton/finite projection methods and in connection to the mesh independence principle in order to develop the cheapest and most efficient mesh refinement strategy [4, 5, 6, 7, 8, 10].

(c) *The results can be also be used to solve equations where the operator* F' *satisfies the autonomous differential equation [4, 5, 6, 7, 8, 10]:*

$$F'(x) = p(F(x)),$$

where p *is a known continuous operator. Since* $F'(p) = p(F(p)) = p(0)$, *we can apply the results without actually knowing the solution* p. *Let as an example* $F(x) = e^x - 1$. *Then, we can choose* $p(x) = x + 1$ *and* $p = 0$.

(d) *It is worth noticing that method (1.3) or method (1.4) are not changing, if we use the new instead of the old conditions [9, 12, 13]. Moreover, for the error bounds in practice we can use the computational order of convergence (COC)*

$$\xi = \frac{\ln \frac{\|x_{n+2} - x_{n+1}\|}{\|x_{n+1} - x_n\|}}{\ln \frac{\|x_{n+1} - x_n\|}{\|x_n - x_{n-1}\|}}, \qquad \text{for all } n = 1, 2, \ldots$$

or the approximate computational order of convergence (ACOC)

$$\xi^* = \frac{\ln \frac{\|x_{n+2} - p\|}{\|x_{n+1} - p\|}}{\ln \frac{\|x_{n+1} - p\|}{\|x_n - p\|}}, \qquad \text{for all } \; n = 0, 1, 2, \ldots$$

instead of the error bounds obtained in Theorem 2.1. Notice that these formulae do not require high order derivatives and in the case of ACOC not even knowledge of p. *The convergence radii are optimum under conditions (A).*

(e) *In view of (a2) and the estimate*

$$\begin{aligned} \|F'(p)^{-1}F'(x)\| &= \|F'(p)^{-1}(F'(x) - F'(p)) + I\| \\ &\leq 1 + \|F'(p)^{-1}(F'(x) - F'(p))\| \leq 1 + L_0\|x - p\| \end{aligned}$$

second condition in (a3) can be dropped and M *can be replaced by*

$$M(t) = 1 + L_0 t$$

or

$$M(t) = M = 2,$$

since $t \in [0, \frac{1}{L_0})$.

3 Numerical examples

Example 3.1. *Let* $\mathcal{B}_1 = \mathcal{B}_2 = \mathbb{R}^3, \Omega = \bar{U}(0, 1), x^* = (0, 0, 0)^T$. *Define function* F *on* Ω *for* $u = (x, y, z)^T$ *by*

$$F(u) = (e^x - 1, \frac{e - 1}{2}y^2 + y, z)^T.$$

Then, the Fréchet-derivative is given by

$$F'(v) = \begin{bmatrix} e^x & 0 & 0 \\ 0 & (e-1)y+1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Notice that using the (2.8)-(2.12), conditions, we get $\omega_0(t) = (e-1)t, \omega(t) = e^{\frac{1}{e-1}}t, \omega_1(t) = e^{\frac{1}{e-1}}$.

Then using the definition of r*, we have that*
$R_1 = 0.15440695135715407082521721804369 = R$
and
$R_2 = 0.17352535186531112265662102345232.$

Example 3.2. *Let* $\mathcal{B}_1 = \mathcal{B}_2 = C[0,1]$*, the space of continuous functions defined on* $[0,1]$ *and be equipped with the max norm. Let* $\Omega = \overline{U}(0,1)$*. Define function* F *on* Ω *by*

$$F(\varphi)(x) = \varphi(x) - 5\int_0^1 x\theta\varphi(\theta)^3 d\theta. \tag{3.1}$$

We have that

$$F'(\varphi(\xi))(x) = \xi(x) - 15\int_0^1 x\theta\varphi(\theta)^2\xi(\theta)d\theta, \text{ for each } \xi \in \Omega.$$

Then, we get that $x^* = 0$*,* $\omega_0(t) = 7.5t, \omega(t) = 15t, \omega_1(t) = 2$*. This way, we have that*
$R_1 = 0.022222222222222222222222222222222 = R$
and
$R_2 = 0.18929637111931424398036938328005.$

Example 3.3. *Let* $\mathcal{B}_1 = \mathcal{B}_2 = \mathbb{R}$*,* $\Omega = [-\frac{1}{2}, \frac{3}{2}]$*. Define* F *on* Ω *by*

$$F(x) = x^3 \log x^2 + x^5 - x^4$$

Then

$$F'(x) = 3x^2 \log x^2 + 5x^4 - 4x^3 + 2x^2,$$

Then, we get that $\omega_0(t) = \omega(t) = 147t, \omega_1(t) = 2$*. So, we obtain*
$R_1 = 0.0015117157974300831443688586545729 = R$
and
$R_2 = 0.012972957123775621934846924432 35.$

Example 3.4. *Let* $\mathcal{B}_1 = \mathcal{B}_2 = C[0,1], \Omega = \bar{U}(x^*, 1)$ *and consider the nonlinear integral equation of the mixed Hammerstein-type [1, 2, 3, 5, 11] defined by*

$$x(s) = \int_0^1 G(s,t)(x(t)^{3/2} + \frac{x(t)^2}{2})dt,$$

where the kernel G *is the Green's function defined on the interval* $[0,1]\times[0,1]$ *by*

$$G(s,t)=\begin{cases}(1-s)t, & t\le s\\ s(1-t), & s\le t.\end{cases}$$

The solution $x^*(s)=0$ *is the same as the solution of equation (1.1), where* $F: C[0,1]\longrightarrow C[0,1])$ *is defined by*

$$F(x)(s)=x(s)-\int_0^1 G(s,t)(x(t)^{3/2}+\frac{x(t)^2}{2})dt.$$

Notice that

$$\|\int_0^1 G(s,t)dt\|\le\frac{1}{8}.$$

Then, we have that

$$F'(x)y(s)=y(s)-\int_0^1 G(s,t)(\frac{3}{2}x(t)^{1/2}+x(t))dt,$$

so since $F'(x^*(s))=I$,

$$\|F'(x^*)^{-1}(F'(x)-F'(y))\|\le\frac{1}{8}(\frac{3}{2}\|x-y\|^{1/2}+\|x-y\|).$$

Then, we get that $\omega_0(t)=\omega(t)=\frac{1}{8}(\frac{3}{2}t^{1/2}+t), \omega_1(t)=1+\omega_0(t)$. *So, we obtain* $R_1=1.2$
and
$R_2=0.827576326349172219920546922367 07=R$.

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The basic ergodic theorems, yet again

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ABSTRACT

A generalization of Rokhlin's Tower Lemma is presented. The Maximal Ergodic Theorem is then obtained as a corollary. We also use the generalized Rokhlin lemma, this time combined with a subadditive version of Kac's formula, to deduce a subadditive version of the Maximal Ergodic Theorem due to Silva and Thieullen.

In both the additive and subadditive cases, these maximal theorems immediately imply that "heavy" points have positive probability. We use heaviness to prove the pointwise ergodic theorems of Birkhoff and Kingman.

RESUMEN

Se presenta una generalización del Lema de la Torre de Rokhlin. El Teorema Ergódico Maximal se obtiene como corolario. También usamos el lema de Rokhlin generalizado, esta vez combinado con una versión subaditiva de la fórmula de Kac, para deducir una versión subaditiva del Teorema Ergódico Maximal obtenida por Silva y Thieullen.

Tanto en el caso aditivo como en el subaditivo, estos teoremas maximales inmediatamente implican que puntos "pesados" tienen probabilidad positiva. Usamos esta pesadez para probar los teoremas ergódicos puntuales de Birkhoff y Kingman.

Keywords and Phrases: Maximal ergodic theorem, Birkhoff's ergodic theorem, Rokhlin lemma, Kingman's subadditive ergodic theorem.

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1 Introduction

Ergodic Theory is the subfield of Dynamical Systems concerned with measure-preserving dynamics, and it has applications throughout Mathematics. Its most fundamental result is the pointwise ergodic theorem proved by Birkhoff [2] in 1931. An important extension was obtained by Kingman [10] in 1968, and is known as the subadditive ergodic theorem. A multitude of other proofs of these basic results were obtained by many authors. One of the most popular methods of proof (especially in the case of Birkhoff's theorem) involves maximal inequalities, which have intrinsic interested by their own.

In this note, we provide self-contained proofs of Birkhoff's and Kingman's theorems by means of maximal inequalities; the only prerequisite is basic measure theory. Our approach has one novelty: it is based on a seemingly new extension of Rokhlin Tower Lemma, which is another basic tool used in many constructions in Ergodic Theory.

Let us proceed directly with the precise statements and proofs. We will provide further connections with the literature in the final Section 5.

Standing hypothesis: Let $(X, \mathcal{A}, \mu)$ be a Lebesgue probability space. Let $T\colon X \to X$ be an automorphism; this means that T and T^{-1} are measurable and preserve the measure μ. We assume that T is aperiodic, that is, the set of periodic points has zero measure.

2 A generalized Rokhlin Lemma

A measurable set $B \subseteq X$ is called *sweeping* if $\mu\left(\bigcup_{n\ge 0} T^n(B)\right) = 1$.

Theorem 2.1 (Generalized Rokhlin Lemma)**.** *For any $\varepsilon > 0$ and any measurable function $N\colon X \to \mathbb{Z}_+$, there exists a sweeping set $B \subseteq X$ such that*

(1) *if $x \in B$ and $1 \le i < N(x)$ then $T^i x \notin B$;*

(2) $\int_B N \, d\mu > 1 - \varepsilon$.

The case of constant N corresponds to the classical Rokhlin Lemma [19].

Let us introduce some useful terminology. If $B \subseteq X$ is any set of positive measure, Poincaré Recurrence Theorem says that a.e. $x \in B$ returns to B under iteration of T. So *return time* function

$$R_B(x) := \min\{k \ge 1 \,;\, T^k x \in B\} \quad \text{is finite for a.e. } x \in B. \tag{2.1}$$

If this function admits a lower bound n then the union $B \cup T(B) \cup \cdots \cup T^{n-1}(B)$ is disjoint; such a set is called a *tower* of *height* n with *base* B and *levels* B, $T(B)$, $\ldots$, $T^{n-1}(B)$. A *skyscraper* is a countable union of disjoint towers; its base is defined as the union of the bases of the towers.

If $B \subseteq X$ is a set of positive measure, the *Kakutani skyscraper* with base B is the union of the towers C_i ($i = 1, 2, \dots$) with respective bases $B_i := \{x \in B \; ; R_B(x) = i\}$. As a set, it equals $\bigcup_{n\geq 0} T^n(B)$. So the set B is sweeping if and only if the Kakutani skyscraper has full measure. In that case,

$$\int_B R_B \, d\mu = \sum_{i=1}^{\infty} i\mu(B_i) = \sum_{i=1}^{\infty} \mu(C_i) = 1; \tag{2.2}$$

this is *Kac's Lemma*.

A set B has property (1) in Theorem 2.1 if and only if $R_B \geq N$ on B. If this property is satisfied and moreover B is a sweeping set, then the following *error set*:

$$E := \left\{T^i x \; ; x \in B \text{ and } N(x) \leq i < R_B(x)\right\}. \tag{2.3}$$

has measure $\int_B (R_B - N) \, d\mu$, which by Kac's Lemma equals $1 - \int_B N \, d\mu$. So we can restate Theorem 2.1 in an equivalent form replacing conclusions (1) and (2) by the following ones:

(i') $R_B \geq N$ on B;

(ii') the error set (2.3) has measure $\mu(E) < \varepsilon$.

Before proving Theorem 2.1, we need a preliminary result which is also used in the proof of the usual Rokhlin Lemma:

Lemma 2.2. *For any integer $m \geq 2$, there exists a sweeping set A with $R_A \geq m$.*

Proof. Since we are working on a non-atomic Lebesgue probability space, we can assume X is the unit interval and μ is Lebesgue measure. Since T is aperiodic, the function

$$\varphi(x) := \inf\{|T^j(x) - x| \; ; 1 \leq j < m\}$$

is positive a.e. As a consequence, any set E of positive measure contains another set F of positive measure such that $R_F \geq m$; indeed, it suffices to take a positive measure set $G \subseteq E$ where φ is bigger than some $\delta > 0$, and the take a positive measure subset $F \subseteq G$ with diameter less than this δ.

Now consider the family $\mathcal{F}$ formed by the sets $F \subseteq X$ of positive measure that satisfy $R_F \geq m$, partially ordered as follows: $F_1 \prec F_2$ if $F_1 \subseteq F_2$ and $\mu(F_2 \setminus F_1) = 0$. Increasing chains are at most countable, and so by Zorn's lemma $\mathcal{F}$ contains a maximal element A.

We claim that A is sweeping. Indeed, the Kakutani skyscraper S with base A is a mod 0 invariant set, and if its complement S^c had positive measure then, using the fact established at the beginning, we could find a positive measure set $F \subseteq S^c$ such that $R_F \geq m$. Then $A \prec A \cup F$, contradicting the maximality of A.[1] $\square$

[1] Incidentally, this construction yields a set A whose Kakutani skyscraper has all towers with heights between m and $2m - 1$.

The following proof is due to Anthony Quas [17]:

Proof of Theorem 2.1. Fix an integer $n > 1$ such that the set $\{x \in X \,;\, N(x) \geq n\}$, which we will call *bad set*, has measure less than $\varepsilon/2$. Fix another integer $m > 2n/\varepsilon$. By Lemma 2.2, there exists a sweeping set A such that $R_A \geq m$. Consider the Kakutani skyscraper with base A. The set $\bigsqcup_{i=1}^{n} T^{-i}(A)$, which we will call *penthouse*, has measure $n\mu(A) < n/m < \varepsilon/2$, and consists on the n topmost levels of the skyscraper.

Let us define the set B. For each point $x \in A$, we follow the steps:

1. If x is in the penthouse, then we stop.
2. If x is in the bad set, then we replace x with $T(x)$, and go back to step 1.
3. Otherwise (i.e. x is neither in the penthouse nor in the bad set), then we put x inside B, replace x with $T^{N(x)}(x)$, and go back to step 1.

The set B constructed in this way is clearly measurable and satisfies $R_B \geq N$. The associated error set (2.3) is contained in the union of the bad set and the penthouse, and therefore has measure less than ε. □

3 The Maximal and Birkhoff's Ergodic Theorems

3.1 Maximal Ergodic Theorem

As a first application of the Generalized Rokhlin Lemma, we will give a short proof of the Maximal Ergodic Theorem.

The Birkhoff sums of f are denoted as:

$$f^{(n)} := f + f \circ T + \cdots + f \circ T^{n-1}\,.$$

Theorem 3.1 (Wiener, Yosida, and Kakutani's Maximal Ergodic Theorem)**.** *Let* $f \in L^1(\mu)$. *Let* P *be the set of* $x \in X$ *such that* $f^{(n)}(x) > 0$ *for some* $n \geq 1$. *Then* $\int_P f \, d\mu \geq 0$.

Proof. Let $L := X \setminus P$ be the set where all Birkhoff sums are non-positive. Define a function $N \colon X \to \mathbb{Z}_+$ as follows: if $x \in P$, let $N(x)$ be the least $n \geq 1$ such that $f^{(n)}(x) > 0$, while if $x \in L$, let $N(x) := 1$. Apply Theorem 2.1 to the function N and a small $\varepsilon > 0$, obtaining a measurable set B whose return time function is $\geq N$, and such that the error set (2.3) has measure $\mu(E) < \varepsilon$. Then:

$$\int_X f = \int_B \left[f^{(N(x))}(x) + \sum_{i=N(x)}^{R_B(x)-1} f(T^i x) \right] d\mu(x) = \int_B f^{(N(x))}(x)\, d\mu(x) + \int_E f\,,$$

by invariance of the measure. The integrand $f^{(N(x))}(x)$ equals $f(x)$ if $x \in L$ and is positive otherwise, and so $\int_X f \geq \int_{B\cap L} f + \int_E f$. But $\int_{B\cap L} f \geq \int_L f$, since $f \leq 0$ on L. So $\int_P f = \int_X f - \int_P f \geq \int_E f$. Since f is integrable and E can be made of arbitrarily small measure, we conclude that $\int_P f \geq 0$. $\square$

3.2 Heaviness

We present a corollary of the Maximal Ergodic Theorem 3.1 that is sufficient for some applications.

Let a, $b \in \mathbb{R}$. We say that a point $x \in X$ is a-*heavy* (resp. b-*light* with respect to a function f if for all $n \geq 1$, we have $f^{(n)}(x) \geq an$ (resp. $f^{(n)}(x) \leq bn$).

A measurable function $f\colon X \to [-\infty, +\infty]$ is called *quasi-integrable* if at least one of the functions f^+ or f^- is integrable (where, as usual, $f^+ \coloneqq \max(f, 0)$ and $f^- \coloneqq \max(-f, 0)$), and so $\int f = \int f^+ - \int f^-$ is defined.

Lemma 3.2 (Heaviness). *Let f be a quasi-integrable function, and let a, $b \in \mathbb{R}$.*

(1) *If $a < \int f \, d\mu$ then the set of a-heavy points has positive measure.*

(2) *If $b > \int f \, d\mu$ then the set of b-light points has positive measure.*

Proof. By symmetry, it is sufficient to prove part (2). Adding a constant to f, we can assume that $b = 0$, so $\int f \, d\mu$ is strictly negative. Let L be the set of points that are 0-light w.r.t. f. First consider the case of integrable f. The set P in the statement of the Maximal Ergodic Theorem 3.1 is the complement of L and therefore $\int_L f = \int_X f - \int_P f$ is strictly negative. In particular, $\mu(L) > 0$, as we wanted to show. Now consider the case that f is not integrable, so $\int f = -\infty$. For sufficiently large $K > 0$, the integrable function $f_* \coloneqq \max(f, -K)$ has $\int f_* < 0$. By the previous case, its set L_* of 0-light points has positive measure. But $L \supseteq L_*$, so L has positive measure as well. $\square$

3.3 Birkhoff's Pointwise Ergodic Theorem

The *conditional expectation* of a quasi-integrable function f with respect to a sub-σ-algebra $\mathcal{B} \subseteq \mathcal{A}$ is the $\mathcal{B}$-measurable quasi-integrable function denoted $\mathbb{E}(f \mid \mathcal{B})$ such that $\int_B \mathbb{E}(f \mid \mathcal{B}) \, d\mu = \int_B f \, d\mu$ for every $B \in \mathcal{B}$. Existence and essential uniqueness are immediate consequences of the Radon–Nikodym Theorem.

Let $\mathcal{I} \subseteq \mathcal{A}$ denote the σ-algebra of T-invariant sets.

Theorem 3.3 (Birkhoff's Ergodic Theorem). *If f is a quasi-integrable function then $\dfrac{f^{(n)}}{n} \to \mathbb{E}(f \mid \mathcal{I})$ a.e.*

Proof. Define functions $g \leq h$ respectively as the the lim inf and the lim sup of the sequence $f^{(n)}/n$. It follows from the equality $f^{(n)} = f_1 + f^{(n-1)} \circ T$ that the functions g and h are invariant. Let

$\varphi \coloneqq \mathbb{E}(f \mid \mathcal{I})$, which by definition is also invariant. We want to prove that $g = \varphi = h$ a.e. Our plan is to prove the following inequalities:

$$g \geq \varphi \geq h \quad \text{a.e.} \tag{3.1}$$

In order to prove the first inequality, it is sufficient to show that for all real numbers $\alpha < \beta$, the set

$$E_{\alpha,\beta} \coloneqq \{x \in X \,;\, g(x) < \alpha < \beta < \varphi(x)\}$$

has zero measure; indeed in that case the functions g and φ coincide over the full-measure set $\bigcap_{\alpha,\beta\in\mathbb{Q},\alpha<\beta} E^c_{\alpha,\beta}$. So let us assume by contradiction that $\mu(E_{\alpha,\beta}) > 0$ for certain numbers $\alpha < \beta$. Let

$$c \coloneqq \frac{1}{\mu(E_{\alpha,\beta})} \int_{E_{\alpha,\beta}} f \, d\mu = \frac{1}{\mu(E_{\alpha,\beta})} \int_{E_{\alpha,\beta}} \varphi \, d\mu \geq \beta \,,$$

where the equality between the integrals is due to the fact that the set $E_{\alpha,\beta}$ is invariant. Applying Lemma 3.2.(1) to the measurable dynamical system $(T|E_{\alpha,\beta}, \frac{\mu|E_{\alpha,\beta}}{\mu(E_{\alpha,\beta})})$, we conclude that for any real $a < c$, there is a positive measure set of points $x \in E_{\alpha,\beta}$ that are a-heavy. Such points clearly satisfy $g(x) \geq a$. Therefore $\alpha > a$. Since this holds for every $a < c$, we conclude that $\alpha \geq c \geq \beta$. This is a contradiction, and the first inequality in (3.1) is therefore proved.

The second inequality in (3.1) follows from the first one applied to $-f$. ☐

4 Subadditive Ergodic Theorems

4.1 Subadditivity

A sequence $(a_n)_{n\geq 1}$ taking values in $\mathbb{R} \cup \{-\infty\}$ is called *subadditive* if

$$a_{n+k} \leq a_n + a_k \quad \text{for all } n,\, k \geq 1.$$

By a well-known exercise (sometimes called Fekete Lemma), the limit $\lim_{n\to\infty} \frac{a_n}{n}$ exists in $\mathbb{R}\cup\{-\infty\}$ and equals $\inf_n \frac{a_n}{n}$. We will denote it by:

$$\operatorname{linf} \frac{a_n}{n} \,.$$

A sequence $(f_n)_{n\geq 1}$ of measurable functions is called *subadditive* with respect to T if, for all n, $k \geq 1$,

$$f_{n+k} \leq f_n + f_k \circ T^n \quad \text{for all } n,\, k \geq 1.$$

Suppose that the positive part f_1^+ is μ-integrable. Then we define the *asymptotic average* of the subadditive sequence by:

$$\lambda \coloneqq \operatorname{linf} \int \frac{f_n}{n} \, d\mu \,.$$

4.2 Subadditive Kac's Formula

Given a sweeping set $B \subseteq X$, let $\hat{T}\colon B \to B$ be the first-return map. It preserves $\hat{\mu} \coloneqq \frac{\mu|B}{\mu(B)}$, the normalized restriction of μ. Kac's formula (2.2) becomes $\int_B R_B \, d\hat{\mu} = \frac{1}{\mu(B)}$.

Now suppose $(f_n)_{n\geq 1}$ is a subadditive sequence with respect to $T\colon (X,\mu) \hookleftarrow$, and $f_1^+ \in L^1(\mu)$. We define a sequence $(\hat{f}_n)_{n\geq 1}$ of functions on B by:

$$\hat{f}_n(y) \coloneqq f_{R_B(y)+R_B(\hat{T}y)+\cdots+R_B(\hat{T}^{n-1}y)}(y) \,.$$

Then $(\hat{f}_n)_{n\geq 1}$ is a subadditive sequence with respect to $\hat{T}$; we call it the *induced subadditive sequence*. Note that $\hat{f}_1^+ \in L^1(\hat{\mu})$. Indeed, considering the Kakutani skyscraper and using invariance of μ, we see that in fact $\int_B \hat{f}_1^+ \, d\hat{\mu} \leq \frac{1}{\mu(B)} \int_X f_1^+ \, d\mu$.

Integrability allows us to define the asymptotic average of the induced subadditive sequence, that is,

$$\hat{\lambda} \coloneqq \operatorname{linf} \int_B \frac{\hat{f}_n}{n} \, d\hat{\mu} \,.$$

The next result relates the asymptotic averages $(\hat{f}_n)$: of the two subadditive sequences:

Theorem 4.1 (Subadditive Kac's Formula). $\hat{\lambda} = \dfrac{\lambda}{\mu(B)}$.

In the case of the additive sequence $f_n \coloneqq n$, the result coincides with the usual Kac's formula.

In the case where the asymptotic average λ is a Lyapunov exponent, Theorem 4.1 appeared in [23, Lemma 2.2]; see also [11, Lemma 2.2].

Actually Theorem 4.1 is a easy consequence of Kingman's Subadditive Ergodic Theorem 4.5; this is the argument used in [23, 11]. Here we go in the opposite direction; our ultimate aim is to prove Kingman's Theorem. So, to avoid circular reasoning, we should provide an independent proof of Theorem 4.1. Though this is possible, we won't do it, because the following weaker version is sufficient for our purposes:

Lemma 4.2 (Subadditive Kac Inequality). $\lambda \leq \displaystyle\int_B \hat{f}_1 \, d\mu$.

Proof. For each positive integer k, let $B_k \coloneqq \{x \in B \,;\, R_B(x) = k\}$, and $C_k \coloneqq \bigsqcup_{j=0}^{k-1} T^j(B_k)$. So C_k is the tower of the Kakutani skyscraper with height k, and B_k is its base. Moreover, the B_k's form a mod 0 partition of B, and the C_k's form a mod 0 partition of X.

We claim that for every $m \geq 1$, the following inequality holds (the integrals being w.r.t. μ):

$$\int_X f_m \leq \sum_{k=1}^{m} (m+1-k) \int_{B_k} \hat{f}_1 + \sum_{\ell=2}^{\infty} \min(\ell-1, m) \int_{C_\ell} f_1^+ \,. \tag{4.1}$$

In order to prove this, fix m and, for each point $x \in X$, consider all times $n_1 < n_2 < \cdots < n_p$ in the interval $\{0, 1, \ldots, m\}$ such that $T^{n_j} x \in B$. If $p \geq 1$ (i.e., the segment of orbit x, …, $T^m x$ hits

B at least once) then we use subadditivity to bound $f_m(x)$ by the following sum:

$$\sum_{i=1}^{n_1-1} f_1^+(T^i x) + \sum_{j=1}^{p-1} \hat{f}_1(T^{n_j} x) + \sum_{i=n_p}^{m-1} f_1^+(T^i x)\,. \tag{4.2}$$

In the case that $p = 0$ (i.e., there are no hits), we bound $f_m(x)$ simply by

$$\sum_{i=0}^{m-1} f_1^+(T^i x)\,. \tag{4.3}$$

Now we analyze the terms that appear in these sums:

- Given $k \geq 1$ and a non-periodic point $y \in B_k$, can a term $\hat{f}_1(y)$ appear in a sum (4.2)? The answer is clearly "no" if $k > m$. On the other hand, if $k \leq m$ then the term $\hat{f}_1(y)$ does appear in a sum (4.2): namely it appears once for each x in the set $\{y, T^{-1}y, \dots, T^{-(m-k)}y\}$, and there is a total of $m - k + 1$ appearances.
- Similarly we ask: given $\ell \geq 1$ and a non-periodic point $z \in C_\ell$, how many times does a term $f_1^+(z)$ appear in a sum (4.2) or in a sum (4.3)? The answer is $\min(\ell - 1, m)$. Indeed, if $\ell \geq m + 1$ then the term $f_1^+(z)$ appears once for each x in the set $\{z, T^{-1}z, \dots, T^{-(m-1)}z\}$, while if $\ell \leq m$ then $m + 1 - \ell$ of these points x do not contribute with a term of the form $f_1^+(z)$ and generate a term of the previous type $\hat{f}_1(z)$ instead.

Using these counts and the T-invariance of the measure μ, we obtain the claimed inequality (4.1). Next, note the following two facts about series, which follow from Fatou's Lemma and Dominated Convergence Theorem, respectively:

$$b_k \in [-\infty, +\infty), \ \sum_{k=1}^{\infty} b_k^+ < +\infty \ \Rightarrow \limsup_{m\to\infty} \frac{1}{m} \sum_{k=1}^{m} (m - k + 1) b_k \leq \sum_{k=1}^{\infty} b_k\,;$$
$$c_\ell \in [0, +\infty), \quad \sum_{\ell=1}^{\infty} c_\ell < +\infty \ \Rightarrow \lim_{m\to\infty} \frac{1}{m} \sum_{\ell=2}^{\infty} \min(\ell - 1, m) c_\ell = 0\,.$$

It follows that the lim sup of the right hand side of (4.1) divided by m is at most $\int_B \hat{f}_1$. But the left hand side of (4.1) divided by m tends to λ. So $\lambda \leq \int_B \hat{f}_1$, as we wanted to show. ☐

4.3 Subadditive Maximal Ergodic Theorem

The following result extends the Maximal Ergodic Theorem 3.1 to the subadditive context:

Theorem 4.3 (Silva and Thieullen's Maximal Subadditive Ergodic Theorem)**.** *Let* (f_n) *be a subadditive sequence of functions satisfying the integrability condition* $f_1^+ \in L^1(\mu)$*, and let* λ *be its asymptotic average. Let* H *be the set of* $x \in X$ *such that* $f_n(x) \geq 0$ *for all* $n \geq 1$*. Then* $\lambda \leq \int_H f_1 \, d\mu$.

Actually, the result is not stated (nor named) exactly in this form by Silva and Thieullen, but it is a corollary of [20, Lemma 2.4(a)].

Using the Generalized Rokhlin Theorem 2.1 and the Subadditive Kac Inequality (Lemma 4.2), Theorem 4.3 becomes almost obvious; its proof is of course similar to the proof of Theorem 3.1:

Proof of Theorem 4.3. Define a function $N\colon X \to \mathbb{Z}_+$ as follows: if $x \notin H$ then $N(x)$ is the least $n \geq 1$ such that $f_n(x) < 0$, while if $x \in H$ then $N(x) := 1$. Apply Theorem 2.1 to the function N and a small $\varepsilon > 0$, obtaining a measurable set B whose return time satisfies $R_B \geq N$ on B, and such that the error set (2.3) has measure $\mu(E) < \varepsilon$. Then (all integrals are w.r.t. μ):

$$\begin{aligned}
\lambda &\leq \int_B \hat{f}_1 && \text{(by Lemma 4.2)}\\
&\leq \int_B \left[f_{N(x)}(x) + \sum_{i=N(x)}^{R_B(x)-1} f_1^+(T^i x) \right] d\mu(x) && \text{(by subadditivity)}\\
&= \int_B f_{N(x)}(x)\, d\mu(x) + \int_E f_1^+ && \text{(by invariance)}\\
&\leq \int_{B\cap H} f_1 + \int_E f_1^+ && \text{(by definition of } N)\\
&\leq \int_H f_1 + \int_E f_1^+ && \text{(since } f_1 \leq 0 \text{ on } H).
\end{aligned}$$

Since $\mu(E)$ can be made arbitrarily small, so can $\int_E f_1^+$. Therefore $\lambda \leq \int_H f_1$. □

4.4 Subadditive Heaviness

Let (f_n) be a subadditive sequence of functions, and let $a, b \in \mathbb{R}$. We say that a point $x \in X$ is a-*heavy*, or respectively *eventually* b-*light*, with respect to the sequence (f_n) if

$$\begin{aligned}
f_n(x) &\geq an \quad \text{for all } n \geq 1, \text{ or respectively}\\
f_n(x) &\leq bn \quad \text{for all sufficiently large } n \geq 1.
\end{aligned}$$

The following is the subadditive version of Lemma 3.2.[2]

Lemma 4.4 (Subadditive Heaviness). *Let* (f_n) *be a subadditive sequence of functions satisfying the integrability condition* $f_1^+ \in L^1(\mu)$*, and let* λ *be its asymptotic average. Let* $a, b \in \mathbb{R}$.

(1) *If* $a < \lambda$ *then the set of* a*-heavy points has positive measure.*

(2) *If* $b > \lambda$ *then the set of eventually* b*-light points has positive measure.*

[2]See the discussion in the Comments (Section 5) about the occurrence of Lemma 4.4 in the literature.

Proof. Replacing f_n by $f_n - an$, we can assume that $a = 0$ and so $\lambda > 0$. Let H be the set of 0-heavy points. By Theorem 4.3, $\int_H f_1 \geq \lambda > 0$; in particular, $\mu(H) > 0$, proving part (1).

Now let us prove part (2). Suppose $b > \lambda$, and take $\varepsilon > 0$ such that $b - \varepsilon > \lambda$. Fix $m \geq 1$ such that $\int \frac{f_m}{m} < b - \varepsilon$. Let $\psi \coloneqq \max(f_1^+, f_2^+, \dots, f_{m-1}^+)$. By subadditivity,

$$f_n(x) \leq \sum_{i=0}^{\lfloor n/m \rfloor - 1} f_m(T^{mi}x) \;+\; \psi(T^{m\lfloor n/m \rfloor}x) \eqqcolon ① \;+\; ②$$

We deal with these two terms as follows:

- Let L be the set of points that that are $(b - \varepsilon)m$-light with respect to the function f_m and the dynamics T^m. So $\mu(L) > 0$ by Lemma 3.2.(2). Note that for all $x \in L$ and all $n \geq 1$ we have ① $\leq (b - \varepsilon)n$.
- Since $\psi \in L^1(\mu)$, by Birkhoff $\lim_{k \to \infty} \frac{1}{k} \varphi \circ T^k = 0$ a.e. [3] In particular, for almost every x and all sufficiently large n, we have ② $\leq \varepsilon n$.

It follows that for almost every $x \in L$ and all sufficiently large n, we have $f_n(x) \leq$ ① $+$ ② $\leq bn$. This proves part (2). ☐

4.5 Kingman's Subadditive Ergodic Theorem

Finally, let us use Lemma 4.4 to prove the following fundamental result:

Theorem 4.5 (Kingman's Subadditive Ergodic Theorem)**.** *Let (f_n) be a subadditive sequence of functions satisfying the integrability condition $f_1^+ \in L^1(\mu)$. Let $\varphi \coloneqq \operatorname{linf} \mathbb{E}\left(\frac{f_n}{n} \,\middle|\, \mathcal{I}\right)$. Then $\frac{f_n}{n} \to \varphi$ a.e.*

Proof. If E is an invariant (or mod 0 invariant) set of positive measure, let $\lambda(E)$ denote the asymptotic average of the subadditive sequence restricted to E, with respect to the restricted system $(T|E, \frac{\mu|E}{\mu(E)})$, that is,

$$\lambda(E) \coloneqq \frac{1}{\mu(E)} \operatorname{linf} \int_E \frac{f_n}{n} \, d\mu \, .$$

We claim that

$$\lambda(E) = \frac{1}{\mu(E)} \int_E \varphi \, d\mu \, . \tag{4.4}$$

Indeed, on one hand, for every n we have $\mathbb{E}(\frac{f_n}{n} \mid \mathcal{I}) \geq \varphi$ and in particular $\int \frac{f_n}{n} \geq \int \varphi$; taking limits we obtain the $\geq$ inequality in (4.4). On the other hand, by subadditivity, each $\mathbb{E}(\frac{f_n}{n} \mid \mathcal{I})$ can be bounded from above by $\mathbb{E}(f_1^+ \mid \mathcal{I})$, which is a integrable function. Using Fatou's Lemma we obtain the $\leq$ inequality in (4.4).

[3] For a simple proof of this fact that does not rely on Birkhoff, see [1, Lemma 2].

The rest of the proof of Kingman's Theorem 4.5 is analogous to our proof of Birkhoff's Theorem 3.3, using Lemma 4.4 instead of Lemma 3.2.

Define functions $g \le h$ respectively as the the lim inf and the lim sup of the sequence f_n/n. It follows from the inequality $f_n \le f_1 + f_{n-1} \circ T$ that the functions g and h are *sub-invariant*, that is, $g \le g \circ T$ and $h \le h \circ T$. By invariance and finiteness of the measure μ, every sub-invariant function is a.e. invariant.

We must prove that $g = \varphi = h$ a.e., and our plan is to show that:

$$g \ge \varphi \ge h \quad \text{a.e.} \tag{4.5}$$

Assume by contradiction that the inequality $g \ge \varphi$ fails on a positive measure set; then there exist real numbers $\alpha < \beta$ such that the (mod 0 invariant) set

$$E_{\alpha,\beta} \coloneqq \{x \in X \,;\, g(x) < \alpha < \beta < \varphi(x)\}$$

has positive measure. Applying Lemma 4.4.(1) to the system restricted to $E_{\alpha,\beta}$, we conclude that for any real $a < \lambda(E_{\alpha,\beta})$, there is a positive measure set of points $x \in E_{\alpha,\beta}$ that are a-heavy. Such points satisfy $g(x) \ge a$. Therefore $\alpha > a$. Since this holds for every $a < \lambda(E_{\alpha,\beta})$, we conclude that α is at least $\lambda(E_{\alpha,\beta})$, which by (4.4) is at least β. So $\alpha \ge \beta$, a contradiction. The first inequality in (4.5) is therefore proved.

The second inequality is proved similarly: Assume by contradiction that there are real numbers $\alpha < \beta$ such that the (mod 0 invariant) set

$$F_{\alpha,\beta} \coloneqq \{x \in X \,;\, \varphi(x) < \alpha < \beta < h(x)\}$$

has positive measure. Applying Lemma 4.4.(2) to the system restricted to $F_{\alpha,\beta}$, we conclude that for any real $b > \lambda(F_{\alpha,\beta})$, there is a positive measure set of points $x \in F_{\alpha,\beta}$ that are eventually b-light. Such points satisfy $h(x) \le a$. Therefore $\beta < b$. Since this holds for every $b > \lambda(F_{\alpha,\beta})$, we conclude that β is at most $\lambda(F_{\alpha,\beta})$, which by (4.4) is at most β. So $\beta \le \beta$, a contradiction. This proves the second inequality in (4.5). ☐

5 Comments

To summarize, our approach to prove Birkhoff's and Kingman's ergodic theorems was:

Generalized Rokhlin Lemma	$\Rightarrow$	maximal inequality	$\Rightarrow$	heaviness lemma	$\Rightarrow$	ergodic theorem

In the subadditive case, the first arrow also relies on a generalization of Kac's Lemma.

These intermediate results are also interesting by themselves. This path to the ergodic theorems is not the shortest one[4], but we hope that it has a gentle slope.

[4]The proof in [7, p. 136] is unbeatable.

There are many extensions and variations of Rokhlin Lemma (see [22, 12]), but nevertheless Theorem 2.1 appears to be new.

There are other proofs of the Maximal Ergodic Theorem 3.1 using towers: see [16, p.27ff]. Garsia's celebrated short proof uses a different idea; see e.g. [16, p.75ff].[5] Quoting Steele [21], *the proof has become a textbook standard, but the inequality and its proof are widely regarded as mysterious.* It is our hope that the Generalized Rokhlin Lemma makes the Maximal Ergodic Theorem more plain to see[6].

Silva and Thieullen deduce their Subadditive Ergodic Theorem [20, Lemma 2.4(a)] (which implies Theorem 3.1 in this note) from a pointwise inequality. This type of proof is probably the shortest in this case, and appears in other proofs of the ergodic theorems [8, 4, 9, 1].[7]

While Kac's formula and Rokhlin Lemma also hold for non-invertible T (see [13, 3]), it turns out that the generalized Rokhlin Lemma introduced here is false for non-invertible T: see Proposition 5.1 below. On the other hand, we can easily drop the invertibility assumption in the heaviness Lemmas 3.2 and 4.4, by considering the natural extension of T [16, p.13].

The "heaviness" terminology comes from Ralston [18] (who used it in a slightly different context). Lessa [14] also uses heaviness (without this terminology) to prove Birkhoff's and Kingman's theorems. Lessa's work is perhaps the first place where the statement of Lemma 4.4 appears explicitly. Lemma 4.4.(1), which as we have seen follows immediately from Silva–Thieullen's result, is also contained in a deeper result by Karlsson and Margulis, namely [6, Lemma 4.1]. Lemma 4.4.(2) is [14, Teorema 3.10].

Neither Karlsson–Margulis [6] nor Lessa [14] use maximal inequalities to obtain heaviness; instead they use Riesz' combinatorial lemma about leaders. See also Karlsson [5] for a related approach.

For another version of heaviness in a subadditive context, see [15, p. 144ff].

We conclude with the following example, also due to Quas, which shows that invertibility of T is necessary for the validity of Theorem 2.1:

Proposition 5.1 (Quas)**.** *Let* T *the shift on the space* $X := \{1, 2\}^{\mathbb{N}}$*, and let* μ *be the* $(\frac{1}{2}, \frac{1}{2})$*-Bernoulli measure. Consider the function* $N(x) := x_0$*. Then for any measurable set* $B \subseteq X$ *such that* $R_B \geq N$ *on* B*, we have* $\int_B (R_B - N)\, d\mu \geq \frac{1}{9}$.

Proof. If $\mu(B) < \frac{4}{9}$, then by Kac's formula $\int_B (R_B - N) d\mu = 1 - \int_B N d\mu \geq 1 - 2\mu(B) > \frac{1}{9}$. So let us assume that $\mu(B) \geq \frac{4}{9}$.

[5] As made clear by Steele, Garsia's proof boils down to a pointwise inequality involving a coboundary: see inequality (3) in [21].

[6] In the case of finite measure, at least.

[7] Incidentally, as remarked by Karlsson [5], Garsia's argument has a minor subadditive extension which is unfortunately insufficient to prove Kingman's theorem.

Consider $S := [1] \cap T^{-1}([2]) \cap T^{-2}(B)$, so that $\mu(S) = \frac{1}{4}\mu(B) \geq \frac{1}{9}$. Note that $T^{-1}(S) \subseteq B^c$, as a consequence of the hypothesis $R_B \geq N$.

Now let $f := \mathbb{1}_B \cdot (R_B - N)$. We claim that for any $x \in S$ that returns to S in finite time $n := R_S(x)$, the Birkhoff sum $f^{(n)}(T(x)) = f(Tx) + \cdots + f(T^n x)$ is at least 1. Indeed, consider the biggest $k \in \{2, 3, \ldots, n\}$ such that $T^k(x) \in B$; such k exists because $n \geq 2$ and $T^2(x) \in B$. Since $T^{n+1}(x) \notin B$ and $T^{n+2}(x) \in B$, we have $R_B(T^k(x)) = n + 2 - k$. Now, if $k = n$ then $N(T^k(x)) = 1$, while if $k < n$ then $R_B(T^k(x)) \geq 3$. In either case, $f(T^k(x)) \geq 1$, proving the claim.

It follows the asymptotic average of f along almost every orbit is at least the frequency that the set S is visited. Since μ is ergodic, this means that $\int f \, d\mu \geq \mu(B) \geq \frac{1}{9}$, as we wanted to prove. $\square$

Acknowledgements. I initially proved Theorem 2.1 under the assumption that N is bounded; this weaker result is sufficient for the proof of maximal inequalities, but an extra step is required. I thank Anthony Quas for removing this assumption, for telling me that invertibility of T cannot be relaxed (Proposition 5.1), and for several comments and corrections. I thank Godofredo Iommi for stimulating conversations. Finally, I thank the referee for suggestions that improved the presentation.

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