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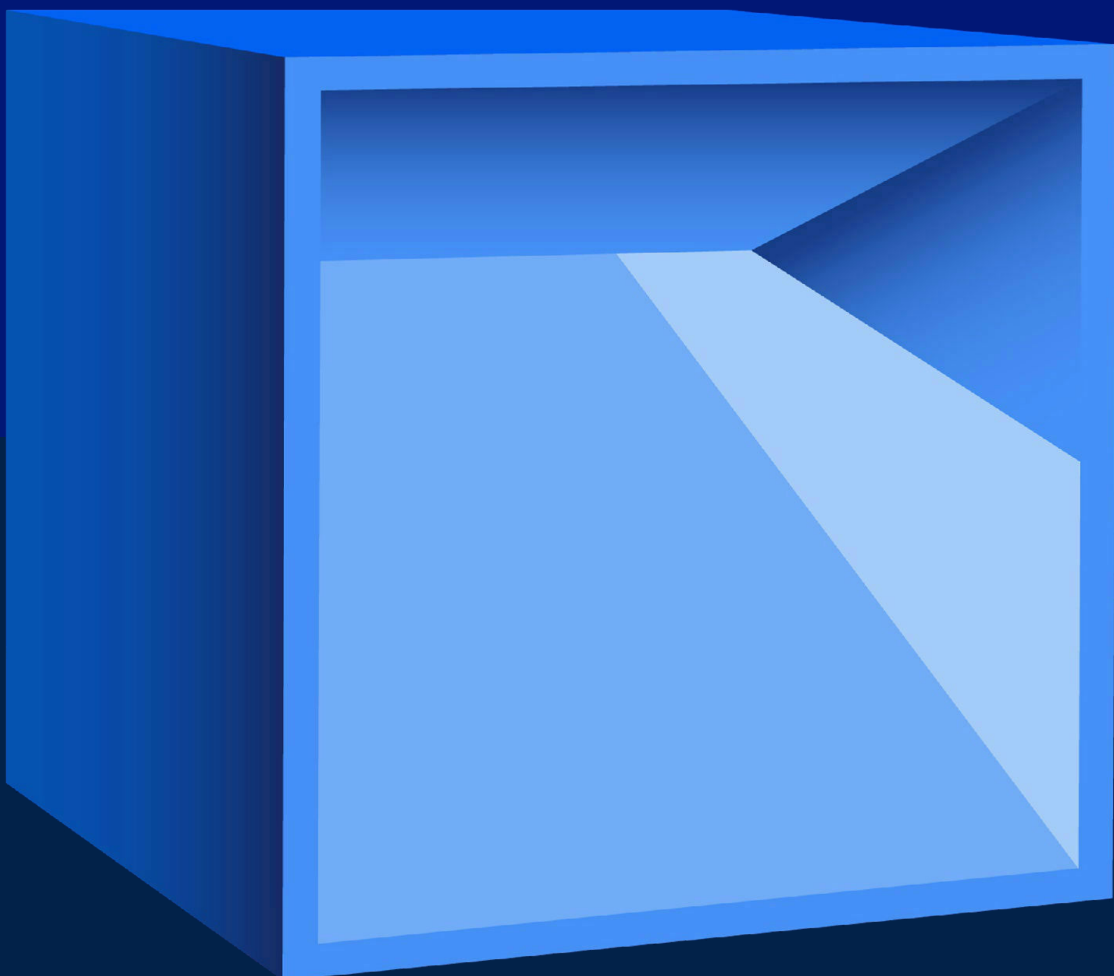

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On algebraic and uniqueness properties of harmonic quaternion fields on 3d manifolds

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ABSTRACT

Let Ω be a smooth compact oriented 3-dimensional Riemannian manifold with boundary. A quaternion field is a pair $q = \{\alpha, u\}$ of a function α and a vector field u on Ω. A field q is *harmonic* if α, u are continuous in Ω and $\nabla\alpha = \operatorname{rot} u$, $\operatorname{div} u = 0$ holds into Ω. The space $\mathscr{Q}(\Omega)$ of harmonic fields is a subspace of the Banach algebra $\mathscr{C}(\Omega)$ of continuous quaternion fields with the point-wise multiplication $qq' = \{\alpha\alpha' - u \cdot u',\ \alpha u' + \alpha' u + u \wedge u'\}$. We prove a Stone-Weierstrass type theorem: the subalgebra $\vee\mathscr{Q}(\Omega)$ generated by harmonic fields is dense in $\mathscr{C}(\Omega)$. Some results on 2-jets of harmonic functions and the uniqueness sets of harmonic fields are provided.

Comprehensive study of harmonic fields is motivated by possible applications to inverse problems of mathematical physics.

RESUMEN

Sea Ω una variedad Riemanniana 3-dimensional suave con borde, orientada y compacta. Un campo cuaterniónico es un par $q = \{\alpha, u\}$ dado por una función α y un campo de vectores u en Ω. Un campo q es *armónico* si α, u son continuos en Ω y $\nabla\alpha = \operatorname{rot} u$, $\operatorname{div} u = 0$ vale en todo Ω. El espacio $\mathscr{Q}(\Omega)$ de campos armónicos es un subespacio del álgebra de Banach $\mathscr{C}(\Omega)$ de campos cuaterniónicos continuos con la multiplicación punto a punto $qq' = \{\alpha\alpha' - u \cdot u',\ \alpha u' + \alpha' u + u \wedge u'\}$. Probamos un teorema de tipo Stone-Weierstrass: la subálgebra $\vee\mathscr{Q}(\Omega)$ generada por campos armónicos es densa en $\mathscr{C}(\Omega)$. Se entregan también algunos resultados acerca de 2-jets de funciones armónicas y los conjuntos de unicidad campos armónicos.

Keywords and Phrases: 3d quaternion harmonic fields, real uniform Banach algebras, Stone-Weierstrass type theorem on density, uniqueness theorems.

2010 AMS Mathematics Subject Classification: 30F15, 35Qxx, 46Jxx.

1 Introduction

Motivation

There is an approach to inverse problems of mathematical physics (the so-called Boundary Control method), which was originally based on the relations between inverse problems and the boundary control theory [4, 7, 9]. The BC-method recovers Riemannian manifolds via spectral and/or dynamical boundary data. Later on, its version that makes use of connections with Banach algebras, was proposed in [2, 5, 6].

The problem of recovering the manifold via its DN-map (the so-called Impedance Tomography Problem) in dimensions $\geqslant 3$ isn't yet properly solved. However, beginning from the papers [3, 10] it becomes clear that harmonic quaternion fields may play the key role in the 3d ITP. It is the reason, which has stimulated the study of their properties [8, 11].

Here we consider certain of algebraic and uniqueness properties of the harmonic quaternion fields with hope for their future application to ITP [8]. In the mean time, our results may be of certain independent interest for functional analysis: namely, the real uniform Banach algebras theory [1, 13, 15].

Main result

• Let Ω be a smooth compact oriented 3-dimensional Riemannian manifold with boundary, $T\Omega_x$ the tangent space at $x \in \Omega$, $u \cdot v$ and $u \wedge v$ the inner and vector products in $T\Omega_x$. Elements of the space $H_x := \mathbb{R} \oplus T\Omega_x$ (the pairs $q = \{\alpha, u\}$) endowed with a multiplication $qq' = \{\alpha\alpha' - u \cdot u',\ \alpha u' + \alpha' u + u \wedge u'\}$ are said to be the *geometric quaternions*. As an algebra, H_x is isometrically isomorphic to the quaternion algebra $\mathbb{H}$.

• A *quaternion field* is a pair $q = \{\alpha, u\}$ of a function α and vector field u on Ω; in other words, q is an H_x-valued function on the manifold. The space $C(\Omega; H)$ of continuous quaternion fields endowed with the point-wise linear operations and multiplication, and the relevant sup-norm, is a real uniform Banach algebra [1, 13, 15].

A field $q = \{\alpha, u\} \in C(\Omega; H)$ is *harmonic* if α, u are continuous in Ω and $\nabla\alpha = \operatorname{rot} u$, $\operatorname{div} u = 0$ holds into Ω. The space $\mathscr{Q}(\Omega)$ of harmonic fields is a subspace of $C(\Omega, H)$ (but not a subalgebra!).

• Let $\mathscr{A}$ be an algebra. For a set $A \subset \mathscr{A}$ by $\vee A$ we denote the minimal subalgebra that contains A. The main result of the paper is a Stone-Weierstrass type Theorem 1 which claims that $\vee\mathscr{Q}(\Omega)$ is dense in $C(\Omega; H)$.

More results and comments

• In the course of proving Theorem 1 we show that $\mathscr{Q}(\Omega)$ (and, hence, $\vee\mathscr{Q}(\Omega)$) separates points of Ω. It is quite evident for $\Omega \subset \mathbb{R}^3$ [11] but far from being evident for a 3d-manifold of arbitrary topology. The separation property is derived from the so-called H-controllability of Ω from the boundary, which is much stronger than separability. The H-controllability is proved by the use of the results [18] on existence of the global Green function and the Landis type uniqueness theorems for the second order elliptic equations [16]. The key step in proving Theorem 1 is to show that $\overline{\vee\mathscr{Q}(\Omega)}$ contains the algebra of scalar fields $\left\{\{\alpha, 0\} \mid \alpha \in C^{\mathbb{R}}(\Omega)\right\}$. The latter resembles the trick applied in [14].

• In sec 4 we prove that the 2-jets of harmonic functions are point-wise controllable from the boundary. The proof also makes use of the elliptic uniqueness theorems. Then this result is applied to show that harmonic functions determine the Riemannian structure of 3d manifold. As we hope, it is a step towards the main prospective goal: application to the 3d impedance tomography problem on Riemannian manifolds.

• One more result, which is of certain independent interest, is the following uniqueness property of harmonic quaternion fields (sec 5). If $q \in \mathscr{Q}(\Omega)$ vanishes on a piece of a smooth surface then it vanishes in Ω identically.

• Everywhere in the paper we deal with *real* functions, fields, spaces, etc. Everywhere *smooth* means C^∞-smooth.

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2 Quaternion fields

Quaternions

• Let E be an oriented 3d euclidean space, $u \cdot v$ and $u \wedge v$ the scalar (inner) and vector products, $|u| = \sqrt{u \cdot u}$. Elements $p = \{\alpha, u\}$ of the space $H := \mathbb{R} \oplus E$ endowed with the norm $|p| = \sqrt{\alpha^2 + |u|^2}$ and a (noncommutative) multiplication

$$pp' := \{\alpha\alpha' - u \cdot u',\ \alpha u' + \alpha' u + u \wedge u'\}, \tag{2.1}$$

are said to be *geometric quaternions*.

The norm obeys $|p^2| = |p|^2$,

• Let $\mathbb{H}$ be the algebra of (standard) quaternions. Recall that it is the real algebra generated by $\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}$ with the unit $\mathbf{1}$ and multiplication defined by the table

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -\mathbf{1}, \quad \mathbf{ij} = \mathbf{k},\ \mathbf{jk} = \mathbf{i},\ \mathbf{ki} = \mathbf{j}\,.$$

• For an orthogonal normalized basis $\varepsilon = \{e_1, e_2, e_3\}$ in E, the correspondence $e_1 \mapsto \mathbf{i}$, $e_2 \mapsto \mathbf{j}$, $e_3 \mapsto \mathbf{k}$ determines an isometric isomorphism $\mu_\varepsilon : H \to \mathbb{H}$,

$$\{\alpha,\ a e_1 + b e_2 + c e_3\} \overset{\mu_\varepsilon}{\mapsto} \alpha \mathbf{1} + a\mathbf{i} + b\mathbf{j} + c\mathbf{k}\,, \tag{2.2}$$

(we write $H \cong \mathbb{H}$). Any isometric isomorphism $\mu : H \to \mathbb{H}$ is of the form (2.2) by proper choice of the basis ε.

Vector analysis

In the sequel, the following assumptions are accepted.

Convention 1. *Ω is a smooth compact oriented Riemannian 3d-manifold with the smooth boundary $\partial\Omega$. It is endowed with the metric tensor $g \in C^2$; $d\mu$ is the Riemannian volume 3-form; $\star$ is the Hodge operator.*

On such a manifold, the intrinsic operations of vector analysis are well defined on smooth functions and vector fields (sections of the tangent bundle $T\Omega$). Following [21], Chapter 10, we recall their definitions.

• For a vector field u, one defines the *conjugate* 1*-form* $u_\flat$ by $u_\flat(v) = g(u, v)$, $\forall v$. For a 1-form f, the *conjugate field* $f^\flat$ is defined by $g(f^\flat, u) = f(u)$, $\forall u$.

• A *scalar product*: $\{\text{fields}\} \times \{\text{fields}\} \xrightarrow{\cdot} \{\text{functions}\}$ is defined point-wise by $u \cdot v = g(u, v)$. A *vector product*: $\{\text{fields}\} \times \{\text{fields}\} \xrightarrow{\wedge} \{\text{fields}\}$ is defined point-wise by $g(u \wedge v, w) = d\mu\,(u, v, w)$, $\forall w$.

• A *gradient*: $\{\text{functions}\} \xrightarrow{\nabla} \{\text{fields}\}$ and a *divergence*: $\{\text{fields}\} \xrightarrow{\text{div}} \{\text{functions}\}$ are defined by $\nabla\alpha = (d\alpha)^\flat$ and $\text{div}\, u = \star\, d \star\, u_\flat$ respectively, where d is the exterior derivative.

• A *rotor*: $\{\text{fields}\} \xrightarrow{\text{rot}} \{\text{fields}\}$ is defined by $\text{rot}\, u = (\star\, d\, u_\flat)^\flat$. Recall the basic identities: $\text{div}\,\text{rot} = 0$ and $\text{rot}\,\nabla = 0$. The equalities

$$\nabla\alpha\ =\ \text{rot}\, u \qquad \text{and} \qquad d\alpha = \star\, d\, u_\flat$$

are equivalent.

• The *Laplacian* $\{\text{functions}\} \xrightarrow{\Delta} \{\text{functions}\}$ is $\Delta = \text{div}\,\nabla$. The *vector Laplacian* $\{\text{fields}\} \xrightarrow{\vec{\Delta}} \{\text{fields}\}$ is $\vec{\Delta} = \nabla\,\text{div}\ - \text{rot}\,\text{rot}\,.$

Remark 1. *Under the above accepted assumptions on the smoothness of* Ω *and* g*, the (harmonic) functions and fields, which obey* $\Delta\alpha = 0$ *and* $\vec{\Delta}u = 0$ *in the relevant weak sense, do belong to the class* $C^2_{\rm loc}$*: see, e.g, [12], Part II, Chapter 1.*

Fields

Let $\dot{\Omega} := \Omega \setminus \partial\Omega$ be the set of the inner points, $C(\Omega)$ and $\vec{C}(\Omega)$ the spaces of continuous functions and vector fields. Let $H_x := \mathbb{R} \oplus T\Omega_x,\ \ x \in \Omega$ be the point-wise geometric quaternion algebras.

- A quaternion *field* is a pair $p = \{\alpha, u\}$ with the components $\alpha \in C(\Omega)$ and $u \in \vec{C}(\Omega)$, the values $p(x) = \{\alpha(x), u(x)\} \in H_x$ being regarded as geometric quaternions.

By $C(\Omega; H)$ we denote the space of continuous quaternion fields. One can regard them as sections of the bundle $C(\Omega; H) = \cup_{x\in\Omega} H_x$.

- Elements of the subspace

$$\mathscr{Q}(\Omega) := \Big\{p \in C(\Omega; H) \,\Big|\, \nabla\alpha = \operatorname{rot} u,\ \operatorname{div} u = 0 \ \text{ in } \ \dot{\Omega}\Big\}$$

are called *harmonic fields*. To be rigorous, here the conditions on the components of p are understood in the relevant sense of distributions but imply $\Delta\alpha = 0$ and $\vec{\Delta}u = 0$, so that α and u are automatically smooth enough by Remark 1.

3 Density theorem

Algebra $C(\Omega; H)$

The space $C(\Omega; H)$ with the point-wise multiplication (2.1) and the norm

$$\|p\| = \sup_{x\in\Omega} |p(x)| = \sup_{x\in\Omega} \sqrt{|\alpha(x)|^2 + |u(x)|^2_{T\Omega_x}}$$

satisfying $\|qp\| \leqslant \|q\|\|p\|$, $\|p^2\| = \|p\|^2$ is a real uniform noncommutative Banach algebra.

- The fields $\{\alpha, 0\}$ constitute a subalgebra $C(\Omega; \mathbb{R})$ of $C(\Omega; H)$, which is isometrically isomorphic to the real continuous function algebra on Ω:

$$C(\Omega; \mathbb{R}) \cong C^{\mathbb{R}}(\Omega)\,. \tag{3.1}$$

We say $\{\alpha, 0\}$ to be the scalar fields and often identify them with functions α via the map $\alpha \mapsto \{\alpha, 0\}$, which embeds $C^{\mathbb{R}}(\Omega)$ in $C(\Omega; H)$.

- The harmonic subspace $\mathscr{Q}(\Omega) \subset C(\Omega; H)$ is not an algebra since, in general, $p, q \in \mathscr{Q}(\Omega)$ does not imply $pq \in \mathscr{Q}(\Omega)$. It is easy to see that

$$\mathscr{Q}(\Omega) \cap C(\Omega; \mathbb{R}) = \{\{c, 0\} \mid c \text{ is a constant function}\},$$

whereas $\{1, 0\}$ is the unit of $C(\Omega; H)$.

Main result

For an algebra $\mathscr{A}$ and a set $S \subset \mathscr{A}$ by $\vee S$ we denote a minimal (sub)algebra in $\mathscr{A}$, which contains S. Our main results is the following.

Theorem 1. *The algebra* $\vee\mathscr{Q}(\Omega)$ *is dense in* $C(\Omega; H)$.

The proof occupies the rest of sec 3.

Green function

• A well-known in geometry fact is that the assumptions of Convention 1, in particular, provide the existence of a compact 3-dimensional C^∞- manifold $\Omega' \Supset \Omega$ endowed with the tensor $g' \in C^2$ such that $g'|_\Omega = g$. This enables one to apply the results by M.Mitrea and M.Taylor [18] (existence of the fundamental solution, Green function, Poisson formula, etc) which are valid for much weaker smoothness restrictions on g and $\partial\Omega$. Also, one can apply the results on the uniqueness of continuation of solutions to the elliptic PDE [12, 16].

• The following results are mostly taken from [18]. Also we use some well-known facts of the elliptic 2-nd order equations theory [17, 12, 16]. By $W^l_p(\Omega)$ we denote the Sobolev space of functions which possess the (generalized) derivatives of the order $l = 1, 2, \dots$ belonging to $L_p(\Omega)$ $(p \geqslant 1)$. Recall that $\dot\Omega = \Omega \setminus \partial\Omega$. Also we put $D := \{(x, y) \in \Omega \times \Omega \,|\, x = y\}$. The distance in Ω is denoted by r_{xy}. Let $\mathscr{D}(\dot\Omega)$ be a space of the smooth compactly supported into Ω functions (test functions) endowed with the standard topology, $\mathscr{D}'(\dot\Omega)$ the corresponding distributions.

For an $h \in L_2(\Omega)$, the Dirichlet problem

$$\begin{aligned} &\Delta v = h && \text{in } \dot\Omega \\ &v = 0 && \text{on } \partial\Omega \end{aligned}$$

has a unique solution $v^h \in W^2_2(\Omega)$ vanishing at the boundary. The solution is represented in the form

$$v^h(x) \;=\; \int_\Omega G(x, y)\, h(y)\, d\mu(y), \qquad x \in \Omega \tag{3.2}$$

via the *Green function* G, which possesses the following properties.

1. $G \in C^2_{\rm loc}\,([\Omega \times \Omega] \setminus D)$; $\;G(x, y) = G(y, x)$, $\;(x, y) \notin D$;

$$G(x, \cdot)|_{\partial\Omega} \;=\; 0, \qquad x \in \dot\Omega\,. \tag{3.3}$$

For the closed sets $K, K' \subset \Omega$ provided $K \cap K' = \emptyset$ the map $y \mapsto G(\cdot, y)$ is continuous from K to $C^2(K')$.

2. The estimates

$$G(x,y) \leqslant \frac{c}{r_{xy}}\,, \qquad |\nabla_y G(x,y)| \leqslant \frac{c}{r^2_{xy}}$$

hold and imply $G(x,\cdot) \in W^1_p(\Omega)$ for $x \in \Omega,\ 1 \leqslant p < \frac{3}{2}$.

3. As a distribution of the class $\mathscr{D}'(\dot{\Omega})$ on the test functions (of the variable y) of the class $\mathscr{D}(\dot{\Omega})$, the Green function satisfies

$$\Delta_y G(x,\cdot) \,=\, \delta_x, \tag{3.4}$$

where δ_x is the Dirac measure supported at x. Note that in (3.4), and below in (3.8), (3.9), the variable $x \in \dot{\Omega}$ plays the role of parameter.

4. For $f \in C^\infty(\partial\Omega)$, the inhomogeneous boundary value problem

$$\Delta w = 0 \qquad \text{in } \dot{\Omega} \tag{3.5}$$

$$w = f \qquad \text{on } \partial\Omega \tag{3.6}$$

has a unique classical solution $w = w^f(x)$, which is represented in the form

$$w^f(x) \,=\, \int_{\partial\Omega} \partial_{\nu_y} G(x,y)\, f(y)\, d\sigma(y), \qquad x \in \dot{\Omega}\,, \tag{3.7}$$

where ν_y is the outward unit normal at the boundary, $d\sigma$ is the boundary surface element. This is a Poisson formula derived from (3.2) by integration by parts. Function f in (3.6) is said to be a *boundary control*.

- Fix a point $x \in \dot{\Omega}$ and a vector $e \in T\Omega_x,\ |e| = 1$. Let γ_e be the geodesic that emanates from x in direction e. Define a functional $\partial^x_e \delta_x \in \mathscr{D}'(\dot{\Omega})$ by

$$\langle \partial^x_e \delta_x, \varphi\rangle := \lim_{\gamma_e \ni x' \to x} \frac{\varphi(x') - \varphi(x)}{r_{xx'}} = \left\langle \lim_{\gamma_e \ni x' \to x} \frac{\delta_{x'} - \delta_x}{r_{xx'}}\,,\ \varphi \right\rangle = e \cdot \nabla\varphi(x)\,.$$

The relevant limit passage in (3.4) determines a derivative $\partial^x_e G(x,\cdot) \in \mathscr{D}'(\dot{\Omega})$ which satisfies

$$\Delta_y[\partial^x_e G(x,\cdot)] \,=\, \partial^x_e \delta_x\,. \tag{3.8}$$

In the mean time, by the properties 1 and 2, $\partial^x_e G(\cdot, y)$ is a (classical) function belonging to $L_p(\Omega)$ for $1 \leqslant p < \frac{3}{2}$. Moreover it is *harmonic* (and hence C^2-smooth) in $\Omega \setminus \{x\}$ and satisfies

$$\partial^x_e G(x,\cdot)|_{\partial\Omega} = 0\,, \qquad x \in \dot{\Omega}. \tag{3.9}$$

- The relevant limit passage in the Poisson formula (3.7) implies

$$e \cdot \nabla w^f(x) \,=\, \int_{\partial\Omega} \partial_{\nu_y}\,[\partial^x_e G(x,y)]\ f(y)\, d\sigma(y), \qquad x \in \dot{\Omega}\,. \tag{3.10}$$

H**-controllability**

• The following result plays the key role in the proof of Theorem 1. Recall that $H_x = \mathbb{R} \oplus T\Omega_x \cong \mathbb{H}$, and Ω obeys Convention 1.

For a set of points $A = \{a_1, \dots, a_N\} \subset \Omega$ define a 4N-dimensional space $H_A := \oplus \sum_{i=1}^N H_{a_i}$ and a map $M_A : C^\infty(\partial\Omega) \to H_A$:

$$f \mapsto \oplus \sum_{i=1}^N \{w^f(a_i), \nabla w^f(a_i)\}$$

(each summand $\{w^f(a_i), \nabla w^f(a_i)\}$ belongs to the corresponding H_{a_i}). We say Ω to be H-*controllable from boundary* if this map is surjective for any finite set A.

Lemma 1. *The manifold* Ω *is* H-*controllable from boundary.*

Proof. The opposite means that $H_A \ominus \operatorname{Ran} M_A \neq \{0\}$, i.e. there is a nonzero element $\oplus \sum_{i=1}^N \{\alpha_i, \beta_i e_i\} \in H_A$ $(\alpha_i, \beta_i \in \mathbb{R},\ |e_i| = 1)$ such that

$$\sum_{i=1}^N \alpha_i w^f(a_i) + \beta_i\, e_i \cdot \nabla w^f(a_i) = 0 \tag{3.11}$$

holds for all $f \in C^\infty(\partial\Omega)$. Show that such an assumption leads to contradiction.

1. Let $A \subset \dot{\Omega}$, i.e., all a_i are the interior points. A function

$$\Phi(y) := \sum_{i=1}^N \alpha_i G(a_i, y) + \beta_i \partial^x_{e_i} G(a_i, y) \tag{3.12}$$

satisfies

$$\Delta\Phi = 0 \qquad \text{in } \Omega \setminus A \tag{3.13}$$

$$\Phi|_{\partial\Omega} = 0 \tag{3.14}$$

by (3.3), (3.4), (3.8), and (3.9).

The relations (3.7), (3.10) and (3.11) easily follow to

$$\int_{\partial\Omega} \partial_\nu \Phi(y)\, f(y)\, d\sigma(y) = 0$$

that implies

$$\partial_\nu \Phi|_{\partial\Omega} = 0 \tag{3.15}$$

by arbitrariness of f.

2. So, Φ is harmonic in $\Omega \setminus A$ and has the *zero* Cauchy data at the boundary: see (3.14) and (3.15). By the well-known uniqueness property of solutions to elliptic PDE (see, e.g., [16], sec. 4.3, Remark 4.17), we get $\Phi = 0$ in $\Omega \setminus A$, i.e., almost everywhere in Ω.

Since $G(a_i, \cdot) \in W^1_p(\Omega)$ and $\partial_{e_i} G(a_i, \cdot) \in L_p(\Omega)$, we have $\Phi \in L_p(\Omega)$ for some $p \geqslant 1$. Therefore, Φ is a summable function equal zero a.e. in Ω. Thus, $\Phi = 0$ as a distribution of the class $\mathscr{D}'(\dot{\Omega})$.

In the mean time, by (3.4) and (3.8) one has

$$\Delta\Phi \,=\, \sum_{i=1}^{N} \alpha_i \delta_{a_i} + \beta_i \partial^x_{e_i} \delta_{a_i} \,\neq\, 0\,,$$

i.e., Φ is a *nonzero* element of $\mathscr{D}'(\dot{\Omega})$. We arrive at the contradiction that proves the Lemma for $A \in \dot{\Omega}$.

3. Let A contain the points of $\partial\Omega$. The smoothness assumptions on Ω enable one to provide Ω', g' obeying Convention 1 and such that $\Omega \Subset \Omega'$ and $g'|_\Omega = g$ holds. Then one has $A \subset \dot{\Omega}'$ that reduces this case to the previous one. $\square$

Note that relations between controllability and uniqueness theorems (like the one used in the proof) are widely exploited in control theory for PDE (see, e.g., [9]).

• Recall that w^f is a harmonic function that solves (3.5), (3.6). As immediate consequence of Lemma 1 we have

Corollary 1. *The algebra* $\vee \left\{|\nabla w^f|^2 \mid f \in C^\infty(\Omega)\right\}$ *is dense in* $C^{\mathbb{R}}(\Omega)$.

Indeed, by Lemma 1, for any $a, b \in \Omega$ there is a smooth f such that $|\nabla w^f(a)|^2 \neq |\nabla w^f(b)|^2$, i.e., the functions $|\nabla w^f(\cdot)|^2$ separate points of Ω. In the mean time, by the same Lemma, there is no $x_0 \in \Omega$, at which all these functions vanish simultaneously. Hence, by the classical Stone-Weierstrass Theorem (see, e.g., [19]), the above mentioned density does hold.

Note that $\{0, \nabla w^f\} \in \mathscr{Q}(\Omega)$ and $\{0, \nabla w^f\}^2 = -\{|\nabla w^f(\cdot)|^2, 0\} \in \vee\mathscr{Q}(\Omega)$. Hence, the algebra $\vee \left\{\{|\nabla w^f|^2, 0\} \mid f \in C^\infty(\Omega)\right\}$ is a subalgebra in $\vee\mathscr{Q}(\Omega)$. By (3.1), Corollary 1 implies that this algebra is dense in $C(\Omega; \mathbb{R})$. As a result, denoting

$$\mathscr{C} \,:=\, \overline{\vee\mathscr{Q}(\Omega)}$$

we arrive at the important relation

$$\mathscr{C} \,\supset\, C(\Omega; \mathbb{R})\,. \tag{3.16}$$

Strong separation

We say that a family $\mathscr{F} \subset C(\Omega; H)$ *strongly separates* points (of Ω) if for any $a, b \in \Omega$ and $h_a \in H_a$, $h_b \in H_b$ there is a $p \in \mathscr{F}$ such that $p(a) = h_a$ and $p(b) = h_b$ holds [13].

Lemma 2. *The space $\mathscr{Q}(\Omega)$ strongly separates points.*

Proof. • Let $\vec{L}_2(\Omega)$ be the space of square-integrable vector fields and $\mathscr{H} := \{v \in \vec{L}_2(\Omega) \mid \operatorname{div} v = 0,\ \operatorname{rot} v = 0\}$ its harmonic subspace. The well-known Hodge-Morrey-Friedrichs decomposition claims that

$$\mathscr{H} = \mathscr{G} \oplus \mathscr{N} = \mathscr{R} \oplus \mathscr{D}, \tag{3.17}$$

where

$$\mathscr{G} := \{v \in \mathscr{H} \mid v = \nabla\alpha\}, \quad \mathscr{N} := \{v \in \mathscr{H} \mid v \cdot \nu = 0\},$$
$$\mathscr{R} := \{v \in \mathscr{H} \mid v = \operatorname{rot} u\}, \quad \mathscr{D} := \{v \in \mathscr{H} \mid v \wedge \nu = 0\}.$$

(see, e.g., [21], Corollary 3.5.2). The subspaces $\mathscr{N}$ and $\mathscr{D}$ determined by the boundary conditions are called the Neumann and Dirichlet spaces respectively. Their *finite* dimensions are equal to the Betti numbers: $\dim \mathscr{N} = \beta_1$, $\dim \mathscr{D} = \beta_2$ [21]. Note that $\mathscr{N} \cap \mathscr{D} = \{0\}$ [3, 21]. Also note that $\dim \mathscr{G} = \dim \mathscr{R} = \infty$.

• As a consequence of (3.17), a field $v \in \mathscr{H}$ is represented in the form $v = \nabla\alpha = \operatorname{rot} u$ if and only if $v \in \mathscr{G} \cap \mathscr{R}$ or, equivalently, $v \perp [\mathscr{N} \dot{+} \mathscr{D}]$.

If $w = w^f(x)$ solves (3.5), (3.6) then for any $d \in \mathscr{D}$ one has

$$(\nabla w^f, d) = \int_\Omega \nabla w^f \cdot d \, d\mu = \int_{\partial\Omega} f \, d \cdot \nu \, d\sigma.$$

In the mean time, since $\nabla w^f \in \mathscr{G}$, the representation $\nabla w^f = \operatorname{rot} u$ holds if and only if $\nabla w^f \perp \mathscr{D}$, which is equivalent to

$$\int_{\partial\Omega} f \, d \cdot \nu \, d\sigma = 0, \qquad d \in \mathscr{D}. \tag{3.18}$$

In particular, taking $f = 1$ one has $w^f = 1$ in Ω and gets

$$\int_{\partial\Omega} d \cdot \nu \, d\sigma = 0, \qquad d \in \mathscr{D}. \tag{3.19}$$

• Now, fix two distinct points $a, b \in \Omega$ and elements $h_a = \{c_a, k_a\} \in H_a$, $h_b = \{c_b, k_b\} \in H_b$. To prove the Lemma we need to show that there is a smooth f, which provides

$$w^f(a) = c_a,\ w^f(b) = c_b; \quad \nabla w^f = \operatorname{rot} u; \quad u(a) = h_a,\ u(b) = h_b\,. \tag{3.20}$$

Step 1. At first assume $a, b \in \dot{\Omega}$. Let $P_x(y) := \partial_{\nu_y} G(x, y)$ be the Poisson kernel. By (3.7) for $f = 1$ we have

$$\int_{\partial\Omega} P_x(y)\, d\sigma(y) \;=\; 1\,, \qquad x \in \Omega\,. \tag{3.21}$$

In accordance with (3.7) and (3.18), to satisfy the relations $w^f(a) = c_a$, $w^f(b) = c_b$; $\nabla w^f = \operatorname{rot} u$ in (3.20) we need to find f provided

$$\int_{\partial\Omega} P_a(y)\, f(y)\, d\sigma(y) = c_a\,, \quad \int_{\partial\Omega} P_b(y)\, f(y)\, d\sigma(y) = c_b\,;$$
$$\int_{\partial\Omega} f(y)\, d(y)\cdot\nu\, d\sigma(y) \;=\; 0\,, \qquad d \in \mathscr{D},$$

or, equivalently,

$$(P_a, f) = c_a,\ (P_b, f) = c_b\,,\ \ f \perp \nu\cdot\mathscr{D} \tag{3.22}$$

(the inner products in $L_2(\partial\Omega)$), where $\nu\cdot\mathscr{D} := \{\nu\cdot d \mid d \in \mathscr{D}\}$.

Comparing (3.19) with (3.21), we conclude that neither P_a nor P_b belong to $\nu\cdot\mathscr{D}$. In the mean time, $P_a \neq P_b$ as elements of $L_2(\partial\Omega)$. Indeed, otherwise we'd have $w^f(a) = w^f(b)$ for any f that is impossible by Lemma 2. Hence, $\operatorname{span}\{P_a, P_b\} \cap \nu\cdot\mathscr{D}$ may consist of $\{c(P_a - P_b) \mid c \in \mathbb{R}\}$ *only*. As a result, to proof the solvability of the linear system (3.22) (with respect to f) in the case of $c_a \neq c_b$ we must show that $P_a - P_b \notin \nu\cdot\mathscr{D}$.

Step 2. Assume the opposite: there is a $d \in \mathscr{D}$ such that $P_a - P_b = d\cdot\nu$, and show that this assumption leads to a contradiction.

Compare the fields $\nabla[G(a,\cdot) - G(b,\cdot)]$ and d. Since $G(a,\cdot) = G(b,\cdot) = 0$ on $\partial\Omega$ both of them are *normal* on the boundary. Hence, by the assumption, they are *equal* on $\partial\Omega$. In the mean time, the field $\nabla[G(a,\cdot) - G(b,\cdot)]$ is harmonic in $\dot{\Omega} \setminus [\{a\} \cup \{b\}]$, whereas d is harmonic in the whole $\dot{\Omega}$. The coincidence at the boundary implies the coincidence in the domain of harmonicity. Hence, $\nabla[G(a,\cdot) - G(b,\cdot)]$ can be extended by continuity to the whole Ω and $\nabla[G(a,\cdot) - G(b,\cdot)] = d$ everywhere. However, the latter is impossible since

$$\operatorname{div}\nabla[G(a,\cdot) - G(b,\cdot)] = \Delta[G(a,\cdot) - G(b,\cdot)] = \delta_a - \delta_b\,,$$

whereas $\operatorname{div} d = 0$ everywhere in $\dot{\Omega}$. This contradiction shows that $P_a - P_b \notin \nu\cdot\mathscr{D}$.

Step 3. The case of a and/or b belonging to the boundary is reduced to the previous one by the collar theorem arguments, which were applied at the end of the proof of Lemma 1. ☐

Corollary 2. *The algebra* $\vee\mathscr{Q}(\Omega) \subset C(\Omega; \mathbb{H})$ *strongly separates points of* Ω.

This property plays important role in proving density theorems [13].

Completing the proof of Theorem 1

Recall that $\mathscr{C} = \overline{\vee \mathscr{Q}(\Omega)}$ and prove that $\mathscr{C} = \mathsf{C}(\Omega; \mathsf{H})$. The fact, which will play the key role, is the embedding $\mathscr{C} \supset \mathsf{C}(\Omega; \mathbb{R}) \cong \mathsf{C}^{\mathbb{R}}(\Omega)$: see (3.16).

• Fix an $x \in \Omega$ and choose the smooth boundary controls f_1^x, f_2^x, f_3^x such that $\nabla w^{f_1^x}(x), \nabla w^{f_2^x}(x), \nabla w^{f_3^x}(x)$ constitute a basis of $T\Omega_x$. It is possible owing to Lemma 1. By continuity, there is a ball $B_{r(x)}[x] \subset \Omega$ centered at x, of (small enough) radius $r(x)$, such that $\nabla w^{f_1^x}(y), \nabla w^{f_2^x}(y), \nabla w^{f_3^x}(y)$ is a basis of $T\Omega_y$ for each $y \in B_{r(x)}[x]$.

Let such a choice be done for each $x \in \Omega$.

• The balls provide an open cover $\Omega = \cup_{x \in \Omega} B_{r(x)}[x]$. By compactness there is a finite subcover $\Omega = \cup_{n=1}^{N} B_{r_n}[x_n]$, where $r_n := r(x_n)$. Let $\eta_1, \dots, \eta_N$ be a partition of unit subordinated to the subcover, so that

$$\eta_1, \dots, \eta_N \in C^\infty(\Omega), \quad \operatorname{supp} \eta_n \subset B_{r_n}[x_n], \quad \sum_{n=1}^{N} \eta_n \equiv 1 \text{ in } \Omega$$

holds.

• Take $p = \{\alpha, u\} \in \mathsf{C}(\Omega; \mathsf{H})$ and represent

$$p = \sum_{n=1}^{N} \eta_n p = \{\sum_{n=1}^{N} \eta_n \alpha, \sum_{n=1}^{N} \eta_n u\} = \sum_{n=1}^{N} \{\eta_n \alpha, 0\} + \sum_{n=1}^{N} \{0, \eta_n u\}$$

with $\{\eta_n \alpha, 0\} \in \mathsf{C}(\Omega; \mathbb{R}) \subset \mathscr{C}$. In the mean time, one has

$$\eta_n u = \sum_{k=1}^{3} \varkappa_k^n \, \nabla w^{f_k^{x_n}}$$

with the certain $\varkappa_k^n \in \mathsf{C}^{\mathbb{R}}(\Omega)$ supported in $B_{r_n}[x_n]$. Note that $\{\varkappa_k^n, 0\} \in \mathsf{C}(\Omega; \mathbb{R}) \subset \mathscr{C}$.

Summarizing, we arrive at the representation

$$p = \sum_{n=1}^{N} \{\eta_n \alpha, 0\} + \sum_{n=1}^{N} \sum_{k=1}^{3} \{\varkappa_k^n, 0\}\{0, \nabla w^{f_k^{x_n}}\},$$

where all cofactors and summands do belong to $\mathscr{C}$. Thus $p \in \mathscr{C}$ and, hence, $\mathsf{C}(\Omega; \mathsf{H}) = \mathscr{C}$.

Theorem 1 is proved.

Remark 2. *Analyzing the proof, it is easy to recognize that the family* $\mathscr{W} := \big\{\{0, \nabla w^f\} \mid f \textit{ is smooth}\big\}$, *which is smaller than* $\mathscr{Q}(\Omega)$, *also generates the whole of the continuous field algebra:* $\overline{\vee \mathscr{W}} = \mathsf{C}(\Omega; \mathsf{H})$.

4 Controllability of 2-jets

Fix an $a \in \dot\Omega$; let x^1, x^2, x^3 be the local coordinates in a neighborhood $\omega \ni a$. With a smooth function ϕ one associates the row of its 0,1,2-order derivatives

$$\begin{aligned} j_a[\phi] &:= \{\phi(a);\ \phi_{x^1}(a), \phi_{x^2}(a), \phi_{x^3}(a);\\ &\phi_{x^1x^1}(a), \phi_{x^1x^2}(a), \phi_{x^1x^3}(a), \phi_{x^2x^2}(a), \phi_{x^2x^3}(a), \phi_{x^3x^3}(a)\} \in \mathbb{R}^{10}, \end{aligned}$$

which provides a coordinate representation of its *second jet* at the point a [20]. For short, we say $j_a[\phi]$ to be a 2-jet of ϕ at a and consider $\mathbb{R}^{10}$ with the (standard) inner product $\langle j, j'\rangle$ as a space of 2-jets.

Recall that in coordinates the Laplacian acts by

$$\Delta\phi = g^{-\frac{1}{2}}[g^{\frac{1}{2}} g^{ik}\phi_{x^k}]_{x^i},$$

where $\{g^{ik}\}$ is the inverse to the metric tensor matrix $\{g_{ik}\}$ and $g = \det\{g_{ik}\}$ (summation over repeating indexes is in the use). We say the row

$$\begin{aligned} &\lambda_a := \\ &= \{0; g^{-\frac{1}{2}}[g^{\frac{1}{2}}g^{i1}]_{x^i}, g^{-\frac{1}{2}}[g^{\frac{1}{2}}g^{i2}]_{x^i}, g^{-\frac{1}{2}}[g^{\frac{1}{2}}g^{i3}]_{x^i}; g^{11}, 2g^{12}, 2g^{13}, g^{22}, 2g^{23}, g^{33}\}\big|_{x=a} \end{aligned}$$

to be the *Laplace jet* and represent $(\Delta\phi)(a) = \langle\lambda_a, j_a[\phi]\rangle$.

The harmonicity $\Delta w = 0$ is equivalent to the orthogonality $\langle j_a[w], \lambda_a\rangle = 0,\ \ a \in \omega$. Therefore one has $j_a[w] \in \mathbb{R}^{10} \ominus \operatorname{span}\lambda_a$. Let us show that the 2-jets of harmonic functions exhaust the subspace $\mathbb{R}^{10} \ominus \operatorname{span}\lambda_a$. This result may be interpreted as a point-wise boundary controllability of 2-jets by harmonic functions. Recall that w^f is a solution to (3.5), (3.6).

Lemma 3. *For any $a \in \Omega$ and $s \in \mathbb{R}^{10} \ominus \operatorname{span}\lambda_a$ there is a smooth f such that $j_a[w^f] = s$.*

Proof. Taking into account the structure of the Laplace jet, we may deal with $s = \{0; s_1, s_2, s_3; s_{11}, \dots, s_{33}\}$, and let it be such that $0 \neq s \in \mathbb{R}^{10} \ominus \operatorname{span}\lambda_a$ but $\langle s, j_a[w^f]\rangle = 0$ for any smooth f. Show that such an assumption leads to contradiction.

• For a differential operator L with smooth coefficients in Ω, by L^* we denote its adjoint by Lagrange that is defined by

$$(L\eta, \zeta)_{L_2(\Omega)} = (\eta, L^*\zeta)_{L_2(\Omega)}, \qquad \eta, \zeta \in \mathscr{D}(\dot\Omega).$$

For a distribution $h \in \mathscr{D}'(\dot\Omega)$ one defines Lh by $(Lh, \eta) := (h, L^*\eta)_{L_2(\Omega)},\ \eta \in \mathscr{D}(\dot\Omega)$.

Let S be a differential operator, which acts by

$$\begin{aligned}
&(Sv)(x) \;= \\
&= \left[s_1 v_{x^1} + s_2 v_{x^2} + s_3 v_{x^3} + s_{11} v_{x^1 x^1} + s_{12} v_{x^1 x^2} + \dots + s_{33} v_{x^3 x^3}\right](x) = \\
&= \langle s, j_x[v]\rangle, \qquad x \in \omega
\end{aligned}$$

in a coordinate neighborhood ω of $a \in \dot\Omega$, where the (constant) coefficients are the components of the above chosen jet s.

- Let $\delta_a \in \mathscr{D}'(\dot\Omega)$ be the Dirac measure supported at the point $a \in \dot\Omega$. Consider the problem

$$\Delta H \;=\; S^*\delta_a \tag{4.1}$$

$$H\big|_{\partial\Omega} \;=\; 0\,. \tag{4.2}$$

The equation is understood as a relation in $\mathscr{D}'(\dot\Omega)$; its r.h.s. is a distribution acting by $(S^*\delta_a, \eta)_{L_2(\Omega)} = (S\eta)(a)$. The boundary condition does make sense since H is harmonic outside $\operatorname{supp} S^*\delta_a = \{a\}$. Also, the normal derivative $\partial_\nu H$ is a smooth function on $\partial\Omega$.

Formally by Green, for a function $v \in C^2(\Omega)$ one has

$$\begin{aligned}
&\langle s, j_a[v]\rangle = (Sv)(a) = \int_\Omega \delta_a\, Sv\, d\mu = \int_\Omega S^*\delta_a\, v\, d\mu \qquad \overset{(4.1)}{=} \int_\Omega \Delta H\, v\, d\mu = \\
&\overset{(4.2)}{=} \int_\Omega H\,\Delta v\, d\mu + \int_{\partial\Omega} \partial_\nu H\, v\, d\sigma\,.
\end{aligned}$$

To justify the final equality

$$\langle s, j_a[v]\rangle \;=\; \int_\Omega H\,\Delta v\, d\mu + \int_{\partial\Omega} \partial_\nu H\, v\, d\sigma \tag{4.3}$$

one can use the standard regularization technique, approximating δ_a by $\delta_a^\varepsilon \in \mathscr{D}(\dot\Omega)$ supported near a.

- By the choice of s, for $v = w^f$ the equality (4.3) provides

$$\int_{\partial\Omega} \partial_\nu H\, w^f\, d\sigma = \int_{\partial\Omega} \partial_\nu H\, f\, d\sigma = 0\,.$$

By arbitrariness of f we get $\partial_\nu H = 0$ on $\partial\Omega$. So, H is harmonic in $\Omega\setminus\{a\}$ and has the zero Cauchy data on the boundary. By the uniqueness theorem, H vanishes everywhere outside a. Hence, the distribution H is supported at a. The well-known fact of the distribution theory is that such an H is a linear combination of δ_a and its derivatives. In the mean time, comparing the orders of singularities in the left and right hand sides of (4.1), one easily concludes that

$$H = c\delta_a$$

with $c = \text{const} \neq 0$. Indeed, otherwise ΔH contains the derivatives of δ_a of the order $\geqslant 3$ that makes the equality (4.1) impossible.

For an $\eta \in \mathscr{D}(\dot{\Omega})$ one has

$$\langle s, j_a[\eta]\rangle = (\delta_a, S\eta) = (S^*\delta_a, \eta) \overset{(4.1)}{=} (\Delta c\delta_a, \eta) = (c\delta_a, \Delta\eta) = \langle c\lambda_a, j_a[\eta]\rangle\,.$$

Comparing the beginning with the end and referring to the evident $\{j_a[\eta] \mid \eta \in \mathscr{D}(\dot{\Omega})\} = \mathbb{R}^{10}$, we arrive at $s = c\lambda_a$ that contradicts to the starting assumption $s \perp \lambda_a$.

• The case $a \in \partial\Omega$ is reduced to the previous one by means of the trick already used at the end of the proof of Lemma 1: embedding $\Omega \Subset \Omega'$. □

As is easy to recognize, Lemma 3 implies the assertion of Lemma 1 for the case of the single point a. However, Lemma 3 may be generalized on the finite set $a_1, \ldots, a_N$ so that the relevant boundary controllability of 2-jets of harmonic functions holds up to the natural defect in $\oplus \sum_i \mathbb{R}^{10}_{a_i}$.

Determination of metric from harmonic functions

The metric on Ω determines the family of harmonic functions. The converse is also true in the following sense.

• Let $c > 0$ be a smooth function on Ω and cg a conformal deformation of the metric g. By Δ_{cg} and Δ_g we denote the corresponding Laplacians. A simple calculation leads to the relation

$$\Delta_{cg} y \;=\; c^{-1}\Delta_g y - 2^{-1}\,\nabla c^{-1}\cdot\nabla y\,, \tag{4.4}$$

which is specific for the 3d case. Taking $y = w^f$, we see that the metrics cg and g have the same reserve of harmonic functions w^f if and only if $\nabla c^{-1}\cdot\nabla w^f = 0$ holds for any smooth f. In the mean time, by Lemma 1 the gradients $\nabla w^f = 0$ constitute the local bases in Ω. Hence, the latter equality implies $\nabla c^{-1} = 0$, i.e., $c = \text{const}$.

• Fix a point a in a coordinate neighborhood $\omega \ni a$. By λ^g_a we denote the Laplace jet of the given metric g. By Lemma 3, the space of jets is

$$\mathbb{R}^{10}_a = \{j_a[\phi] \mid \phi \text{ is smooth}\} = \{j_a[w^f] \mid f \text{ is smooth}\} \oplus \operatorname{span}\lambda^g_a\,. \tag{4.5}$$

Therefore, writing $(\Delta w^f)(a) = 0$ in the form

$$\langle\lambda^g_a, j_a[w^f]\rangle = 0, \qquad f \text{ is smooth}$$

and varying $f = f_1, f_2, \ldots$, we get a linear homogeneous algebraic system with respect to the components of the jet λ^g_a, which determines them up to a factor, which may depend on a. Along with the components, we determine the tensor g up to a factor, possibly depending on a. However, by the above mentioned geometric reasons, this factor is a constant.

Thus, the family $\{w^f \mid f \text{ is smooth}\}$ determines the metric g up to a constant positive factor. If g is known at least at a single point $x_0 \in \Omega$, then it is uniquely determined everywhere.

Notice in addition that in two-dimensional case relation (4.4) is of the form $\Delta_{cg} y = c^{-1}\Delta_g y$, so that the metrics cg and g determine the same reserve of harmonic functions. It is the reason, because of which in 2d impedance tomography problem the metric is recovered up to conformal equivalence [2].

• Here we describe a trick, which is used in dynamical/spectral inverse problems and 2d impedance tomography problem, for recovering the metric via boundary data[9]. The hope is that it may be useful in future investigation of 3d ITP.

Assume that a topological space $\tilde{\Omega}$ is *homeomorphic* to Ω via a homeomorphism $\beta : \Omega \to \tilde{\Omega}$. Also assume that the family of functions

$$\{\tilde{w}^f = w^f \circ \beta^{-1} \mid f \text{ is smooth}\}$$

is given. The following procedure enables one to determine the metric $\tilde{g} = \beta_* g$ in $\tilde{\Omega}$.

1. Fix a point $a \in \tilde{\Omega}$ and choose its neighborhood $\tilde{\omega}$ with the coordinates x^1, x^2, x^3. By the way, Lemma 1 enables one to use the images $\tilde{w}^f$ as local coordinates.

2. Find $\operatorname{span}\lambda_a^{\tilde{g}}$ by (4.5) (replacing functions w^f on ω with $\tilde{w}^f$ on $\tilde{\omega}$). As was shown above, the family of these subspaces given for $a \in \tilde{\omega}$ determines the metric up to a constant factor. So, $c\tilde{g}$ is recovered. Assuming $\tilde{g}$ to be known at least at a single point $a_0 \in \tilde{\omega}$, one recovers $\tilde{g}$ uniquely.

3. Covering $\tilde{\Omega}$ by the coordinate neighborhoods and repeating the previous steps, we determine $\tilde{g}$ in $\tilde{\Omega}$.

5 Uniqueness properties of harmonic fields

Roughly speaking, the following result means that the set of zeros of a harmonic quaternion field may be at most of dimension 1.

Lemma 4. *Let $\Sigma \in \Omega$ be a C^2-smooth surface (2-dim submanifold). If $p \in \mathscr{Q}(\Omega)$ obeys $p|_\Sigma = 0$ then $p = 0$ in the whole Ω.*

Proof. Since the claimed result is of local character, we assume Σ to be a both-side surface endowed with a smooth field of the unit normals ν. Also, Σ possesses the (induced) Riemannian metric and is provided with the corresponding operations on vector fields. In particular, a divergence, which is denoted by $\operatorname{div}_\Sigma$, is well defined.

- For a point $x \in \Sigma$ and vector $v \in T\Omega_x$ we represent

$$v = v_\theta + v_\nu : \qquad v_\nu = v \cdot \nu\, \nu, \quad v_\theta = v - v_\nu$$

and, by default, identify v_θ with the proper vector of $T\Sigma_x$. By the latter, for a smooth vector field v given in a neighborhood of Σ, the value $[\operatorname{div}_\Sigma v_\theta](x)$ is of clear meaning. Also, recall the well-known vectot analysis relation

$$\nu \cdot \operatorname{rot} v \,=\, \operatorname{div}_\Sigma \nu \wedge v_\theta \qquad \text{on} \;\; \Sigma \tag{5.1}$$

(see, e.g. [21]).

- Begin with the case $\Sigma \subset \dot{\Omega}$. Let $p = \{\alpha, u\} \in \mathscr{Q}(\Omega)$, so that

$$\nabla\alpha \,=\, \operatorname{rot} u, \quad \operatorname{div} u \,=\, 0 \qquad \text{in} \;\; \dot{\Omega} \tag{5.2}$$

holds. Let $p|_\Sigma = 0$. Since $\alpha|_\Sigma = 0$, we have $(\nabla\alpha)_\theta|_\Sigma = 0$ that implies $(\operatorname{rot} u)_\theta\big|_\Sigma = 0$ by (5.2). In the mean time, $u|_\Sigma = 0$ is equivalent to $u_\theta = u_\nu = 0$ on Σ; hence $(\operatorname{rot} u)_\nu|_\Sigma = \operatorname{div}_\Sigma \nu \wedge u_\theta = 0$ by virtue of (5.1). Thus we get $(\operatorname{rot} u)_\theta|_\Sigma = (\operatorname{rot} u)_\nu|_\Sigma = 0$, i.e. $\operatorname{rot} u|_\Sigma = 0$.

The latter equality and (5.2) lead to $(\nabla\alpha)|_\Sigma = 0$ (along with $\alpha|_\Sigma = 0$). So, α is a harmonic function with the zero Cauchy data on Σ. Therefore $\alpha = 0$ in Ω by the elliptic uniqueness theorems [16].

As a result, $\operatorname{rot} u = \nabla\alpha = 0$ everywhere in Ω. Since $\operatorname{div} u = 0$, the vector field u is harmonic in Ω and vanishes on Σ. Therefore, locally near the points $x \in \Sigma$ one represents $u = \nabla\varphi$ with a *harmonic* function φ provided $\nabla\varphi|_\Sigma = 0$. Such a function is a constant; hence $u = 0$ near Σ. By its harmonicity, u vanishes globally in Ω.

So, we have $p = 0$ in Ω.

- The case $\Sigma \subset \partial\Omega$ is reduced to the previous one by means of the trick already used at the end of the proof of Lemma 1: embedding $\Omega \Subset \Omega'$. $\square$

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Some New Simple Inequalities Involving Exponential, Trigonometric and Hyperbolic Functions

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ABSTRACT

The prime goal of this paper is to establish sharp lower and upper bounds for useful functions such as the exponential functions, with a focus on $\exp(-x^2)$, the trigonometric functions (cosine and sine) and the hyperbolic functions (cosine and sine). The bounds obtained for hyperbolic cosine are very sharp. New proofs, refinements as well as new results are offered. Some graphical and numerical results illustrate the findings.

RESUMEN

El objetivo principal de este artículo es establecer cotas inferiores y superiores precisas para funciones útiles tales como las funciones exponenciales, con énfasis especial en $\exp(-x^2)$, las funciones trigonométricas (coseno y seno) y las funciones hiperbólicas (coseno y seno). Las cotas obtenidas para el coseno hiperbólico son muy precisas. Se presentan, tanto nuevas demostraciones y refinamientos, como resultados nuevos. Algunos resultados numéricos y gráficos ilustran los resultados encontrados.

Keywords and Phrases: Exponential function; trigonometric function; hyperbolic function.

2010 AMS Mathematics Subject Classification: 26D07, 33B10, 33B20.

1 Introduction

Sharp bounds for useful functions play a central role in many areas of mathematics and theoretical physics. They aim to provide some properties of functions of interest, possibly complex, by dealing with more tractable functions (in the context). The literature on the bounds dealing with the special functions such as e^{-x^2}, $\cos(x)$, $\sin(x)$, $\mathrm{sinc}(x)$, $\cosh(x)$, $\sinh(x)$ and $\tanh(x)$, is very vast. Recent developments can be found in [10, 11, 7, 5, 1, 20, 17, 4, 15, 6, 21, 16, 3, 8, 14, 13, 18, 19] and the references therein. In this paper, we offer new simple tight (lower and upper) bounds involving these functions, with a high potential of interest for many researchers in mathematics or theoretical physics. Some proofs of our results are based on the so-called l'Hospital's rule of monotonicity, the others used recent results with a new approach. The sharpness of our bounds are highlighted by some graphics and numerical studies using a global L_2 error as benchmark.

The result below shows bounds for e^{-x^2} defined with the cosine function and well-chosen constants.

Proposition 1.1. *For $x \in (0, \pi/2)$, the best possible constants α and β in the following inequalities*

$$\frac{\cos(x) - 1 + \alpha}{\alpha} \leqslant e^{-x^2} \leqslant \frac{\cos(x) - 1 + \beta}{\beta} \tag{1.1}$$

are $1/2$ and ≈ 1.092663 respectively.

The interest of Proposition 1.1 is the simplicity of the bounds, with very tractable expressions. It can be useful to evaluate complex functions depending on e^{-x^2} (Gaussian probability density function, error function etc.). The bounds of Proposition 1.1 are illustrated in Figure 1. We see that the lower bound is sharp for small values for x.

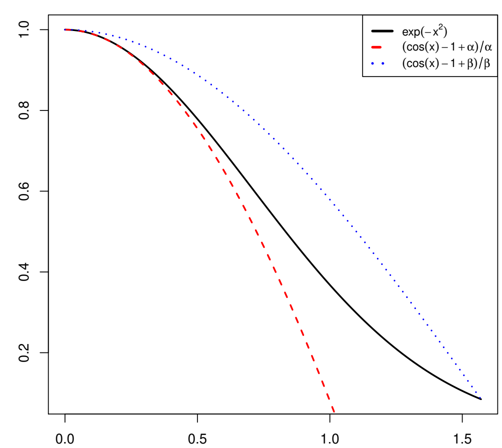

Figure 1: Graphs of the functions of the bounds (1.1) for $x \in (0, \pi/2)$.

Note: Using exponential and cosine series, Proposition 1.1 can be expressed in terms of alternating series as follows.
For $x \in (-\pi/2, \pi/2)$, we have

$$\frac{1}{\alpha}\sum_{k=1}^{\infty}\frac{(-1)^k x^{2k}}{(2k)!} \leqslant \sum_{k=1}^{\infty}\frac{(-1)^k x^{2k}}{k!} \leqslant \frac{1}{\beta}\sum_{k=1}^{\infty}\frac{(-1)^k x^{2k}}{(2k)!},$$

where α and β are as defined above.

Now let us recall that the $sinc$ function is defined by

$$\operatorname{sinc}(x) = \begin{cases} \frac{\sin(x)}{x} & x \neq 0, \\ 1 & x = 0. \end{cases} \tag{1.2}$$

It is of importance due to it's frequent occurrence in Fourier analysis. So the interest of finding the bounds of this type of functions is increasing. In the next proposition, we give new bounds to sinc function using hyperbolic tangent.

Proposition 1.2. *For $x \in (0, \pi/2)$, we have*

$$\left(\frac{\tanh(x)}{x}\right)^{\delta} < \frac{\sin(x)}{x} < \left(\frac{\tanh(x)}{x}\right)^{\eta} \tag{1.3}$$

with the best possible constants $\delta = 0.839273$ and $\eta = 1/2$.

In the following propositions, the inequalities presented are somewhat Cusa-Huygen's type [13, 18]. Proposition 1.3 below provides bounds for the sinc function using e^{-x^2} or hyperbolic cosine.

Proposition 1.3. *For $x \in (0, \pi/2)$, the inequalities*

$$\left(\frac{2+e^{-x^2}}{3}\right)^{a} < \frac{\sin(x)}{x} < \left(\frac{2+e^{-x^2}}{3}\right)^{b} \tag{1.4}$$

and

$$\left(\frac{3}{2+\cosh(x)}\right)^{c} < \frac{\sin(x)}{x} < \left(\frac{3}{2+\cosh(x)}\right)^{d} \tag{1.5}$$

are true with the best possible constants $a \approx 1.240827$, $b = 1/2$, $c \approx 1.108171$ and $d = 1$.

In view of Propositions 1.2 and 1.3, it is natural to address the following question: Which bounds for sinc are the best ? We provide the answer by doing a numerical study. We investigate the global L_2 error defined by

$$e(u) = \int_0^{\pi/2} \left(\frac{\sin x}{x} - u(x) \right)^2 dx,$$

where $u(x)$ denotes bound (lower or upper) in (1.3), (1.4) and (1.5). The results are summarized in Table 1.

Table 1: Global L_2 errors $e(u)$ for $\mathrm{sinc}(x)$ and the functions $u(x)$ in the bounds of (1.3), (1.4) and (1.5) for $x \in (0, \pi/2)$.

	Inequality (1.3)	
$u(x)$	lower	upper
$e(u)$	≈ 0.001421437	≈ 0.003648618
	Inequality (1.4)	
$u(x)$	lower	upper
$e(u)$	≈ 0.006242974	≈ 0.008628254
	Inequality (1.5)	
$u(x)$	lower	upper
$e(u)$	$\approx 6.53313 \times 10^{-5}$	≈ 0.0001542441

It follows from Table 1 that the bounds (1.5) are more sharp. This sharpness is illustrated in Figure 2.

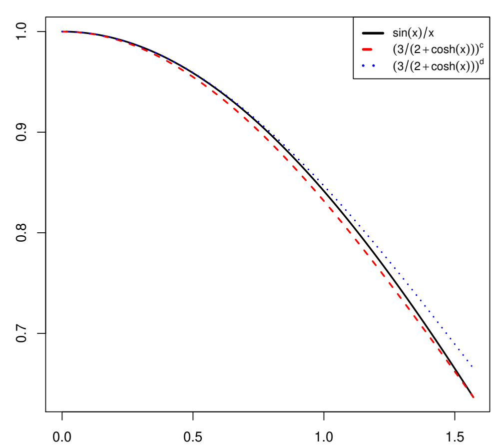

Figure 2: Graphs of the functions of the bounds (1.5) for $x \in (0, \pi/2)$.

The next result provides bounds for $x/\sinh(x)$ using cosine function.

Proposition 1.4. *If* $x \in (0, \pi/2)$ *then we have*

$$\left(\frac{2+\cos(x)}{3}\right)^m < \frac{x}{\sinh(x)} < \left(\frac{2+\cos(x)}{3}\right)^n \tag{1.6}$$

with the constants $m \approx 1.014227$ *and* $n \approx 0.928648$.

The obtained bounds are illustrated in Figure 3.

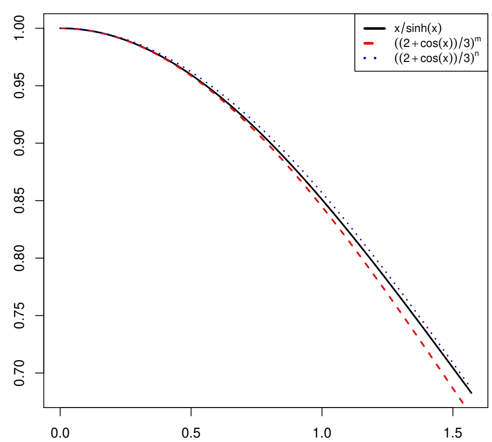

Figure 3: Graphs of the functions of the bounds (1.6) for $x \in (0, \pi/2)$.

Note: The inequality

$$\frac{2+\cos(x)}{3} < \frac{x}{\sinh(x)}$$

is more sharp version of left inequality of (1.6). It is appeared in [19, Theorem 6].

Proposition 1.5 below presents sharp bounds for $\sinh(x)/x$ using hyperbolic cosine.

Proposition 1.5. *For* $x \in (0, \pi/2)$ *one has*

$$\left(\frac{2+\cosh(x)}{3}\right)^p < \frac{\sinh(x)}{x} < \left(\frac{2+\cosh(x)}{3}\right)^q \tag{1.7}$$

with the constants $p \approx 0.928648$ *and* $q \approx 1.009155$.

The bounds are illustrated in Figure 4.

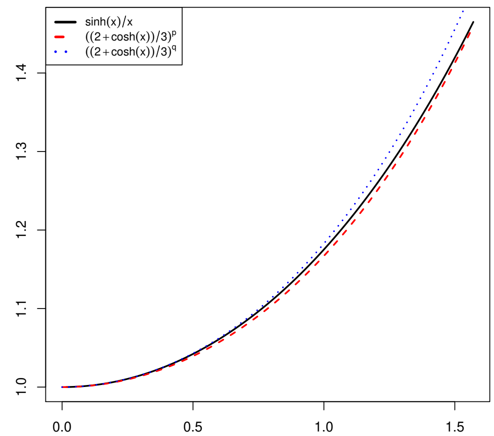

Figure 4: Graphs of the functions of the bounds (1.7) for $x \in (0, \pi/2)$.

Note: The hyperbolic Cusa-Huygen's inequality[16]

$$\frac{\sinh(x)}{x} < \frac{2+\cosh(x)}{3}$$

is however more sharp than right inequality of (1.7).

The rest of the study is devoted to new bounds for $\cosh(x)$, with discussion. A well-known upper bound for $\cosh(x)$ is given by $e^{x^2/2}$. This result was recently completed by Yogesh Bagul[3, Theorem 2.1] who finds a sharp lower bound, i.e.

$$e^{ax^2} < \cosh(x) < e^{x^2/2}, \quad x \in (0,1), \tag{1.8}$$

with the best possible constants $a \approx 0.433781$ and $1/2$. We now aim to refine the inequalities of (1.8) in Proposition 1.6 below.

Proposition 1.6. *For* $x \in (0,1)$, *we have*

$$\exp\left(\frac{3}{2}\left(1-e^{-x^2/3}\right)\right) \leqslant \cosh(x) \leqslant \exp\left(\frac{1}{2\theta}\left(1-e^{-\theta x^2}\right)\right) \tag{1.9}$$

with $\theta \approx 0.272342$.

Note: Using the well-known inequality $e^y \geqslant 1+y$ for $y \in \mathbb{R}$, we obtain $\exp\left(\left(1-e^{-\theta x^2}\right)/(2\theta)\right) \leqslant e^{x^2/2}$. This proves that the upper bound in (1.9) is sharper to the one in (1.8).

Alternative bounds are given in Proposition 1.7 below, with discussion.

Proposition 1.7. *For* $x \in (0,1)$*, we have*

$$\left(1+\frac{x^2}{3}\right)^{3/2} \leqslant \cosh(x) \leqslant \left(1+\frac{x^2}{\xi}\right)^{\xi/2} \tag{1.10}$$

with $\xi \approx 3.194528$.

Note: Again, using the well-known inequality $e^y \geqslant 1+y$ for $y \in \mathbb{R}$, we get $\left(1+x^2/\xi\right)^{\xi/2} \leqslant e^{x^2/2}$. This shows that the upper bound in (1.10) is sharper to the one in (1.8).

We now claim that the bounds obtained in (1.10) are better than those in (1.8) and (1.9). Numerical results support this claim. Indeed, by considering the global L_2 error defined by

$$e_*(u) = \int_0^1 (\cosh(x) - u(x))^2 \, dx,$$

where $u(x)$ denotes bound (lower or upper) in (1.8), (1.9) and (1.10), Table 1 indicates that (1.10) are the best.

Table 2: Global L_2 errors $e_*(u)$ for $\cosh(x)$ and the functions $u(x)$ in the bounds of (1.8), (1.9) and (1.10) for $x \in (0,1)$.

	Inequality (1.8)	
$u(x)$	lower	upper
$e_*(u)$	≈ 0.0001352084	≈ 0.001139289
	Inequality (1.9)	
$u(x)$	lower	upper
$e_*(u)$	$\approx 1.335929 \times 10^{-5}$	$\approx 7.004029 \times 10^{-6}$
	Inequality (1.10)	
$u(x)$	lower	upper
$e_*(u)$	$\approx 9.456552 \times 10^{-7}$	$\approx 6.895902 \times 10^{-7}$

The sharpness of the obtained bounds is illustrated in Figures 5 and 6 (for a zoom on the interval $(0.95, 1)$, where the hierarchy of the bounds is more clear).

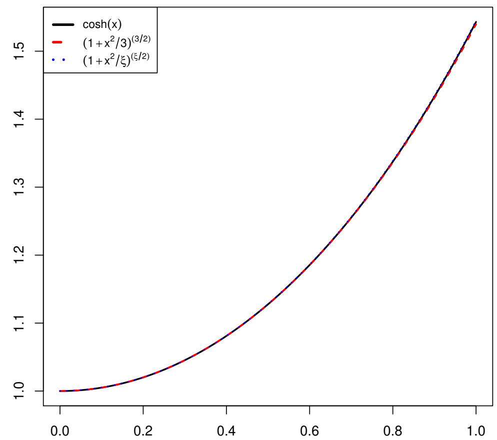

Figura 5: Graphs of the functions of the bounds (1.10) for $x \in (0, 1)$.

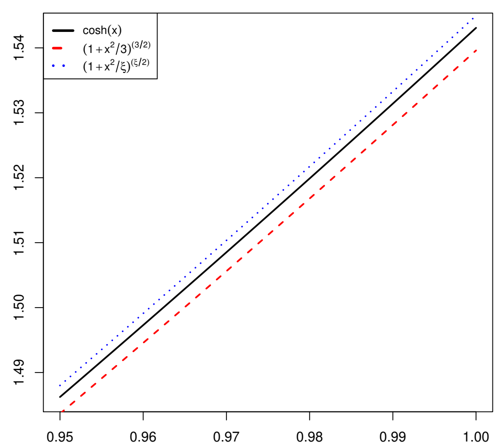

Figura 6: Graphs of the functions of the bounds (1.10) for $x \in (0.95, 1)$.

Note: To prove the inequalities (1.5), (1.6) and (1.7), we will simply use the results of [7, 5, 12]. We stress on the fact that it is not difficult to verify that all the results in [5] are also true in $(0, \pi/2)$ with the respective best possible constants obtained accordingly (see [12]). Propositions 1.6 and 1.7 will be proved by the techniques of integration on some known results[4, 6]. For proving Proposition 1.1, Proposition 1.2 and Proposition 1.3, we need the Lemmas presented in the next section.

2 Lemmas

The following Lemma is known as l'Hospital's rule of monotonicity. The details are given in [9] and [2].

Lemma 2.1. *([2]) Let f, g be two real valued functions which are continuous on $[a, b]$ and differentiable on (a, b), where $-\infty < a < b < \infty$ and $g'(x) \neq 0$, for $\forall x \in (a, b)$. Let,*

$$A(x) = \frac{f(x) - f(a)}{g(x) - g(a)}$$

and

$$B(x) = \frac{f(x) - f(b)}{g(x) - g(b)}.$$

Then,
I) $A(x)$ and $B(x)$ are increasing on (a, b) if f'/g' is increasing on (a, b) and
II) $A(x)$ and $B(x)$ are decreasing on (a, b) if f'/g' is decreasing on (a, b).
The strictness of the monotonicity of $A(x)$ and $B(x)$ depends on the strictness of monotonicity of f'/g'.

Lemma 2.2. $H(x) = \frac{\sin(x) - x\cos(x)}{x^2 \sin(x)}$ *is strictly positive increasing in $(0, \pi/2)$.*

Proof: $H(x)$ is positive as $\cos(x) < \frac{\sin(x)}{x}$ on $(0, \pi/2)$.
Consider,

$$H(x) = \frac{\sin(x) - x\cos(x)}{x^2 \sin(x)} = \frac{H_1(x)}{H_2(x)},$$

where $H_1(x) = \sin(x) - x\cos(x)$ and $H_2(x) = x^2 \sin(x)$ are such that $H_1(0) = 0$ and $H_2(0) = 0$. By differentiating

$$\frac{H_1'(x)}{H_2'(x)} = \frac{\sin(x)}{x\cos(x) + 2\sin(x)} = \frac{H_3(x)}{H_4(x)},$$

where $H_3(x) = \sin(x)$ and $H_4(x) = x\cos(x) + 2\sin(x)$ with $H_3(0) = 0$ and $H_4(0) = 0$. Again differentiating we get

$$\frac{H_3'(x)}{H_4'(x)} = \frac{\cos(x)}{-x\sin(x) + 3\cos(x)} = \frac{1}{-x\tan(x) + 3}.$$

Now, it is well known that $-x\tan(x)$ is decreasing in $(0, \pi/2)$ and so is $-x\tan(x) + 3$. By Lemma 1, $H(x)$ is a strictly increasing function in $(0, \pi/2)$.

3 Proofs of the Main Results

This section is devoted to the proofs of our main results.

Proof of Proposition 1.1: Clearly, the equalities hold at $x = 0$. Consider

$$f(x) = \frac{\cos(x) - 1}{e^{-x^2} - 1} = \frac{f_1(x)}{f_2(x)},$$

where $f_1(x) = \cos(x) - 1$ and $f_2(x) = e^{-x^2} - 1$ with $f_1(0) = 0$ and $f_2(0) = 0$. By differentiation, we obtain

$$\frac{f_1'(x)}{f_2'(x)} = \frac{\sin(x)e^{x^2}}{2x} = \frac{f_3(x)}{f_4(x)},$$

where $f_3(x) = \sin(x)e^{x^2}$ and $f_4(x) = 2x$ with $f_3(0) = 0$ and $f_4(0) = 0$. Again differentiating we get

$$\begin{aligned}\frac{f_3'(x)}{f_4'(x)} &= \frac{e^{x^2}}{2}[\cos(x) + 2x\sin(x)] \\ &= \frac{e^{x^2}}{2}F(x),\end{aligned}$$

where $F(x) = \cos(x) + 2x\sin(x)$. Differentiation gives

$$F'(x) = 2x\cos(x) + \sin(x) > 0$$

in $(0, \pi/2)$, which implies that $F(x)$ is increasing. Thus $\frac{f_3'(x)}{f_4'(x)}$ being a product of two positive increasing functions is a positive increasing. By Lemma 2.1, $f(x)$ is also increasing in $(0, \pi/2)$. So $\alpha = f(0+) = 1/2$ and $\beta = f(\pi/2-) = -1/[e^{-(\pi/2)^2} - 1] \approx 1.092663$. $\square$

Proof of Proposition 1.2: Let us set

$$h(x) = \frac{\log(\sin(x)/x)}{\log(\tanh(x)/x)} = \frac{h_1(x)}{h_2(x)},$$

where $h_1(x) = \log(\sin(x)/x)$ and $h_2(x) = \log(\tanh(x)/x)$ with $h_1(0+) = 0$ and $h_2(0+) = 0$. Differentiating we get

$$\frac{h_1'(x)}{h_2'(x)} = \frac{\sin(x) - x\cos(x)}{x^2\sin(x)} \frac{x^2\tanh(x)}{\tanh(x) - x\,\mathrm{sech}^2(x)} = H(x)\,J(x),$$

where $H(x) = \frac{\sin(x)-x\cos(x)}{x^2\sin(x)}$ and $J(x) = \frac{x^2\tanh(x)}{\tanh(x)-x\,\mathrm{sech}^2(x)}$. Now set

$$J(x) = \frac{J_1(x)}{J_2(x)},$$

where $J_1(x) = x^2\tanh(x)$ and $J_2(x) = \tanh(x) - x\,\mathrm{sech}^2(x)$ with $J_1(0) = 0$ and $J_2(0) = 0$. Differentiation gives

$$\begin{aligned}\frac{J_1'(x)}{J_2'(x)} &= \frac{x\,\mathrm{sech}^2(x) + 2\tanh(x)}{2\,\mathrm{sech}^2(x)\tanh(x)} \\ &= \frac{1}{2}\frac{x}{\tanh(x)} + \cosh^2(x),\end{aligned}$$

which is clearly increasing as both $x/\tanh(x)$ and $\cosh^2(x)$ are increasing. By Lemma 2.1, $J(x)$ is also increasing in $(0, \pi/2)$. Moreover, $J(x)$ is positive as $x/\sinh(x) < \cosh(x)$. By Lemma 2.2, $H(x)$ is strictly positive increasing in $(0, \pi/2)$. $h_1'(x)/h_2'(x)$, being product of two positive increasing functions is positive increasing. Again by Lemma 2.1, $h(x)$ is strictly increasing in $(0, \pi/2)$. So $\delta = \log(2/\pi)/\log(2\tanh(\pi/2)/\pi) \approx 0.839273$ and $\eta = f(0+) = 1/2$, by l'Hospital's rule. This completes the assertion. $\square$

Proof of Proposition 1.3:

- *Proof of* (1.4). Let

$$f(x) = \frac{\log(\sin(x)/x)}{\log\left(2 + e^{-x^2}\right) - \log 3} = \frac{f_1(x)}{f_2(x)},$$

where $f_1(x) = \log(\sin(x)/x)$ and $f_2(x) = \log\left(2 + e^{-x^2}\right) - \log 3$ such that $f_1(0+) = 0$ and $f_2(0) = 0$. Differentiation gives

$$\begin{aligned} \frac{f_1'(x)}{f_2'(x)} &= \frac{1}{2}\frac{(\sin(x) - x\cos(x))}{x^2\sin(x)}(2e^{x^2} + 1) \\ &= \frac{1}{2}H(x)\,G(x), \end{aligned}$$

where $H(x) = \frac{\sin(x)-x\cos(x)}{x^2\sin(x)}$ is strictly positive increasing in $(0, \pi/2)$ by Lemma 2.2 and $G(x) = 2e^{x^2} + 1$ is also clearly positive increasing. Therefore $H(x)\,G(x)$ is strictly increasing. By making use of Lemma 2.1, we conclude that $f(x)$ is strictly increasing in $(0, \pi/2)$. So

$$f(0+) < f(x) < f(\pi/2);\ x \in (0, \pi/2).$$

Hence, $a = f(\pi/2) = \log(2/\pi)/[\log(2 + e^{-(\pi/2)^2}) - \log 3] \approx 1.240827$ and $b = f(0+) = 1/2$ by l'Hospital's rule.

- *Proof of* (1.5). Utilizing [5, Theorem 2], [12, Proposition 3] we have

$$e^{-kx^2} < \frac{\sin(x)}{x} < e^{-x^2/6},$$

where $k = \frac{-\log(2/\pi)}{(\pi/2)^2}$. After rearrangement, it can be written as

$$\left(\frac{\sin(x)}{x}\right)^6 < e^{-x^2} < \left(\frac{\sin(x)}{x}\right)^{1/k}. \tag{3.1}$$

By virtue of [7, Theorem 2] we write

$$\left(\frac{3}{2 + \cosh(x)}\right)^{\gamma} < e^{-x^2} < \left(\frac{3}{2 + \cosh(x)}\right)^6, \tag{3.2}$$

where $\gamma = \frac{(\pi/2)^2}{\log[(2+\cosh(\pi/2))/3]}$. Combining (3.1) and (3.2), we get

$$\left(\frac{3}{2+\cosh(x)}\right)^c < \frac{\sin(x)}{x} < \left(\frac{3}{2+\cosh(x)}\right),$$

where $c = k\gamma = \frac{-\log(2/\pi)}{\log[(2+\cosh(\pi/2))/3]} \approx 1.108171.$ □

Proof of Proposition 1.4: According to [5, Theorem 3] and [12] we have

$$e^{-x^2/6} < \frac{x}{\sinh(x)} < e^{-tx^2},\ x \in (0, \pi/2),$$

where $t = \frac{-\log[\pi/(2\sinh(\pi/2))]}{(\pi/2)^2}$. It is equivalent to

$$\left(\frac{x}{\sinh(x)}\right)^{1/t} < e^{-x^2} < \left(\frac{x}{\sinh(x)}\right)^6. \tag{3.3}$$

Similarly, using [7, Theorem 1] we have

$$\left(\frac{2+\cos(x)}{3}\right)^{\lambda} < e^{-x^2} < \left(\frac{2+\cos(x)}{3}\right)^6, \tag{3.4}$$

where $\lambda = \frac{-(\pi/2)^2}{\log(2/3)}$. Combining (3.3) and (3.4) we get

$$\left(\frac{2+\cos(x)}{3}\right)^m < \frac{x}{\sinh(x)} < \left(\frac{2+\cos(x)}{3}\right)^n,$$

where $m = \frac{\lambda}{6} = \frac{-(\pi/2)^2}{6\log(2/3)} \approx 1.014227$ and $n = 6t = \frac{-6\log[\pi/(2\sinh(\pi/2))]}{(\pi/2)^2} \approx 0.928648.$ □

Proof of Proposition 1.5: The proof follows easily by combining inequalities (3.2) and (3.3) to get

$p = \frac{-6\log[\pi/(2\sinh(\pi/2))]}{(\pi/2)^2} \approx 0.928648$ and $q = \frac{(\pi/2)^2}{6\log[(2+\cosh(\pi/2))/3]} \approx 1.009155.$ □

Proof of Proposition 1.6: For $x = 0$ equalities hold obviously. Rearranging [4, Theorem 5], for any $t \in (0, 1)$, we have

$$te^{-t^2/3} < \tanh(t) < te^{-\theta t^2}$$

with $\theta \approx 0.272342$. Therefore by integration, for $x \in (0, 1)$, we get

$$\int_0^x te^{-t^2/3}dt < \int_0^x \tanh(t)dt < \int_0^x te^{-\theta t^2}dt,$$

which yields

$$\frac{3}{2}\left(1 - e^{-x^2/3}\right) < \log(\cosh(x)) < \frac{1}{2\theta}\left(1 - e^{-\theta x^2}\right).$$

By composing with the exponential function, we get the required result. □

Proof of Proposition 1.7: Clearly, the equalities hold at $x = 0$. Rearranging [6, Theorem 4], for any $t \in (0, 1)$, we have

$$\frac{3t}{3+t^2} < \tanh(t) < \frac{\xi t}{\xi + t^2}$$

with $\xi \approx 3.194528$. On integration, for $x \in (0, 1)$, we have

$$\int_0^x \frac{3t}{3+t^2} dt < \int_0^x \tanh(t) dt < \int_0^x \frac{\xi t}{\xi + t^2} dt$$

which implies that

$$\frac{3}{2}\log\left(1+\frac{x^2}{3}\right) < \log(\cosh(x)) < \frac{\xi}{2}\log\left(1+\frac{x^2}{\xi}\right).$$

The desired result follows by composing with the exponential function. □

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Commutator criteria for strong mixing II. More general and simpler

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ABSTRACT

We present a new criterion, based on commutator methods, for the strong mixing property of unitary representations of topological groups equipped with a proper length function. Our result generalises and unifies recent results on the strong mixing property of discrete flows $\{U^N\}_{N\in\mathbb{Z}}$ and continuous flows $\{e^{-itH}\}_{t\in\mathbb{R}}$ induced by unitary operators U and self-adjoint operators H in a Hilbert space. As an application, we present a short alternative proof (not using convolutions) of the strong mixing property of the left regular representation of σ-compact locally compact groups.

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RESUMEN

Presentamos un nuevo criterio, basado en métodos de conmutadores, para la propiedad de mezcla fuerte de representaciones unitarias de grupos topológicos dotados de una función de longitud propia. Nuestro resultado generaliza y unifica resultados recientes acerca de la propiedad de mezcla fuerte de flujos discretos $\{U^N\}_{N\in\mathbb{Z}}$ y flujos continuos $\{e^{-itH}\}_{t\in\mathbb{R}}$ inducidos por operadores unitarios U y operadores autoadjuntos H en un espacio de Hilbert. Como aplicación, presentamos una demostración corta alternativa (sin usar convoluciones) de la propiedad de mezcla fuerte de la representación regular de grupos localmente compactos σ-compactos.

Keywords and Phrases: Strong mixing, unitary representations, commutator methods.

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1 Introduction

In the recent paper [14], itself motivated by the previous papers [8, 12, 13, 15], it has been shown that commutator methods for unitary and self-adjoint operators can be used to establish strong mixing. The main results of [14] are the following two commutator criteria for strong mixing. First, given a unitary operator U in a Hilbert space $\mathcal{H}$, assume there exists an auxiliary self-adjoint operator A in $\mathcal{H}$ such that the commutators $[A, U^N]$ exist and are bounded in some precise sense, and such that the strong limit

$$D_1 := \operatorname*{s-lim}_{N\to\infty} \frac{1}{N}[A, U^N]U^{-N} \tag{1.1}$$

exists. Then, the discrete flow $\{U^N\}_{N\in\mathbb{Z}}$ is strongly mixing in $\ker(D_1)^\perp$. Second, given a self-adjoint operator H in $\mathcal{H}$, assume there exists an auxiliary self-adjoint operator A in $\mathcal{H}$ such that the commutators $[A, e^{-itH}]$ exist and are bounded in some precise sense, and such that the strong limit

$$D_2 := \operatorname*{s-lim}_{t\to\infty} \frac{1}{t}[A, e^{-itH}]\, e^{itH} \tag{1.2}$$

exists. Then, the continuous flow $\{e^{-itH}\}_{t\in\mathbb{R}}$ is strongly mixing in $\ker(D_2)^\perp$. These criteria were then applied to skew products of compact Lie groups, Furstenberg-type transformations, time changes of horocycle flows and adjacency operators on graphs.

The purpose of this note is to unify these two commutator criteria into a single, more general, commutator criterion for strong mixing of unitary representations of topological groups, and also to remove an unnecessary invariance assumption made in [14].

Our main result is the following. We consider a topological group X equipped with a proper length function $\ell : X \to \mathbb{R}_+$, a unitary representation $U : X \to \mathscr{U}(\mathcal{H})$, and a net $\{x_j\}_{j\in J}$ in X with $x_j \to \infty$ (see Section 2 for precise definitions). Also, we assume there exists an auxiliary self-adjoint operator A in $\mathcal{H}$ such that the commutators $[A, U(x_j)]$ exist and are bounded in some precise sense, and such that the strong limit

$$D := \operatorname*{s-lim}_{j} \frac{1}{\ell(x_j)}[A, U(x_j)]U(x_j)^{-1} \tag{1.3}$$

exists. Then, under these assumptions we show that the unitary representation U is strongly mixing in $\ker(D)^\perp$ along the net $\{x_j\}_{j\in J}$ (Theorem 2.3). As a corollary, we obtain criteria for strong mixing in the cases of unitary representations of compactly generated locally compact Hausdorff groups (Corollary 2.5) and the Euclidean group $\mathbb{R}^d$ (Corollary 2.7). These results generalise the commutator criteria of [14] for the strong mixing of discrete and continuous flows, as well as the strong limit (1.3) generalises the strong limits (1.1) and (1.2) (see Remarks 2.6 and 2.8). To conclude, we present in Example 2.9 an application which was not possible to cover with the results of [14]: a short alternative proof (not using convolutions) of the strong mixing property of the left regular representation of σ-compact locally compact Hausdorff groups.

We refer the reader to [4, 6, 9, 10, 11, 16] for references on strong mixing properties of unitary representations of groups.

2 Commutator criteria for strong mixing

We start with a short review of basic facts on commutators of operators and regularity classes associated with them. We refer to [1, Chap. 5-6] for more details.

Let $\mathcal{H}$ be an arbitrary Hilbert space with scalar product $\langle \cdot, \cdot \rangle$ antilinear in the first argument, denote by $\mathscr{B}(\mathcal{H})$ the set of bounded linear operators on $\mathcal{H}$, and write $\|\cdot\|$ both for the norm on $\mathcal{H}$ and the norm on $\mathscr{B}(\mathcal{H})$. Let A be a self-adjoint operator in $\mathcal{H}$ with domain $\mathcal{D}(A)$, and take $S \in \mathscr{B}(\mathcal{H})$. For any $k \in \mathbb{N}$, we say that S belongs to $C^k(A)$, with notation $S \in C^k(A)$, if the map

$$\mathbb{R} \ni t \mapsto e^{-itA} S e^{itA} \in \mathscr{B}(\mathcal{H}) \tag{2.1}$$

is strongly of class C^k. In the case $k = 1$, one has $S \in C^1(A)$ if and only if the quadratic form

$$\mathcal{D}(A) \ni \varphi \mapsto \langle \varphi, iSA\varphi \rangle - \langle A\varphi, iS\varphi \rangle \in \mathbb{C}$$

is continuous for the topology induced by $\mathcal{H}$ on $\mathcal{D}(A)$. We denote by $[iS, A]$ the bounded operator associated with the continuous extension of this form, or equivalently the strong derivative of the map (2.1) at $t = 0$. Moreover, if we set $A_\varepsilon := (i\varepsilon)^{-1}(e^{i\varepsilon A} - 1)$ for $\varepsilon \in \mathbb{R} \setminus \{0\}$, we have (see [1, Lemma 6.2.3(a)]):

$$\operatorname*{s-lim}_{\varepsilon \searrow 0} [iS, A_\varepsilon] = [iS, A]. \tag{2.2}$$

Now, if H is a self-adjoint operator in $\mathcal{H}$ with domain $\mathcal{D}(H)$ and spectrum $\sigma(H)$, we say that H is of class $C^k(A)$ if $(H - z)^{-1} \in C^k(A)$ for some $z \in \mathbb{C} \setminus \sigma(H)$. In particular, H is of class $C^1(A)$ if and only if the quadratic form

$$\mathcal{D}(A) \ni \varphi \mapsto \langle \varphi, (H - z)^{-1} A\varphi \rangle - \langle A\varphi, (H - z)^{-1}\varphi \rangle \in \mathbb{C}$$

extends continuously to a bounded form with corresponding operator denoted by $[(H - z)^{-1}, A] \in \mathscr{B}(\mathcal{H})$. In such a case, the set $\mathcal{D}(H) \cap \mathcal{D}(A)$ is a core for H and the quadratic form

$$\mathcal{D}(H) \cap \mathcal{D}(A) \ni \varphi \mapsto \langle H\varphi, A\varphi \rangle - \langle A\varphi, H\varphi \rangle \in \mathbb{C}$$

is continuous in the topology of $\mathcal{D}(H)$ (see [1, Thm. 6.2.10(b)]). This form then extends uniquely to a continuous quadratic form on $\mathcal{D}(H)$ which can be identified with a continuous operator $[H, A]$ from $\mathcal{D}(H)$ to the adjoint space $\mathcal{D}(H)^*$. In addition, the following relation holds in $\mathscr{B}(\mathcal{H})$ (see [1, Thm. 6.2.10(b)]):

$$[(H - z)^{-1}, A] = -(H - z)^{-1}[H, A](H - z)^{-1}. \tag{2.3}$$

With this, we can now present our first result, which is at the root of the new commutator criterion for strong mixing. For it, we recall that a net $\{x_j\}_{j\in J}$ in a topological space X diverges to infinity, with notation $x_j \to \infty$, if $\{x_j\}_{j\in J}$ has no limit point in X. This implies that for each compact set $K \subset X$, there exists $j_K \in J$ such that $x_j \notin K$ for $j \geq j_K$. In particular, X is not compact. We also fix the notations $\mathscr{U}(\mathcal{H})$ for the set of unitary operators on $\mathcal{H}$ and $\mathbb{R}_+ := [0, \infty)$.

Proposition 2.1. *Let $\{U_j\}_{j\in J}$ be a net in $\mathscr{U}(\mathcal{H})$, let $\{\ell_j\}_{j\in J} \subset \mathbb{R}_+$ satisfy $\ell_j \to \infty$, assume there exists a self-adjoint operator A in $\mathcal{H}$ such that $U_j \in C^1(A)$ for each $j \in J$, and suppose that the strong limit*

$$D := \operatorname*{s-lim}_j \frac{1}{\ell_j}[A, U_j]U_j^{-1}$$

exists. Then, $\lim_j \langle \varphi, U_j\psi \rangle = 0$ for all $\varphi \in \ker(D)^\perp$ and $\psi \in \mathcal{H}$.

Before the proof, we note that for $j \in J$ large enough (so that $\ell_j \neq 0$) the operators $\frac{1}{\ell_j}[A, U_j]U_j^{-1}$ are well-defined, bounded and self-adjoint. Therefore, their strong limit D is also bounded and self-adjoint.

Proof. Let $\varphi = D\widetilde{\varphi} \in D\mathcal{D}(A)$ and $\psi \in \mathcal{D}(A)$, take $j \in J$ large enough, and set

$$D_j := \frac{1}{\ell_j}[A, U_j]U_j^{-1}.$$

Since U_j and U_j^{-1} belong to $C^1(A)$ (see [1, Prop. 5.1.6(a)]), both $U_j\psi$ and $U_j^{-1}\widetilde{\varphi}$ belong to $\mathcal{D}(A)$. Thus,

$$\begin{aligned}
&\left|\langle \varphi, U_j\psi \rangle\right| \\
&= \left|\langle (D - D_j)\widetilde{\varphi}, U_j\psi \rangle + \langle D_j\widetilde{\varphi}, U_j\psi \rangle\right| \\
&\leq \left\|(D - D_j)\widetilde{\varphi}\right\| \|\psi\| + \frac{1}{\ell_j}\left|\langle [A, U_j]U_j^{-1}\widetilde{\varphi}, U_j\psi \rangle\right| \\
&\leq \left\|(D - D_j)\widetilde{\varphi}\right\| \|\psi\| + \frac{1}{\ell_j}\left|\langle A\widetilde{\varphi}, U_j\psi \rangle\right| + \frac{1}{\ell_j}\left|\langle U_jAU_j^{-1}\widetilde{\varphi}, U_j\psi \rangle\right| \\
&\leq \left\|(D - D_j)\widetilde{\varphi}\right\| \|\psi\| + \frac{1}{\ell_j}\left\|A\widetilde{\varphi}\right\| \|\psi\| + \frac{1}{\ell_j}\left\|\widetilde{\varphi}\right\| \|A\psi\|.
\end{aligned}$$

Since $D = \operatorname{s-lim}_j D_j$ and $\ell_j \to \infty$, we infer that $\lim_j \langle \varphi, U_j\psi \rangle = 0$, and thus the claim follows by the density of $D\mathcal{D}(A)$ in $\overline{D\mathcal{H}} = \ker(D)^\perp$ and the density of $\mathcal{D}(A)$ in $\mathcal{H}$. □

In the sequel, we assume that the unitary operators U_j are given by a unitary representation of a topological group X. We also assume that the scalars ℓ_j are given by a proper length function on X, that is, a function $\ell : X \to \mathbb{R}_+$ satisfying the following properties (e denotes the identity of X):

(L1) $\ell(e) = 0$,

(L2) $\ell(x^{-1}) = \ell(x)$ for all $x \in X$,

(L3) $\ell(xy) \le \ell(x) + \ell(y)$ for all $x, y \in X$,

(L4) if $K \subset \mathbb{R}_+$ is compact, then $\ell^{-1}(K) \subset X$ is relatively compact.

Remark 2.2 (Topological groups with a proper left-invariant pseudo-metric)**.** *Let X be a Hausdorff topological group equipped with a proper left-invariant pseudo-metric $d : X \times X \to \mathbb{R}_+$ (see [7, Def. 2.A.5 & 2.A.7]). Then, simple calculations show that the associated length function $\ell : X \to \mathbb{R}_+$ given by $\ell(x) := d(e, x)$ satisfies the properties (L1)-(L4) above. Examples of groups admitting a proper left-invariant pseudo-metric are σ-compact locally compact Hausdorff groups [7, Prop. 4.A.2], as for instance compactly generated locally compact Hausdorff groups with the word metric [7, Prop. 4.B.4(2)].*

The next theorem provides a general commutator criterion for the strong mixing property of a unitary representation of a topological group. Before stating it, we recall that if a topological group X is equipped with a proper length function ℓ, and if $\{x_j\}_{j\in J}$ is a net in X with $x_j \to \infty$, then $\ell(x_j) \to \infty$ (this can be shown by absurd using the property (L4) above).

Theorem 2.3 (Topological groups)**.** *Let X be a topological group equipped with a proper length function ℓ, let $U : X \to \mathscr{U}(\mathcal{H})$ be a unitary representation of X, let $\{x_j\}_{j\in J}$ be a net in X with $x_j \to \infty$, assume there exists a self-adjoint operator A in $\mathcal{H}$ such that $U(x_j) \in C^1(A)$ for each $j \in J$, and suppose that the strong limit*

$$D := \operatorname*{s-lim}_{j} \frac{1}{\ell(x_j)} [A, U(x_j)] U(x_j)^{-1} \tag{2.4}$$

exists. Then,

(a) $\lim_j \langle \varphi, U(x_j)\psi \rangle = 0$ *for all* $\varphi \in \ker(D)^\perp$ *and* $\psi \in \mathcal{H}$,

(b) U *has no nontrivial finite-dimensional unitary subrepresentation in* $\ker(D)^\perp$.

Proof. The claim (a) follows from Proposition 2.1 and the fact that $\ell(x_j) \to \infty$. The claim (b) follows from (a) and the fact that matrix coefficients of finite-dimensional unitary representations of a group do not vanish at infinity (see for instance [3, Rem. 2.15(iii)]). ☐

Remark 2.4. *(i) The result of Theorem 2.3(a) amounts to the strong mixing property of the unitary representation U in $\ker(D)^\perp$ along the net $\{x_j\}_{j\in J}$, as mentioned in the introduction. If the strong limit (2.4) exists for all nets $\{x_j\}_{j\in J}$ with $x_j \to \infty$, then Theorem 2.3(a) implies the usual strong mixing property of the unitary representation U in $\ker(D)^\perp$.*

(ii) One can easily see that Theorem 2.3 remains true if the scalars $\ell(x_j)$ in (2.4) are replaced by $(f \circ \ell)(x_j)$, with $f : \mathbb{R}_+ \to \mathbb{R}_+$ any proper function. For simplicity, we decided to present only the case $f = \mathrm{id}_{\mathbb{R}_+}$, but we note this additional freedom might be useful in applications.

Theorem 2.3 and Remark 2.2 imply the following result in the particular case of a compactly generated locally compact group X:

Corollary 2.5 (Compactly generated locally compact groups)**.** *Let X be a compactly generated locally compact Hausdorff group with generating set Y and word length function ℓ, let $U : X \to \mathscr{U}(\mathcal{H})$ be a unitary representation of X, let $\{x_j\}_{j\in J}$ be a net in X with $x_j \to \infty$, assume there exists a self-adjoint operator A in $\mathcal{H}$ such that $U(y) \in C^1(A)$ for each $y \in Y$, and suppose that the strong limit*

$$D := \operatorname*{s-lim}_j \frac{1}{\ell(x_j)}[A, U(x_j)]U(x_j)^{-1} \tag{2.5}$$

exists. Then,

(a) $\lim_j \langle \varphi, U(x_j)\psi\rangle = 0$ *for all* $\varphi \in \ker(D)^\perp$ *and* $\psi \in \mathcal{H}$,

(b) U *has no nontrivial finite-dimensional unitary subrepresentation in* $\ker(D)^\perp$.

Proof. In order to apply Theorem 2.3, we first note from Remark 2.2 that the word length function ℓ is a proper length function. Second, we note that $X = \bigcup_{n\ge1}(Y \cup Y^{-1})^n$. Therefore, for each $x \in X$ there exist $n \ge 1$, $y_1, \dots, y_n \in Y$ and $m_1, \dots, m_n \in \{\pm1\}$ such that $x = y_1^{m_1} \cdots y_n^{m_n}$. Thus,

$$U(x) = U\big(y_1^{m_1} \cdots y_n^{m_n}\big) = U(y_1)^{m_1} \cdots U(y_n)^{m_n},$$

and it follows from the inclusions $U(y_1), \dots, U(y_n) \in C^1(A)$ and standard results on commutator methods [1, Prop. 5.1.5 & 5.1.6(a)] that $U(x) \in C^1(A)$. Thus, we have $U(x_j) \in C^1(A)$ for each $j \in J$, and the commutators $[A, U(x_j)]$ appearing in (2.5) make sense. So, we can apply Theorem 2.3 to conclude. □

Remark 2.6. *Corollary 2.5 is a generalisation of [14, Thm. 3.1] to the case of unitary representations of compactly generated locally compact Hausdorff groups. Indeed, if we let X be the additive group $\mathbb{Z}$ with generating element 1, take the trivial net $\{x_j = j\}_{j\in\mathbb{N}^*} = \{N \mid N \in \mathbb{N}^*\}$, and set $U := U(1)$ in Corollary 2.5, then the strong limit* (2.5) *reduces to*

$$D = \operatorname*{s-lim}_{N\to\infty} \frac{1}{N}\big[A, U^N\big]U^{-N} = \operatorname*{s-lim}_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} U^n\big([A, U]U^{-1}\big)U^{-n},$$

which is the strong limit appearing in [14, Thm. 3.1]. In Corollary 2.5 we also removed the unnecessary invariance assumption $\eta(D)\mathcal{D}(A) \subset \mathcal{D}(A)$ for each $\eta \in C_c^\infty(\mathbb{R}\setminus\{0\})$. So, the strong mixing properties for skew products and Furstenberg-type transformations established in [14, Sec. 3] and [5, Sec. 3] can be obtained more directly using Corollary 2.5.

In the next corollary we consider the case of a strongly continuous unitary representation $U : \mathbb{R}^d \to \mathscr{U}(\mathcal{H})$ of the Euclidean group $\mathbb{R}^d$, $d \ge 1$. In such a case Stone's theorem implies the existence of a family of mutually commuting self-adjoint operators $H_1,\dots,H_d$ such that $U(x) = e^{-i\sum_{k=1}^d x_k H_k}$ for each $x=(x_1,\dots,x_d)\in\mathbb{R}^d$. Therefore, we give a criterion for strong mixing in terms of the operators $H_1,\dots,H_d$. We use the shorthand notations

$$H := (H_1,\dots,H_d),\quad \Pi(H) := (H_1+i)^{-1}\cdots(H_d+i)^{-1}\quad\text{and}\quad x\cdot H := \sum_{k=1}^d x_k H_k.$$

Corollary 2.7 (Euclidean group $\mathbb{R}^d$). *Let $\mathbb{R}^d$, $d\ge 1$, be the Euclidean group with Euclidean length function ℓ, let $U:\mathbb{R}^d\to\mathscr{U}(\mathcal{H})$ be a strongly continuous unitary representation of $\mathbb{R}^d$, let $\{x_j\}_{j\in J}$ be a net in $\mathbb{R}^d$ with $x_j\to\infty$, assume there exists a self-adjoint operator A in $\mathcal{H}$ such that $(H_k-i)^{-1}\in C^1(A)$ for each $k\in\{1,\dots,d\}$, and suppose that the strong limit*

$$D := \operatorname*{s-lim}_j \frac{1}{\ell(x_j)}\int_0^1 ds\; e^{-is(x_j\cdot H)}\,\Pi(H)\big[i(x_j\cdot H),A\big]\Pi(H)^*\,e^{is(x_j\cdot H)} \tag{2.6}$$

exists. Then,

(a) $\lim_j\langle\varphi,U(x_j)\psi\rangle = 0$ *for all* $\varphi\in\ker(D)^\perp$ *and* $\psi\in\mathcal{H}$*,*

(b) U *has no nontrivial finite-dimensional unitary subrepresentation in* $\ker(D)^\perp$*.*

Proof. The proof consists in applying Theorem 2.3 with A replaced by a new operator $\widetilde{A}$ that we now define.

The inclusions $(H_1-i)^{-1},\dots,(H_d-i)^{-1}\in C^1(A)$ and the standard result on commutator methods [1, Prop. 5.1.5] imply that $\Pi(H)^*\in C^1(A)$. So, we have $\Pi(H)^*\mathcal{D}(A)\subset\mathcal{D}(A)$, and the operator

$$\widetilde{A}\,\varphi := \Pi(H)A\Pi(H)^*\varphi,\quad \varphi\in\mathcal{D}(A),$$

is essentially self-adjoint (see [1, Lemma 7.2.15]). Take $\varphi\in\mathcal{D}(A)$ and $j_0\in J$ such that $\ell(x_j)>0$ for all $j\ge j_0$, and define for $\varepsilon\in\mathbb{R}\setminus\{0\}$ the operator $A_\varepsilon := (i\varepsilon)^{-1}(e^{i\varepsilon A}-1)$. Then, we have

$$\begin{aligned}
&\big\langle\widetilde{A}\,\varphi,U(x_j)\varphi\big\rangle - \big\langle\varphi,U(x_j)\widetilde{A}\,\varphi\big\rangle\\
&= \lim_{\varepsilon\searrow0}\Big(\big\langle\varphi,\Pi(H)A_\varepsilon\Pi(H)^*\,e^{-i(x_j\cdot H)}\,\varphi\big\rangle - \big\langle\varphi,e^{-i(x_j\cdot H)}\,\Pi(H)A_\varepsilon\Pi(H)^*\varphi\big\rangle\Big)\\
&= \lim_{\varepsilon\searrow0}\int_0^{\ell(x_j)} dq\,\frac{d}{dq}\big\langle\varphi,e^{i(q-\ell(x_j))(x_j\cdot H)/\ell(x_j)}\,\Pi(H)A_\varepsilon\Pi(H)^*\,e^{-iq(x_j\cdot H)/\ell(x_j)}\,\varphi\big\rangle\\
&= \frac{1}{\ell(x_j)}\lim_{\varepsilon\searrow0}\int_0^{\ell(x_j)} dq\,\big\langle\varphi,e^{i(q-\ell(x_j))(x_j\cdot H)/\ell(x_j)}\,\Pi(H)\big[i(x_j\cdot H),A_\varepsilon\big]\Pi(H)^*\,e^{-iq(x_j\cdot H)/\ell(x_j)}\,\varphi\big\rangle.
\end{aligned} \tag{2.7}$$

But, $(H_1-i)^{-1},\dots,(H_d-i)^{-1}\in C^1(A)$. Therefore, (2.2) and (2.3) imply that

$$\operatorname*{s-lim}_{\varepsilon\searrow0}\Pi(H)\big[i(x_j\cdot H),A_\varepsilon\big]\Pi(H)^* = \Pi(H)\big[i(x_j\cdot H),A\big]\Pi(H)^*,$$

and we can exchange the limit and the integral in (2.7) to obtain

$$
\begin{aligned}
&\big\langle \widetilde{A}\,\varphi, U(x_j)\varphi\big\rangle - \big\langle \varphi, U(x_j)\widetilde{A}\,\varphi\big\rangle \\
&= \frac{1}{\ell(x_j)}\int_0^{\ell(x_j)} \mathrm{d}q\,\big\langle \varphi, \mathrm{e}^{i(q-\ell(x_j))(x_j\cdot H)/\ell(x_j)}\,\Pi(H)\big[i(x_j\cdot H), A\big]\Pi(H)^*\,\mathrm{e}^{-iq(x_j\cdot H)/\ell(x_j)}\,\varphi\big\rangle \\
&= \frac{1}{\ell(x_j)}\int_0^{\ell(x_j)} \mathrm{d}r\,\big\langle \varphi, \mathrm{e}^{-ir(x_j\cdot H)/\ell(x_j)}\,\Pi(H)\big[i(x_j\cdot H), A\big]\Pi(H)^*\,\mathrm{e}^{i(r-\ell(x_j))(x_j\cdot H)/\ell(x_j)}\,\varphi\big\rangle \\
&= \int_0^1 \mathrm{d}s\,\big\langle \varphi, \mathrm{e}^{-is(x_j\cdot H)}\,\Pi(H)\big[i(x_j\cdot H), A\big]\Pi(H)^*\,\mathrm{e}^{is(x_j\cdot H)}\,U(x_j)\,\varphi\big\rangle \\
&= \big\langle \varphi, \ell(x_j)D_j U(x_j)\,\varphi\big\rangle
\end{aligned}
$$

with

$$D_j := \frac{1}{\ell(x_j)}\int_0^1 \mathrm{d}s\ \mathrm{e}^{-is(x_j\cdot H)}\,\Pi(H)\big[i(x_j\cdot H), A\big]\Pi(H)^*\,\mathrm{e}^{is(x_j\cdot H)}\,.$$

Since $\mathcal{D}(A)$ is a core for $\widetilde{A}$, this implies that $U(x_j) \in C^1(\widetilde{A})$ with $\big[\widetilde{A}, U(x_j)\big] = \ell(x_j)D_j\,U(x_j)$. Therefore, we have

$$D_j = \frac{1}{\ell(x_j)}\big[\widetilde{A}, U(x_j)\big]U(x_j)^{-1},$$

and all the assumptions of Theorem 2.3 are satisfied with A replaced by $\widetilde{A}$. □

Remark 2.8. *Corollary 2.7 is a generalisation of [14, Thm. 4.1] to the case of strongly continuous unitary representations of* $\mathbb{R}^d$ *for an arbitrary* $d \geq 1$*. Indeed, if we set* $d = 1$*, write* H *for* H_1*, and take the trivial net* $\{x_j = j\}_{j\in(0,\infty)} = \{t \mid t > 0\}$ *in Corollary 2.7, then the strong limit* (2.6) *reduces to*

$$
\begin{aligned}
D &= \operatorname*{s-lim}_{t\to\infty} \frac{1}{t}\int_0^1 \mathrm{d}s\ \mathrm{e}^{-is(t\cdot H)}(H+i)^{-1}\big[itH, A\big](H-i)^{-1}\,\mathrm{e}^{is(t\cdot H)} \\
&= \operatorname*{s-lim}_{t\to\infty} \frac{1}{t}\int_0^t \mathrm{d}s\ \mathrm{e}^{-isH}(H+i)^{-1}\big[iH, A\big](H-i)^{-1}\,\mathrm{e}^{isH},
\end{aligned}
$$

which is (up to a sign) the strong limit appearing in [14, Thm. 4.1]. In Corollary 2.7, we also removed the unnecessary invariance assumption $\eta(D)\mathcal{D}(A) \subset \mathcal{D}(A)$ *for each* $\eta \in C_c^\infty(\mathbb{R}\setminus\{0\})$*. So, the strong mixing properties for adjacency operators, time changes of horocycle flows, etc., established in [14, Sec. 4] can be obtained more directly using Corollary 2.7.*

To conclude, we add to the list of examples presented in [14] an application which was not possible to cover with the results of [14]. It is a short alternative proof, not using convolutions, of the strong mixing property of the left regular representation of σ-compact locally compact Hausdorff groups (see for instance [2, Sec. C.4] for the proof using convolutions):

Example 2.9 (Left regular representation)**.** *Let* X *be a* σ*-compact locally compact Hausdorff group with left Haar measure* μ *and proper length function* ℓ *(see Remark 2.2). Let* $\mathscr{D} \subset \mathcal{H}$ *be the set of functions* $X \to \mathbb{C}$ *with compact support, and let* $U : X \to \mathscr{U}(\mathcal{H})$ *be the left regular representation of* X *on* $\mathcal{H} := L^2(X, \mu)$ *given by*

$$U(x)\varphi := \varphi(x^{-1}\,\cdot), \quad x \in X, \ \varphi \in \mathcal{H}.$$

Let finally A *be the maximal multiplication operator in* $\mathcal{H}$ *given by*

$$A\varphi := \ell\,\varphi \equiv \ell(\cdot)\varphi, \quad \varphi \in \mathcal{D}(A) := \big\{\varphi \in \mathcal{H} \mid \|\ell\,\varphi\| < \infty\big\}.$$

For $\varphi \in \mathscr{D}$ *and* $x \in X$*, one has*

$$AU(x)\varphi - U(x)A\varphi = \big(\ell(\,\cdot\,) - \ell(x^{-1}\,\cdot\,)\big)U(x)\varphi.$$

Furthermore, the properties (L2)-(L3) of a length function imply that

$$\Big|\big(\ell(\,\cdot\,) - \ell(x^{-1}\,\cdot\,)\big)\Big| \le \ell(x). \tag{2.8}$$

Therefore, since $\mathscr{D}$ *is dense in* $\mathcal{D}(A)$*, it follows that* $U(x) \in C^1(A)$ *with*

$$[A, U(x)]U(x)^{-1} = \ell(\,\cdot\,) - \ell(x^{-1}\,\cdot\,).$$

Now, we take $\{x_j\}_{j\in J}$ *a net in* X *with* $x_j \to \infty$*, and show that*

$$D := \operatorname*{s-lim}_j \frac{1}{\ell(x_j)}[A, U(x_j)]U(x_j)^{-1} = -1. \tag{2.9}$$

For this, we first note that for $\varphi \in \mathcal{H}$ *we have*

$$\left(\frac{1}{\ell(x_j)}[A, U(x_j)]U(x_j)^{-1} + 1\right)\varphi = \frac{\ell(\,\cdot\,) - \ell(x_j^{-1}\,\cdot\,) + \ell(x_j)}{\ell(x_j)}\,\varphi.$$

Next, we note that (2.8) *implies that*

$$\left|\frac{\ell(\,\cdot\,) - \ell(x_j^{-1}\,\cdot\,) + \ell(x_j)}{\ell(x_j)}\,\varphi\right|^2 \le 4|\varphi|^2 \in L^1(X, \mu),$$

and that the properties (L2)-(L3) imply that

$$\lim_j \left|\frac{\ell(\,\cdot\,) - \ell(x_j^{-1}\,\cdot\,) + \ell(x_j)}{\ell(x_j)}\,\varphi\right|^2 \le \lim_j \left|\frac{2\,\ell(\,\cdot\,)}{\ell(x_j)}\,\varphi\right|^2 = 0 \quad \mu\text{-almost everywhere.}$$

Therefore, we can apply Lebesgue dominated convergence theorem to get the equality

$$\operatorname*{s-lim}_j \left(\frac{1}{\ell(x_j)}[A, U(x_j)]U(x_j)^{-1} + 1\right)\varphi = 0,$$

which proves (2.9)*. So, Theorem 2.3 applies with* $D = -1$*, and thus* $\lim_j \big\langle\varphi, U(x_j)\psi\big\rangle = 0$ *for all* $\varphi, \psi \in \mathcal{H}$*.*

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Certain integral Transforms of the generalized Lommel-Wright function

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ABSTRACT

The aim of this article is to establish some integral transforms of the generalized Lommel-Wright functions, which are expressed in terms of Wright Hypergeometric function. Some integrals involving trigonometric, generalized Bessel and Struve functions are also indicated as special cases of our main results.

RESUMEN

El objetivo de este artículo es establecer algunas transformadas integrales de las funciones generalizadas de Lommel-Wright, que se expresan en términos de la función hipergeométrica de Wright. Algunas integrales que involucran funciones trigonométricas, de Bessel generalizadas y de Struve también se obtienen como casos especiales de nuestros resultados principales.

Keywords and Phrases: Gamma function, generalized Wright hypergeometric function ${}_p\psi_q$, generalized Lommel-Wright functions $J^{\mu,m}_{\nu,\lambda}(z)$, Integral Transforms.

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1 Introduction

The k-Pochhammer symbol $(\lambda)_{\nu,k}$ is defined (for $\nu, \lambda \in \mathbb{C}; k \in \mathbb{R}$) by [4]

$$(\lambda)_{\nu,k} = \frac{\Gamma_k(\lambda + \nu\ k)}{\Gamma_k(\lambda)} \quad (\lambda \in \mathbb{C}/0) \tag{1.1}$$

and the k-gamma function has the relation

$$\Gamma_k(z) = k^{z/k-1}\Gamma(z/k), \tag{1.2}$$

is such that $\Gamma_k(z) \to \Gamma(z)$ if $k \to 1$.

The Wright hypergeometric function defined by the series [21]

$${}_p\psi_q\left[\begin{array}{c}(\alpha_1, A_1), ..., (\alpha_p, A_p);\\ \\ (\beta_1, B_1), ..., (\beta_q, B_q)\end{array} z\right] = \sum_{k=0}^{\infty}\frac{\prod_{j=1}^{p}\Gamma(\alpha_j + A_j k)z^k}{\prod_{j=1}^{q}\Gamma(\beta_j + B_j k)k!}, \tag{1.3}$$

where the coefficients $A_1, ..., A_p$ and $B_1, ..., B_q$ are positive real numbers such that

$$1 + \sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_j \geq 0. \tag{1.4}$$

can be slightly generalized (1.3) as given below.

$${}_p\psi_q\left[\begin{array}{c}(\alpha_1, 1), ..., (\alpha_p, 1);\\ \\ (\beta_1, 1), ..., (\beta_q, 1);\end{array} z\right] = \frac{\prod_{j=1}^{p}\Gamma(\alpha_j)}{\prod_{j=1}^{q}\Gamma(\beta_j)}\,{}_pF_q\left[\begin{array}{c}\alpha_1, .., \alpha_p;\\ \\ \beta_1, ..., \beta_q;\end{array} z\right], \tag{1.5}$$

where ${}_pF_q$ is the generalized hypergeometric function defined by [19, 21]

$${}_pF_q\left[\begin{array}{c}\alpha_1, ..., \alpha_p;\\ \\ \beta_1, ..., \beta_q\end{array} z\right] = \sum_{k=0}^{\infty}\frac{(\alpha_1)_n, ..., (\alpha_p)_n z^n}{(\beta_1)_n, ..., (\beta_q)_n n!} = {}_pF_q(\alpha_1, ..., \alpha_p; \beta_1, ..., \beta_q; z), \tag{1.6}$$

where $(\lambda)_n$ is the well known Pochhammer symbol [21].
The generalization of $(\lambda)_n$ is given as

$$(\lambda)_n = \lambda(\lambda+1)(\lambda+2), ..., (\lambda+n-1)) \quad , n > 0 \tag{1.7}$$

$$(\lambda)_n = \prod_{m=1}^{n}(\lambda + m - 1), \quad (\lambda)_0 = 1, \ \lambda \neq 0$$

$$(\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)}$$

Generalized Bessel, Lommel, Struve and Lommel-Wright function have originated from concrete problems in Mechanics, Physics, Engineering and Astronomy.

The series representation of the generalized Lommel Wright function as [8];

$$J^{\mu,m}_{\nu,\lambda}(z) = \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(k+1)(\frac{z}{2})^{2k+\nu+2\lambda}}{\Gamma(\lambda+k+1)^m \Gamma(\nu+k\mu+\lambda+1)k!}, \tag{1.8}$$

$$(z \in \mathbb{N}/(-\infty,0] \ m \in \mathbb{N}, \ \nu,\lambda \in \mathbb{C}, \mu > 0).$$

Also, we have the following relations of generalized Lommel Wright functions with trigonometric functions and the generalized Bessel function and Struve function:

$$J^{1,1}_{1/2,0}(z) = \sqrt(\frac{2}{\pi z}) \sin(z) \tag{1.9}$$

$$J^{1,1}_{-1/2,0}(z) = \sqrt(\frac{2}{\pi z}) \cos(z) \tag{1.10}$$

$$J^{\mu,1}_{\nu,\lambda}(z) = \mathbb{J}^{\mu}_{\nu,\lambda}(z) \tag{1.11}$$

$$J^{1,1}_{\nu,1/2}(z) = H_\nu(z) \tag{1.12}$$

Further, we recall the following results [5].

$$\int_0^\infty t^{u-1} \ \exp(-t/2) W_{\lambda,\mu}(t)dt = \frac{\Gamma(1/2+\mu+u)\Gamma(1/2-\mu+u)}{\Gamma(1-\lambda+u)}, \tag{1.13}$$

$$(\mathrm{Re}(u \pm \ \mu) > -1/2),$$

where the Whittaker function $\mathbb{W}_{\lambda,\mu}(t)$ is given in[5, 11].

$$W_{\lambda,\mu}(t) = \frac{\Gamma(-2\mu)}{\Gamma(1/2-\mu-\lambda)} M_{\lambda,\mu}(t) + \frac{\Gamma(2\mu)}{\Gamma(1/2+\mu-\lambda)} M_{\lambda,-\mu}(t)$$

where $M_{\lambda,\mu}(t)$ is defined as

$$M_{\lambda,\mu}(t) = z^{1/2+\mu} \exp(-t/2) \ {}_1F_1\left(1/2+\mu+u; 2\mu+1; t\right)$$

Definition 1.1. Euler Transform:

Let $\rho, \sigma \in \mathbb{C}$ and $\mathrm{Re}(\rho), \mathrm{Re}(\sigma) > 0$, then the Euler transform of the function $f(z)$ is defined by

$$\mathbb{B}(f(z); \rho, \sigma) = \int_0^1 z^{\rho-1}(1-z)^{\sigma-1} f(z)dz \tag{1.14}$$

Definition 1.2. Laplace Transform:
The Laplace transform of the function $f(t)$ is defined as

$$F(\delta) = L(f(t);\delta) = \int_0^\infty \exp(-t\delta)f(t)dt, \qquad \mathrm{Re}(\delta) > 0 \tag{1.15}$$

Definition 1.3. Fourier Transform:
The following integral gives the Fourier transform

$$u = \mathrm{Im}[u](w) = \int_R u(t)\exp(iwt)dt, \tag{1.16}$$

where $u = u(t)$ be a function of the space $S(R)$ Shwartzian space of the function that decay rapidly at ∞ together with all derivatives.

Definition 1.4. The Fractional Fourier Transform (FFT):
Let u be the function belonging to $\phi(R)$, the Lizorkin space of function, where
$\phi(R) = \{\phi \in \ S(R)\} : \mathrm{Im}[\phi] \in \ V(R)$
and $V(R)$ is the set of functions defined by
$V(R) = \{v \in \ S(R)\} : V_0^u = 0, n = 0, 1, 2, ...$
then FFT of order α, $0 \leq \ \alpha \leq \ 1$ is given by

$$U_\alpha(w) = \mathrm{Im}_\alpha(w) = \int_R \exp(i\ w^{\alpha\ t})u(t)dt \tag{1.17}$$

particularly, if $\alpha = 1$ (1.17) reduces to FT and for $w > 0$ (1.17) reduces to FFT given by Luchko et al [10].
The aim of this paper is to obtain the Euler, Laplace, Whittaker and Fractional Fourier transforms of Lommel-Wright function.
Various generalizations, integrals, transforms and fractional calculus of special functions have been investigated by many researchers (see, for details, [1, 2, 6, 7, 9, 12, 13, 14, 15, 16, 17, 18, 20]). In this sequel, here, we aim at establishing certain new generalized integral formula involving the generalized Lommel-Wright function $J_{\nu,\lambda}^{\mu,m}(z)$ interesting integral formulas which are derived as special cases.

2 Main Results

This section deals with some integral formulas involving Lommel-Wright function.

Theorem 2.1. *For* $t \in \mathbb{N}/(-\infty, 0]$ $m \in \mathbb{N}$, $\nu, \lambda \in \mathbb{C}$ *and* $\mu > 0$, *the following integral formula holds true*

$$\int_0^1 t^{\alpha-1}(1-t)^{\beta-1} J_{\nu,\lambda}^{\mu,m}(x\ t^{\sigma})dt = \left(\frac{x}{2}\right)^{\nu+2\lambda} \Gamma(\beta)$$
$$\times\ {}_2\psi_{m+2}\left[\begin{array}{c} (1,1),(\alpha+\nu\sigma+2\lambda\sigma,2\sigma); \\ (\lambda+1,1),...,(\lambda+1,1),(\nu+\lambda+1,\mu),(\alpha+\beta+\nu\sigma+2\lambda\sigma,2\sigma); \end{array} -\frac{x^2}{4}\right]. \tag{2.1}$$

Proof. On using (1.8) in the integrand of (2.1) and then interchanging the order of integral sign and summation which is verified by uniform convergence of the involved series under the given conditions we get

$$\int_0^1 t^{\alpha-1}(1-t)^{\beta-1} J_{\nu,\lambda}^{\mu,m}(x\ t^{\sigma})dt$$
$$= \left(\frac{x}{2}\right)^{\nu+2\lambda} \sum_{k=0}^{\infty} \frac{\Gamma(k+1)(-x^2/4)^k}{\Gamma(\lambda+k+1)^m \Gamma(\nu+k\mu+\lambda+1)k!}$$
$$\times \int_0^1 t^{\alpha+\sigma(2k+\nu+2\lambda)-1}(1-t)^{\beta-1}dt. \tag{2.2}$$

Now using (1.14) in the above equation we get

$$\int_0^1 t^{\alpha-1}(1-t)^{\beta-1} J_{\nu,\lambda}^{\mu,m}(x\ t^{\sigma})dt = \Gamma(\beta)\left(\frac{x}{2}\right)^{\nu+2\lambda}$$
$$\times \sum_{k=0}^{\infty} \frac{\Gamma(k+1)\Gamma(\alpha+\nu\sigma+2\lambda\sigma+2k\sigma)(\frac{-x^2}{4})^k}{\Gamma(\lambda+k+1)^m \Gamma(\alpha+\beta+\nu\sigma+2\lambda\sigma+2k\sigma)\Gamma(\nu+k\mu+\lambda+1)k!}. \tag{2.3}$$

Finally, using (1.3) in the above equation, we get our assertion (2.1). This completes the proof of Theorem 2.1. □

Theorem 2.2. *For* $t \in \mathbb{N}/(-\infty, 0]$ $m \in \mathbb{N}$, $\nu, \lambda \in \mathbb{C}$ *and* $\mu > 0$, *the following integral formula holds true*

$$\int_0^{\infty} t^{\alpha-1} \exp(-t\delta) J_{\nu,\lambda}^{\mu,m}(x\ t^{\sigma})dt = \left(\frac{x}{2\ \delta^{-\alpha}}\right)^{\nu+2\lambda} (\delta)^{-\alpha}$$
$$\times\ {}_2\psi_{m+1}\left[\begin{array}{c} (1,1),(\alpha+\nu\sigma+2\lambda\sigma,2\sigma); \\ (\lambda+1,1),...,(\lambda+1,1),(\nu+\lambda+1,\mu); \end{array} -\frac{x^2}{4\ \delta^{2\sigma}}\right]. \tag{2.4}$$

Proof. On using (1.8) in the integrand of (2.4) and then interchanging the order of integral sign and summation which is verified by uniform convergence of the involved series under the given conditions we get

$$\int_0^\infty t^{\alpha-1} \exp(-\delta\ t) J_{\nu,\lambda}^{\mu,m}(x\ t^\sigma)dt$$
$$= \left(\frac{x}{2}\right)^{\nu+2\lambda} \sum_{k=0}^\infty \frac{\Gamma(k+1)(-x^2/4)^k}{\Gamma(\lambda+k+1)^m \Gamma(\nu+k\mu+\lambda+1)k!}$$
$$\times \int_0^\infty t^{\alpha+\sigma(2k+\nu+2\lambda)-1} \exp(-\delta\ t)dt. \tag{2.5}$$

Now using (1.15) in the above equation we get

$$\int_0^\infty t^{\alpha-1} \exp(-\delta\ t) J_{\nu,\lambda}^{\mu,m}(x\ t^\sigma)dt = (\delta)^{-\alpha} \left(\frac{x}{2\delta^\sigma}\right)^{\nu+2\lambda}$$
$$\times \sum_{k=0}^\infty \frac{\Gamma(k+1)\Gamma(\alpha+\nu\sigma+2\lambda\sigma+2k\sigma)(\frac{-x^2}{4\ \delta^{2\sigma}})^k}{\Gamma(\lambda+k+1)^m \Gamma(\nu+k\mu+\lambda+1)k!}. \tag{2.6}$$

Finally, using (1.3) in the above equation, we get our assertion (2.6). This completes the proof of Theorem 2.2. □

Theorem 2.3. *For* $t \in \mathbb{N}/(-\infty, 0]$ $m \in \mathbb{N}$, $\nu, \lambda \in \mathbb{C}$ *and* $\mu > 0$ *, the following integral formula holds true*

$$\int_0^\infty t^{\eta-1} \exp(-p\ t)/2\ W_{\lambda,\mu}(p\ t) J_{\nu,\lambda}^{\mu,m}(w\ t^\delta)dt = \left(\frac{w}{p^\delta}\right)^{\nu+2\lambda}$$
$$\times\ {}_3\psi_{m+2}\left[\begin{array}{c} (1,1),(1/2+\mu+\eta+\delta\ \nu+2\delta\lambda,2\delta),(1/2-\mu+\eta+\delta\ \nu+2\delta\lambda,2\delta); \\ (\lambda+1,1),...,(\lambda+1,1),(\nu+\lambda+1,\mu),(1-\lambda+\eta+\nu\delta+2\delta\lambda,2\delta); \end{array} -\frac{w^2}{4\ p^{2\delta}}\right]. \tag{2.7}$$

Proof. On using (1.8) in the integrand of (2.7) and then interchanging the order of integral sign and summation which is verified by uniform convergence of the involved series under the given conditions we get

$$\int_0^\infty (u/p)^{\eta-1} \exp(-u/2) W_{\lambda,\mu}(u) J_{\nu,\lambda}^{\mu,m}(w\ (u/p)^\delta)du$$
$$= \left(\frac{w}{p^\delta}\right)^{\nu+2\lambda} \sum_{k=0}^\infty \frac{\Gamma(k+1)(-w^2/4\ p^{2\delta})^k}{\Gamma(\lambda+k+1)^m \Gamma(\nu+k\mu+\lambda+1)k!}$$
$$\times \int_0^\infty u^{\eta+\delta(2k+\nu+2\lambda)-1} \exp(-u/2) W_{\lambda,\mu}(u)du. \tag{2.8}$$

Now using (1.13) in the above equation we get

$$\int_0^\infty t^{\eta-1} \exp(-p\ t)/2\ W_{\lambda,\mu}(p\ t) J_{\nu,\lambda}^{\mu,m}(w\ t^\delta)dt = \left(\frac{w}{p^\delta}\right)^{\nu+2\lambda}$$
$$\times \sum_{k=0}^\infty \frac{\Gamma(k+1)\Gamma(1/2+\mu+\eta+2k\delta+\delta\nu+2\delta\lambda)\Gamma(1/2-\mu+\eta+2k\delta+\delta\nu+2\delta\lambda)(\frac{-w^2}{4\ p^{2\delta}})^k}{\Gamma(\lambda+k+1)^m \Gamma(\nu+k\mu+\lambda+1)\Gamma(1-\lambda+\eta+2k\delta+\delta\nu+2\delta\lambda)k!}. \tag{2.9}$$

Finally, using (1.3) in the above equation, we get our assertion (2.9). This completes the proof of Theorem 2.3. $\square$

3 Special Cases

In this section, we get some integral formulas involving trigonometric function and generalized Lommel-Wright function as follows:

Corollary 3.1. *If we take* $m = 1, \mu = 1, \lambda = 0$ *and* $\nu = 1/2$ *in* (2.1) *and then by using* (1.9)*, we derive the following integral formula:*

$$\int_0^1 t^{\alpha-\sigma/2-1}(1-t)^{(\beta-1)}\sin(x\ t^{\sigma})dt$$
$$= \sqrt{\pi}\left(\frac{x}{2}\right)\Gamma(\beta)\ {}_1\psi_2\left[\begin{array}{c}(\alpha+\sigma/2, 2\sigma);\\ \\ (3/2,1),(\alpha+\beta+\sigma/2,2\sigma);\end{array} -\frac{x^2}{4}\right] \tag{3.1}$$

Corollary 3.2. *If we take* $m = 1, \mu = 1, \lambda = 0$ *and* $\nu = 1/2$ *in* (2.4) *and then by using* (1.9)*, we derive the following integral formula:*

$$\int_0^\infty t^{\alpha-\sigma/2-1}\exp(-\delta\ t)\sin(x\ t^{\sigma})dt$$
$$= \delta^{-\alpha}\sqrt{\frac{\pi}{\delta^{\sigma}}}\left(\frac{x}{2}\right)\Gamma(\beta)\ {}_1\psi_1\left[\begin{array}{c}(\alpha+\sigma/2, 2\sigma);\\ \\ (3/2,1);\end{array} -\frac{x^2}{4\ \delta^{2\sigma}}\right] \tag{3.2}$$

Corollary 3.3. *Further if we take* $m = 1, \mu = 1, \lambda = 0$ *and* $\nu = 1/2$ *in* (2.7) *and then by using* (1.9)*, we derive the following integral formula:*

$$\int_0^\infty t^{\eta-\delta/2-1}\exp(-pt/2)W_{\lambda,\mu}(p\ t)\sin(w\ t^{\delta})dt$$
$$= w\sqrt{\frac{\pi}{2\ p^{\delta}}}\ {}_2\psi_2\left[\begin{array}{c}(\eta+\delta/2+3/2, 2\delta)(\eta+\delta/2-1/2,2\delta),;\\ \\ (3/2,1),(\eta+\delta/2+1,2\delta);\end{array} -\frac{w^2}{4\ p^{2\delta}}\right] \tag{3.3}$$

Corollary 3.4. *If we take* $m = 1, \mu = 1, \lambda = 0$ *and* $\nu = -1/2$ *in* (2.1) *and then by using* (1.10)*, we derive the following integral formula:*

$$\int_0^1 t^{\alpha-\sigma/2-1}(1-t)^{(\beta-1)}\cos(x\ t^{\sigma})dt$$
$$= \sqrt{\pi}\Gamma(\beta)\ {}_1\psi_2\left[\begin{array}{c}(\alpha-\sigma/2, 2\sigma);\\ \\ (1/2,1),(\alpha+\beta-\sigma/2,2\sigma);\end{array} -\frac{x^2}{4}\right] \tag{3.4}$$

Corollary 3.5. *If we take* $m = 1, \mu = 1, \lambda = 0$ *and* $\nu = -1/2$ *in* (2.4) *and then by using* (1.10), *we derive the following integral formula:*

$$\int_0^\infty t^{\alpha-\sigma/2-1} \exp(-\delta\ t) \cos(x\ t^\sigma) dt$$
$$= \delta^{(\sigma-\alpha)} \sqrt{\pi}\ {}_1\psi_1 \left[\begin{array}{cc} (\alpha - \sigma/2, 2\sigma); & \\ & -\dfrac{x^2}{4\ \delta^{2\sigma}} \\ (1/2, 1); & \end{array} \right] \tag{3.5}$$

Corollary 3.6. *Further if we take* $m = 1, \mu = 1, \lambda = 0$ *and* $\nu = -1/2$ *in* (2.7) *and then by using* (1.10), *we derive the following integral formula:*

$$\int_0^\infty t^{\eta-\delta/2-1} \exp(-pt/2) W_{\lambda,\mu}(p\ t) \cos(w\ t^\delta) dt$$
$$= w\sqrt{\frac{\pi}{2}}\ {}_2\psi_2 \left[\begin{array}{cc} (\eta - \delta/2 + 3/2, 2\delta)(\eta - \delta/2 - 1/2, 2\delta), ; & \\ & -\dfrac{w^2}{4\ p^{2\delta}} \\ (1/2, 1), (\eta - \delta/2 + 1, 2\delta); & \end{array} \right] \tag{3.6}$$

Corollary 3.7. *If we take* $m = 1$ *in* (2.1) *and then by using* (1.11), *we derive the following integral formula:*

$$\int_0^1 t^{\alpha-1}(1-t)^{(\beta-1)} \mathbb{J}^\mu_{\nu,\lambda}(x\ t^\sigma) dt = \left(\frac{x}{2}\right)^{\nu+2\lambda} \Gamma(\beta)$$
$$\times {}_2\psi_3 \left[\begin{array}{cc} (1, 1), (\alpha + \nu\sigma + 2\lambda\sigma, 2\sigma); & \\ & -\dfrac{x^2}{4} \\ (\lambda + 1, 1), (\nu + \lambda + 1, \mu), (\alpha + \beta + \nu\sigma + 2\lambda\sigma, 2\sigma); & \end{array} \right] \tag{3.7}$$

Corollary 3.8. *If we take* $m = 1$ *in* (2.4) *and then by using* (1.11), *we derive the following integral formula:*

$$\int_0^\infty t^{\alpha-1} \exp(-\delta\ t) \mathbb{J}^\mu_{\nu,\lambda}(x\ t^\sigma) dt$$
$$= \left(\frac{x}{2}\right)^{\nu+2\lambda} \delta^{-\alpha}\ {}_2\psi_2 \left[\begin{array}{cc} (1, 1), (\alpha + \nu\sigma + 2\lambda\sigma, 2\sigma); & \\ & -\dfrac{x^2}{4\ \delta^{2\sigma}} \\ (\lambda + 1, 1), (\nu + \lambda + 1, \mu); & \end{array} \right] \tag{3.8}$$

Corollary 3.9. *Further if we take* $m = 1$ *in* (2.7) *and then by using* (1.11), *we derive the following integral formula:*

$$\int_0^\infty t^{\eta-1} \exp(-pt/2) W_{\lambda,\mu}(p\ t) \mathbb{J}^\mu_{\nu,\lambda}(w\ t^\delta) dt = \left(\frac{w}{p^\delta}\right)^{\nu+2\lambda}$$
$$\times {}_3\psi_3 \left[\begin{array}{cc} (1, 1), (1/2 + \mu + \eta + \nu\delta + 2\lambda\delta, 2\delta), (1/2 - \mu + \eta + \nu\delta + 2\lambda\delta, 2\delta); & \\ & -\dfrac{w^2}{4\ p^{2\delta}} \\ (\lambda + 1, 1), (\nu + \lambda + 1, \mu), (1 - \lambda + \eta + \delta\nu + 2\delta\lambda, 2\delta); & \end{array} \right] \tag{3.9}$$

Corollary 3.10. *If we take* $\mu = 1, m = 1$ *and* $\lambda = 1/2$ *in* (2.1) *and then by using* (1.12), *we derive the following integral formula:*

$$\int_0^1 t^{\alpha-1}(1-t)^{(\beta-1)}\mathbb{H}_\nu(x\ t^\sigma)dt = \left(\frac{x}{2}\right)^{\nu+1}\Gamma(\beta)$$
$$\times {}_2\psi_3\left[\begin{array}{c} (1,1),(\alpha+\nu\sigma+\sigma,2\sigma); \\ \\ (3/2,1),(\nu+3/2,1),(\alpha+\beta+\nu\sigma+\sigma,2\sigma); \end{array} -\frac{x^2}{4}\right] \tag{3.10}$$

Corollary 3.11. *If we take* $\mu = 1, m = 1$ *and* $\lambda = 1/2$ *in* (2.4) *and then by using* (1.12), *we derive the following integral formula:*

$$\int_0^\infty t^{\alpha-1}\exp(-\delta\ t)\mathbb{H}_\nu(x\ t^\sigma)dt = \left(\frac{x}{2\ \delta^\sigma}\right)^{\nu+1}\delta^{-\alpha}$$
$$\times {}_2\psi_2\left[\begin{array}{c} (1,1),(\alpha+\nu\sigma+\sigma,2\sigma); \\ \\ (3/2,1),(\nu+3/2,1); \end{array} -\frac{x^2}{4\ \delta^{2\sigma}}\right] \tag{3.11}$$

Corollary 3.12. *Further if we take* $\mu = 1, m = 1$ *and* $\lambda = 1/2$ *in* (2.7) *and then by using* (1.12), *we derive the following integral formula:*

$$\int_0^\infty t^{\eta-1}\exp(-pt/2)W_{\lambda,\mu}(p\ t)\mathbb{H}_\nu(w\ t^\delta)dt = \left(\frac{w}{p^\delta}\right)^{\nu+1}$$
$$\times {}_3\psi_3\left[\begin{array}{c} (1,1),(\eta+\nu\delta+\delta+3/2,2\delta),(\eta+\nu\delta+\delta-1/2,2\delta); \\ \\ (3/2,1),(\nu+3/2,1),(\eta+\delta\nu+\delta+1/2,2\delta); \end{array} -\frac{w^2}{4\ p^{2\delta}}\right] \tag{3.12}$$

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On Fractional Integro-differential Equations with State-Dependent Delay and Non-Instantaneous Impulses

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ABSTRACT

In this paper, we prove the existence of mild solution of the fractional integro-differential equations with state-dependent delay with not instantaneous impulses. The existence results are obtained under the conditions in respect of Kuratowski's measure of non-compactness. An example is also given to illustrate the results.

RESUMEN

En este artículo, demostramos la existencia de soluciones mild de ecuaciones integro-diferenciales fraccionarias con retardo dependiente del estado e impulsos no instantáneos. Los resultados de existencia se obtienen bajo condiciones respecto de la medida de Kuratowski de no compacidad. También se entrega un ejemplo para ilustrar los resultados.

Keywords and Phrases: Non-instantaneous impulsive conditions, fractional integro-differential equations, Caputo fractional derivative, mild solution, fixed point, state-dependent delay.

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1 Introduction

Fractional differential equations play the crucial and significant role in the field of science and engineering. Most importantly non-integer order differential equations have ability to describe the real behavior and memory effects of the system and processes. For more details about fractional differential equations and its applications refer the monographs of Abbas *et al.* [1, 2, 3], Baleanu *et al.* [12], Diethelm [18], Hilfer [24], Kilbas *et al.* [26], Miller and Ross [32], Samko *et al.* [37], Tarasov [38], and Zhou [39] and the references therein.

Most of the research papers deal with the existence of solutions for differential equations with instantaneous impulsive conditions see [6, 7, 10, 11, 14, 28, 31]. But many times it has seen that certain dynamics of evolution processes cannot describe by instantaneous impulses, For instance: Pharmacotherapy, high or low levels of glucose, this situation can be interpreted as an impulsive action which starts abruptly at certain point of time and continue with a finite time interval. Such type of systems are known as non-instantaneous impulsive systems which are more suitable to study the dynamics of evolution processes [4].

This theory of a new class of impulsive differential equation was initiated by Hernández *et al.* [23]. Afterwards, Pierri *et al.* [35] continued the work in this field and extend the theory of [23] in a PC_α normed Banach space. The existence of solutions for non-instantaneous impulsive fractional differential equations have also been discussed in [8, 19, 27, 29, 34].

Recently, Benchohra *et al.* [15] investigated the existence and uniqueness of solutions on a compact interval for non-linear fractional integro-differential equations with state-dependent delay and noninstantaneous impulses. Anguraj and Kanjanadevi [9] studied the existence and uniqueness of fractional neutral differential equations with state-dependent delay subject to non-instantaneous impulsive conditions.

Motivated by the papers cited above, in this paper, we consider the existence of mild solutions for fractional integro-differential equations with state-dependent delay and non instantaneous impulses described by the form

$$\begin{aligned} {}^{C}D_t^q x(t) + Ax(t) &= \int_0^t a(t,s) f(s, x_{\rho(s,x_s)}, x(s))ds, \ \ a.e. \quad t \in (s_i, t_{i+1}] \subset J, i = 0, \dots, N, \\ x(t) &= h_i(t, x_{\rho(t,x_t)}, x(t)), \qquad\qquad t \in (t_i, s_i], i = 1, \dots, N, \\ x_0 &= \phi \in \mathcal{B}, \end{aligned} \tag{1.1}$$

where ${}^{C}D_t^q$ is the Caputo fractional derivative of order $0 < q < 1$, $A : D(A) \subset X \to X$ is the infinitesimal generator of an analytic semigroup $\{S(t)\}_{t\geq 0}$ of uniformly bounded linear operators on X, $f : J \times \mathcal{B} \times X \longrightarrow X$, $J = [0, T]$, $T > 0$, and $\rho : J \times \mathcal{B} \to (-\infty, T]$ are appropriate functions, $a : D \to \mathbb{R}$ ($D = \{(t, s) \in J \times J : t \geq s\}$). Here $0 = t_0 = s_0 < t_1 \leq s_1 \leq t_2 < \dots < t_{N-1} \leq s_N \leq t_N \leq t_{N+1} = T$ are pre-fixed numbers, and $h_i \in C((t_i, s_i] \times \mathcal{B} \times X, X)$, for all $i = 1, 2, \dots, N$. For any continuous function x defined on $(-\infty, T]$ and any $t \in J$, we denote by x_t the element of $\mathcal{B}$ defined by

$$x_t(\theta) = x(t+\theta), \qquad \theta \in (-\infty, 0].$$

Here x_t represents the history of the state up to the present time t and $\phi \in \mathcal{B}$ to be specified later.

2 Preliminaries

Let $(X, \|\cdot\|)$ be a real Banach space.
$C = C(J, X)$ be the space of all X-valued continuous functions on J.
$L(X)$ be the Banach space of all linear and bounded operators on X.
$L^1(J, X)$ the space of $X-$valued Bochner integrable functions on J with the norm

$$\|y\|_{L^1} = \int_0^T \|y(t)\| dt.$$

$L^\infty(J, \mathbb{R})$ is the Banach space of measurable functions which are essentially bounded, normed by

$$\|y\|_{L^\infty} = \inf\{d > 0 : |y(t)| \leq d, \ a.e. \ t \in J\}.$$

We need some basic definitions of the fractional calculus theory which are used in this paper.

Definition 2.1. *Let $\alpha > 0$ and $f : \mathbb{R}_+ \to X$ be in $L^1(\mathbb{R}_+, X)$. Then the Riemann–Liouville integral is given by:*

$$I_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s)}{(t-s)^{1-\alpha}} ds,$$

where $\Gamma(\cdot)$ is the Euler gamma function.

For more details on the Riemann–Liouville fractional derivative, we refer the reader to [17].

Definition 2.2. *[36] The Caputo derivative of order α for a function $f : [0, +\infty) \to X$ can be written as*

$$D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{\alpha+1-n}} ds = I^{n-\alpha} f^{(n)}(t), \quad t > 0, \ n-1 \leq \alpha < n.$$

If $0 \leq \alpha < 1$, then

$$D_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f^{(1)}(s)}{(t-s)^\alpha} ds.$$

Obviously, The Caputo derivative of a constant is equal to zero.

Definition 2.3. *A function $f : J \times \mathcal{B} \times X \longrightarrow X$ is said to be an Carathéodory function if it satisfies :*

(i) for each $t \in J$ the function $f(t, \cdot, \cdot) : \mathcal{B} \times X \longrightarrow X$ is continuous;

(ii) for each $(v, w) \in \mathcal{B} \times X$ *the function* $f(\cdot, v, w) : J \to X$ *is measurable .*

Next we give the concept of a measure of noncompactness [13].

Definition 2.4. *Let* B *be a bounded subset of a Banach space* Y*. The Kuratowski measure of noncompactness of* B *is defined as*

$$\alpha(B) = \inf\{d > 0 : B \text{ has a finite cover by sets of diameter } \leq d\}.$$

We note that this measure of noncompactness satisfies the properties ([13]).

Lemma 2.5.

1. *If* $A \subseteq B$ *then* $\alpha(A) \leq \alpha(B)$*,*
2. $\alpha(A) = \alpha(\overline{A})$*, where* $\overline{A}$ *denotes the closure of* A*,*
3. $\alpha(A) = 0 \Leftrightarrow \overline{A}$ *is compact (*A *is relatively compact),*
4. $\alpha(\lambda A) = |\lambda| A,\ with\ \lambda \in \mathbb{R}$,
5. $\alpha(A \cup B) = \max\{\alpha(A), \alpha(B)\}$,
6. $\alpha(A + B) \leq \alpha(A) + \alpha(B)$, *where*
$$A + B = \{x + y : x \in A, y \in B\},$$
7. $\alpha(A + a) = \alpha(A)$ *for any* $a \in X$,
8. $\alpha(\overline{conv} A) = \alpha(A)$, *where* $\overline{conv} A$ *is the closed convex hull of* A.

For $H \subset C(J, X)$, we define

$$\int_0^t H(s)ds = \left\{ \int_0^t u(s)ds : u \in H \right\} \quad \text{for } t \in J, \tag{2.1}$$

where $H(s) = \{u(s) \in X : u \in H\}$.

Lemma 2.6. *[13] If* $H \subset C(J, X)$ *is a bounded, equicontinuous set, then*

$$\alpha_C(H) = \sup_{t \in J} \alpha(H(t)). \tag{2.2}$$

Lemma 2.7. *[21] If* $\{u_n\}_{n=1}^{\infty} \subset L^1(J, X)$ *and there exists* $m \in L^1(J, \mathbb{R}^+)$ *such that* $\|u_n(t)\| \leq m(t)$*, a.e.* $t \in J$*, then* $\alpha(\{u_n(t)\}_{n=1}^{\infty})$ *is integrable and*

$$\alpha\left(\left\{ \int_0^t u_n(s)ds \right\}_{n=1}^{\infty} \right) \leq 2 \int_0^t \alpha(\{u_n(s)\}_{n=1}^{\infty})ds. \tag{2.3}$$

In this paper, we will employ an axiomatic definition for the phase space $\mathcal{B}$ which is similar to those introduced by Hale and Kato [20]. Specifically, $\mathcal{B}$ will be a linear space of functions mapping $(-\infty, 0]$ into X endowed with a seminorm $\|\cdot\|_{\mathcal{B}}$, and satisfies the following axioms:

(A1) If $x : (-\infty, T\,] \longrightarrow X$ is continuous on J and $x_0 \in \mathcal{B}$, then $x_t \in \mathcal{B}$ and x_t is continuous in $t \in J$ and
$$\|x(t)\| \leq C\|x_t\|_{\mathcal{B}}, \tag{2.4}$$
where $C \geq 0$ is a constant.

(A2) There exist a continuous function $C_1(t) > 0$ and a locally bounded function $C_2(t) \geq 0$ in $t \geq 0$ such that
$$\|x_t\|_{\mathcal{B}} \leq C_1(t) \sup_{s\in[0,t]} \|x(s)\| + C_2(t)\|x_0\|_{\mathcal{B}}, \tag{2.5}$$
for $t \in [0, T]$ and x as in $(A1)$.

(A3) The space $\mathcal{B}$ is complete.

Remark 2.8. *Condition (2.4) in (A1) is equivalent to* $\|\phi(0)\| \leq C\|\phi\|_{\mathcal{B}}$, *for all* $\phi \in \mathcal{B}$.

Example 2.9. ***The phase space*** $C_r \times L^p(g, X)$.

Let $r \geq 0, 1 \leq p < \infty$, *and let* $g : (-\infty, -r) \to \mathbb{R}$ *be a nonnegative measurable function which satisfies the conditions* $(g-5), (g-6)$ *in the terminology of [25]. Briefly, this means that* g *is locally integrable and there exists a nonnegative, locally bounded function* Λ *on* $(-\infty, 0]$, *such that* $g(\xi + \theta) \leq \Lambda(\xi)g(\theta)$, *for all* $\xi \leq 0$ *and* $\theta \in (-\infty, -r)\backslash N_\xi$, *where* $N_\xi \subseteq (-\infty, -r)$ *is a set with Lebesgue measure zero.*

The space $C_r \times L^p(g, X)$ *consists of all classes of functions* $\varphi : (-\infty, 0] \to X$, *such that* φ *is continuous on* $[-r, 0]$, *Lebesgue-measurable, and* $g\|\varphi\|^p$ *on* $(-\infty, -r)$. *The seminorm in* $\|.\|_{\mathcal{B}}$ *is defined by*
$$\|\varphi\|_{\mathcal{B}} = \sup_{\theta\in[-r,0]} \|\varphi(\theta)\| + \left(\int_{-\infty}^{-r} g(\theta)\|\varphi(\theta)\|^p d\theta\right)^{\frac{1}{p}}.$$
The space $\mathcal{B} = C_r \times L^p(g, X)$ *satisfies axioms* ***(A1), (A2), (A3)***. *Moreover, for* $r = 0$ *and* $p = 2$, *this space coincides with* $C_0 \times L^2(g, X), H = 1, M(t) = \Lambda(-t)^{\frac{1}{2}}, K(t) = 1 + \left(\int_{-r}^{0} g(\tau)d\tau\right)^{\frac{1}{2}}$, *for* $t \geq 0$ *(see [25], Theorem 1.3.8 for details).*

For our purpose we will only need the following fixed point theorems.

Theorem 2.10. *[5, 33] Let* U *be a bounded, closed and convex subset of a Banach space, and let* N *be a continuous mapping of* U *into itself. If the implication*
$$V = \overline{\text{conv}}N(V) \text{ or } V = N(V) \cup \{0\} \Longrightarrow \alpha(V) = 0$$
holds for every subset V *of* U, *then* N *has a fixed point.*

A continuous map $N : D \subseteq E \to E$ is said to be a α-contraction if there exists a constant $\nu \in [0, 1)$ such that $\alpha(N(C)) \leq \nu\alpha(C)$ for any bounded closed subset $C \subseteq D$.

Theorem 2.11. *(Darbo–Sadovskii)[13] Let E be a Banach space. If $D \subseteq E$ is bounded closed and convex, the continuous map $N : D \to D$ is a α-contraction, then the map N has at least one fixed point in D.*

Consider the space

$$\begin{aligned} PC(J, X) = &\Big\{x : J \to X, x \in C\left(J \cap \left(\cup_{k=0}^{N}(t_k, s_k]\right), X\right), \\ &\text{and } x(t_k^+), x(s_k^-) \text{ exist with, } x(s_k^-) = x(s_k), k = 1, \dots, N\Big\}. \end{aligned}$$

Obviously, $PC(J, X)$ is a Banach space with the norm

$$\|x\|_{PC} = \sup_{t \in J} \|x(t)\|.$$

3 Existence Results

In this section, we prove the existence of mild solution of (1.1).

Definition 3.1. *A function $x : (-\infty, T] \to X$ is said to be a mild solution of the equation (1.1) if $x_0 = \phi$ on $(-\infty, T]$, $x|_{[0,T]} \in PC([0, T], X)$ and x satisfies*

$$x(t) = \begin{cases} Q(t)\phi(0) + \displaystyle\int_0^t \int_0^s R(t-s)a(s,\tau)f(\tau, x_{\rho(\tau,x_\tau)}, x(\tau))d\tau ds, & t \in [0, t_1], \\ h_i(t, x_{\rho(t,x_t)}, x(t)), & t \in (t_i, s_i], i = 1, 2, \dots, N, \\ Q(t-s_i)h_i(s_i, x_{\rho(s_i,x_{s_i})}, x(s_i)) & \\ + \displaystyle\int_0^t \int_0^s R(t-s)a(s,\tau)f(\tau, x_{\rho(\tau,x_\tau)}, x(\tau))d\tau ds, & t \in (s_i, t_{i+1}], \end{cases} \tag{3.1}$$

where

$$Q(t) = \int_0^\infty \xi_q(\sigma)S(t^q\sigma)d\sigma, \qquad R(t) = q\int_0^\infty \sigma t^{q-1}\xi_q(\sigma)S(t^q\sigma)d\sigma$$

and ξ_q is a probability density function defined on $(0, \infty)$ such that

$$\xi_q(\sigma) = \frac{1}{q}\sigma^{-1-(\frac{1}{q})}\varpi_q(\sigma^{-\frac{1}{q}}) \geq 0,$$

where

$$\varpi_q(\sigma) = \frac{1}{\pi}\sum_{k=1}^{\infty}(-1)^{k-1}\sigma^{-qk-1}\frac{\Gamma(kq+1)}{k!}\sin(k\pi q), \quad \sigma \in (0,\infty).$$

Remark 3.2. *Note that* $\{S(t)\}_{t\geq 0}$ *is a uniformly bounded i.e*

$$\text{there exists a constant} \quad M > 0 \quad \text{such that} \quad \|S(t)\|_{L(X)} \leq M \quad \text{for all} \quad t \geq 0.$$

Remark 3.3. *According to [30], direct calculation gives that*

$$\|R(t)\| \leq C_{q,M} t^{q-1}, \quad t > 0, \tag{3.2}$$

where $C_{q,M} = \dfrac{qM}{\Gamma(1+q)}$.

Set

$$\mathcal{R}(\rho^-) = \{\rho(s,\varphi) : (s,\varphi) \in J \times \mathcal{B}, \rho(s,\varphi) \leq 0\}.$$

We always assume that $\rho : J \times \mathcal{B} \to (-\infty, T]$ is continuous. Additionally, we introduce following hypothesis:

(H_φ) The function $t \to \varphi_t$ is continuous from $\mathcal{R}(\rho^-)$ into $\mathcal{B}$ and there exists a continuous and bounded function $L^\phi : \mathcal{R}(\rho^-) \to (0,\infty)$ such that

$$\|\phi_t\|_{\mathcal{B}} \leq L^\phi(t)\|\phi\|_{\mathcal{B}} \quad \text{for every } t \in \mathcal{R}(\rho^-).$$

Remark 3.4. *Condition* (H_φ), *is frequently verified by the continuous and bounded functions. For more details see [25].*

Remark 3.5. *In the rest of this section,* C_1^* *and* C_2^* *are the constants*

$$C_1^* = \sup_{s\in J} C_1(s) \text{ and } C_2^* = \sup_{s\in J} C_2(s).$$

Lemma 3.6. *[22] If* $x : \mathbb{R} \to X$ *is a function such that* $x_0 = \phi$, *then*

$$\|x_s\|_{\mathcal{B}} \leq (C_2^* + L^\phi)\|\phi\|_{\mathcal{B}} + C_1^* \sup\{|x(\theta)|; \theta \in [0, \max\{0,s\}]\}, \; s \in \mathcal{R}(\rho^-) \cup J,$$

where $L^\phi = \sup\limits_{t\in\mathcal{R}(\rho^-)} L^\phi(t)$.

Let us introduce the following hypotheses:

(H1) $f : J \times \mathcal{B} \times X \longrightarrow X$ satisfies the Carathéodory conditions.

(H2) There exist functions $\mu, \mu^* \in L^1(J, \mathbb{R}^+)$ and continuous nondecreasing functions $\psi, \psi^* : \mathbb{R}^+ \to (0, +\infty)$ such that

$$\|f(t,x,y)\| \leq \mu(t)\psi\left(\|x\|_{\mathcal{B}} + \|y\|\right), \quad (t,x,y) \in J \times \mathcal{B} \times X,$$

$$\|h_i(t,x,y)\| \leq \mu^*(t)\psi^*\left(\|x\|_{\mathcal{B}} + \|y\|\right), \quad (t,x,y) \in J \times \mathcal{B} \times X,$$

(H3) For any bounded sets $D_1 \subset \mathcal{B}, D_2 \subset X$, and $0 \leq s \leq t \leq T$, there exists an integrable positive function η such that

$$\alpha\left(R(t-s)f(\tau, D_1, D_2)\right) \leq \eta_t(s,\tau)\left(\alpha(D_2) + \sup_{-\infty<\theta\leq 0}\alpha(D_1(\theta))\right),$$

where $\eta_t(s,\tau) = \eta(t,s,\tau)$ and $\sup_{t\in J}\int_0^t\int_0^s \eta_t(s,\tau)d\tau ds = \eta^* < \infty$.

(H4) There exists a constant $L > 0$ such that, for each bounded sets $D_1 \subset \mathcal{B}, D_2 \subset X$,

$$\alpha\left(h_i(\tau, D_1, D_2)\right) \leq L\left(\alpha(D_2) + \sup_{-\infty<\theta\leq 0}\alpha(D_1(\theta))\right).$$

(H5) For each $t \in J$, $a(t,s)$ is measurable on $[0,t]$ and $a(t) = ess\sup\{|a(t,s)|, 0 \leq s \leq t\}$ is bounded on J. The map $t \to a_t$ is continuous from J to $L^\infty(J,\mathbb{R})$, here, $a_t(s) = a(t,s)$.

Set $a = \sup_{t\in J} a(t)$.

Our first result is based on the Mönch fixed point theorem.

Theorem 3.7. *Suppose that the assumptions* $(H_\varphi), (H1)-(H5)$ *hold, and if*

$$2ML + 16\ a\ \eta^* < 1, \tag{3.3}$$

then the problem (1.1) has at least one mild solution.

Proof. Let $Y = \{u \in PC(X) : u(0) = \phi(0) = 0\}$ endowed with the uniform convergence topology and define the operator $P : Y \to Y$ by

$$P(x)(t) = \begin{cases} Q(t)\phi(0) + \displaystyle\int_0^t\int_0^s R(t-s)a(s,\tau)f(\tau, \overline{x}_{\rho(\tau,\overline{x}_\tau)}, \overline{x}(\tau))d\tau ds, & t \in [0,t_1], \\ h_i(t, \overline{x}_{\rho(t,\overline{x}_t)}, \overline{x}(t)), & t \in (t_i, s_i], i = 1,2,\ldots,N, \\ Q(t-s_i)h_i(s_i, \overline{x}_{\rho(s_i,\overline{x}_{s_i})}, \overline{x}(s_i)) & \\ + \displaystyle\int_{s_i}^t\int_0^s R(t-s)a(s,\tau)f(\tau, \overline{x}_{\rho(\tau,\overline{x}_\tau)}, \overline{x}(\tau))d\tau ds, & t \in (s_i, t_{i+1}], \end{cases}$$

where $x : (-\infty, T] \to X$ is such that $x_0 = \phi$ and $\overline{x} = x$ on J. Let $\overline{\phi} : (-\infty, T] \longrightarrow X$ be the extension of ϕ to $(-\infty, T]$ such that $\overline{\phi}(\theta) = \phi(0) = 0$ on J.

Choose

$$r \geq M\|\mu^*\|_{L^1}\psi^*\left((C_2^* + L^\phi)\|\phi\|_{\mathcal{B}} + (C_1^*+1)r\right) + aC_{q,M}\|\mu\|_{L^1}\frac{T^q}{q}\psi\left((C_2^* + L^\phi)\|\phi\|_{\mathcal{B}} + (C_1^*+1)r\right),$$

and define the set

$$B_r = \{x \in Y : \|x\|_{PC} \leq r\},$$

then B_r is a bounded, closed-convex subset in Y.

Step 1: P is continuous.

Let $\{x^k\}_{k\in\mathbb{N}}$ be a sequence such that $x^k \to x$ in B_r as $k \to \infty$.

Case 1. For each $t \in [0, t_1]$, we have

$$\begin{aligned}
\|P(x^k)(t) - P(x)(t)\| &\leq \int_0^t \int_0^s \|R(t-s)\| \|a(s,\tau)\| \|f(\tau, \overline{x}^k_{\rho(\tau,\overline{x}^k_\tau)}, \overline{x}^k(\tau)) \\
&- f(\tau, \overline{x}_{\rho(\tau,\overline{x}_\tau)}, \overline{x}(\tau))\| d\tau ds \\
&\leq aC_{q,M} \int_0^t \int_0^s (t-s)^{q-1} \|f(\tau, \overline{x}^k_{\rho(\tau,\overline{x}^k_\tau)}, \overline{x}^k(\tau)) \\
&- f(\tau, \overline{x}_{\rho(\tau,\overline{x}_\tau)}, \overline{x}(\tau))\| d\tau ds.
\end{aligned}$$

Case 2. For each $t \in [t_i, s_i), i = 1, 2, \ldots, N$, we have

$$\begin{aligned}
\|P(x^k)(t) - P(x)(t)\| &= \|h_i(t, \overline{x}^k_{\rho(t,\overline{x}^k_t)}, \overline{x}^k(t)) - h_i(t, \overline{x}_{\rho(t,\overline{x}_t)}, \overline{x}(t))\| \\
&\to 0 \quad k \to \infty.
\end{aligned}$$

Case 3. For each $t \in (s_i, t_{i+1}], i = 1, 2, \ldots, N$, we obtain

$$\begin{aligned}
\|P(x^k)(t) - P(x)(t)\| &\leq \|Q(t-s_i)\| \|h_i(s_i, \overline{x}^k_{\rho(s_i,\overline{x}^k_{s_i})}, \overline{x}^k(s_i)) - h_i(s_i, \overline{x}_{\rho(s_i,\overline{x}_{s_i})}, \overline{x}(s_i))\| \\
&+ \int_{s_i}^t \int_0^s \|R(t-s)\| \|a(s,\tau)\| \|f(\tau, \overline{x}^k_{\rho(\tau,\overline{x}^k_\tau)}, \overline{x}^k(\tau)) \\
&- f(\tau, \overline{x}_{\rho(\tau,\overline{x}_\tau)}, \overline{x}(\tau))\| d\tau ds \\
&\leq M \|h_i(s_i, \overline{x}^k_{\rho(s_i,\overline{x}^k_{s_i})}, \overline{x}^k(s_i)) - h_i(s_i, \overline{x}_{\rho(s_i,\overline{x}_{s_i})}, \overline{x}(s_i))\| \\
&+ aC_{q,M} \int_{s_i}^t \int_0^s (t-s)^{q-1} \|f(\tau, \overline{x}^k_{\rho(\tau,\overline{x}^k_\tau)}, \overline{x}^k(\tau)) \\
&- f(\tau, \overline{x}_{\rho(\tau,\overline{x}_\tau)}, \overline{x}(\tau))\| d\tau ds.
\end{aligned}$$

Since the function h_i is continuous and f is of Carathéodory type, we have by the Lebesgue dominated convergence theorem that

$$\|P(x^k)(t) - P(x)(t)\| \to 0 \quad \text{as } k \to \infty,$$

which shows the operator P is continuous.

Step 2: P maps B_r into itself.

Case 1. For all $t \in [0, t_1]$, we get

$$\begin{aligned}
\|P(x)(t)\| &\leq \|Q(t)\phi(0)\| + \int_0^t \int_0^s \|R(t-s)a(s,\tau)f(\tau, \overline{x}_{\rho(\tau,\overline{x}_\tau)}, \overline{x}(\tau))\| d\tau ds \\
&\leq MC\|\phi\|_{\mathcal{B}} + aC_{q,M} \int_0^t \int_0^s (t-s)^{q-1}\mu(\tau)\psi(\|\overline{x}_{\rho(\tau,\overline{x}_\tau)}\|_{\mathcal{B}} + \|\overline{x}\|) d\tau ds \\
&\leq MC\|\phi\|_{\mathcal{B}} + aC_{q,M} \int_0^t \int_0^s (t-s)^{q-1}\mu(\tau) \\
&\times \psi\left((C_2^* + L^\phi)\|\phi\|_{\mathcal{B}} + C_1^* r + r\right) d\tau ds \\
&\leq MC\|\phi\|_{\mathcal{B}} + aC_{q,M}\|\mu\|_{L^1} \frac{T^q}{q} \psi\left((C_2^* + L^\phi)\|\phi\|_{\mathcal{B}} + (C_1^* + 1)r\right) \\
&\leq r.
\end{aligned}$$

Case 2. For all $t \in [t_i, s_i), i = 1, 2, \ldots, N$, we have

$$\begin{aligned}
\|P(x)(t)\| &\leq \|h_i(t, \overline{x}_{\rho(t,\overline{x}_t)}, \overline{x}(t))\| \\
&\leq \mu^*(t)\psi^*\left(\|\overline{x}_{\rho(t,\overline{x}_t)}\|_{\mathcal{B}} + \|\overline{x}\|\right) \\
&\leq \|\mu^*\|_{L^1}\psi^*\left((C_2^* + L^\phi)\|\phi\|_{\mathcal{B}} + (C_1^* + 1)r\right) \\
&\leq r.
\end{aligned}$$

Case 3. For all $t \in (s_i, t_{i+1}], i = 1, 2, \ldots, N$, we obtain

$$\begin{aligned}
\|P(x)(t)\| &\leq \|Q(t-s_i)h_i(s_i, \overline{x}_{\rho(s_i,\overline{x}_{s_i})}, \overline{x}(s_i))\| \\
&+ \int_{s_i}^t \int_0^s \|R(t-s)a(s,\tau)f(\tau, \overline{x}_{\rho(\tau,\overline{x}_\tau)}, \overline{x}(\tau))\| d\tau ds, \\
&\leq M\|\mu^*\|_{L^1}\psi^*\left((C_2^* + L^\phi)\|\phi\|_{\mathcal{B}} + (C_1^* + 1)r\right) \\
&+ aC_{q,M}\|\mu\|_{L^1}\frac{T^q}{q}\psi\left((C_2^* + L^\phi)\|\phi\|_{\mathcal{B}} + (C_1^* + 1)r\right) \\
&\leq r.
\end{aligned}$$

Step 3: $P(B_r)$ is bounded and equicontinuous.
Case 1. For each $t \in [0, t_1], 0 \leq \tau_2 \leq \tau_1 \leq t_1$, and $x \in B_r$. Then we have

$$\|P(x)(\tau_1) - P(x)(\tau_2)\| \leq I_1 + I_2 + I_3,$$

where

$$\begin{aligned}
I_1 &= \|Q(\tau_1) - Q(\tau_2)\|\|\phi(0)\| \\
I_2 &= \left\|\int_0^{\tau_2}\int_0^s [R(\tau_1 - s) - R(\tau_2 - s)]a(s,\tau)f(\tau, \overline{x}_{\rho(\tau,\overline{x}_\tau)}, \overline{x}(\tau)) d\tau ds\right\| \\
I_3 &= \left\|\int_{\tau_2}^{\tau_1}\int_0^s R(\tau_1 - s)a(s,\tau)f(\tau, \overline{x}_{\rho(\tau,\overline{x}_\tau)}, \overline{x}(\tau)) d\tau ds\right\|.
\end{aligned}$$

I_1 tends to zero as $\tau_2 \to \tau_1$, since $S(t)$ is uniformly continuous operator.
For I_2, using (3.2) and (H2), we have

$$\begin{aligned}
I_2 &\leq a\psi\left((C_2^* + L^\phi)\|\phi\|_{\mathcal{B}} + (C_1^*+1)r\right)\|\mu\|_{L^1}\int_0^{\tau_2}[R(\tau_1-s)-R(\tau_2-s)]ds \\
&\leq a\psi\left((C_2^* + L^\phi)\|\phi\|_{\mathcal{B}} + (C_1^*+1)r\right)\|\mu\|_{L^1} \\
&\quad \times \int_0^{\tau_2}\left[q\int_0^\infty \sigma(\tau_1-s)^{q-1}\xi_q(\sigma)S((\tau_1-s)^q\sigma)d\sigma\right. \\
&\quad \left. -q\int_0^\infty \sigma(\tau_2-s)^{q-1}\xi_q(\sigma)S((\tau_2-s)^q\sigma)d\sigma\right] ds \\
&\leq a\psi\left((C_2^* + L^\phi)\|\phi\|_{\mathcal{B}} + (C_1^*+1)r\right)\|\mu\|_{L^1} \\
&\quad \times \left[q\int_0^{\tau_2}\int_0^\infty \sigma\|[(\tau_1-s)^{q-1}-(\tau_2-s)^{q-1}]\xi_q(\sigma)S((\tau_1-s)^q\sigma)\right. \\
&\quad \left. +q\int_0^{\tau_2}\int_0^\infty \sigma(\tau_2-s)^{q-1}\xi_q(\sigma)\|S((\tau_1-s)^q\sigma)-S((\tau_2-s)^q\sigma)\|\right] \\
&\leq a\psi\left((C_2^* + L^\phi)\|\phi\|_{\mathcal{B}} + (C_1^*+1)r\right)\|\mu\|_{L^1} \\
&\times \quad [C_{q,M}\int_0^{\tau_2}\left|(\tau_1-s)^{q-1}-(\tau_2-s)^{q-1}\right| ds \\
&+ \quad q\int_0^{\tau_2}\int_0^\infty \sigma(\tau_2-s)^{q-1}\xi_q(\sigma)\|S((\tau_1-s)^q\sigma)-S((\tau_2-s)^q\sigma)\|d\sigma ds].
\end{aligned}$$

Clearly, the first term on the right-hand side of the above inequality tends to zero as $\tau_2 \to \tau_1$. From the continuity of $S(t)$ in the uniform operator topology for $t > 0$, The second term on the right-hand side of the above inequality tends to zero as $\tau_2 \to \tau_1$.
In view of (H2), we have

$$\begin{aligned}
I_3 &\leq a\ C_{q,M}\int_{\tau_2}^{\tau_1}\int_0^s(\tau_1-s)^{q-1}\|f(\tau,\overline{x}_{\rho(\tau,\overline{x}_\tau)},\overline{x}(\tau))\|d\tau ds \\
&\leq a\ C_{q,M}\psi\left((C_2^* + L^\phi)\|\phi\|_{\mathcal{B}} + (C_1^*+1)r\right)\|\mu\|_{L^1}\int_{\tau_2}^{\tau_1}(\tau_1-s)^{q-1}ds.
\end{aligned}$$

As $\tau_2 \to \tau_1$, I_3 tends to zero.
Case 2. For each $t \in [t_i, s_i), i = 1, 2, \ldots, N, t_i \leq \tau_2 \leq \tau_1 \leq s_i$, and $x \in B_r$. Then we have

$$\begin{aligned}
\|P(x)(\tau_1) - P(x)(\tau_2)\| &= \|h_i(\tau_1, \overline{x}_{\rho(\tau_1,\overline{x}_{\tau_1})}, \overline{x}(\tau_1)) - h_i(\tau_2, \overline{x}_{\rho(\tau_2,\overline{x}_{\tau_2})}, \overline{x}(\tau_2))\| \\
&\to 0 \quad as \quad \tau_2 \to \tau_1.
\end{aligned}$$

Case 3. For each $t \in (s_i, t_{i+1}], i = 1, 2, \ldots, N, s_i \leq \tau_2 \leq \tau_1 \leq t_{i+1}$, and $x \in B_r$. Then we have

$$\|P(x)(\tau_1) - P(x)(\tau_2)\| \leq \|Q(\tau_1 - s_i) - Q(\tau_2 - s_i)\|\|h_i(s_i, \overline{x}_{\rho(s_i,\overline{x}_{s_i})}, \overline{x}(s_i))\| + I_1 + I_2 + I_3.$$

Since $S(t)$ is uniformly continuous operator, so

$$\lim_{\tau_2 \to \tau_1}\|Q(\tau_1 - s_i) - Q(\tau_2 - s_i)\| = 0,\ i = 1, \ldots, N.$$

Consequently

$$\lim_{\tau_2 \to \tau_1} \|P(x)(\tau_1) - P(x)(\tau_2)\| = 0.$$

Thus, $P(B_r)$ is equicontinuous.

Now let V be a subset of B_r such that $V \subset \overline{conv}(P(V) \cup \{0\})$. Moreover, for any $\varepsilon > 0$ and bounded set D, we can take a sequence $\{v_n\}_{n=1}^{\infty} \subset D$ such that $\alpha(D) \leq 2\alpha(\{v_n\}) + \varepsilon$ ([16], P. 125). Thus, for $\{v_n\}_{n=1}^{\infty} \subset V$, and using lemmas 2.5-2.7 and (H3), we have, for $t \in [0, t_1]$,

$$\begin{aligned}
\alpha(PV) &\leq 2\alpha(\{Pv_n\}) + \varepsilon \\
&= 2\sup_{t\in J} \alpha(\{Pv_n(t)\}) + \varepsilon \\
&= 2\sup_{t\in J} \alpha\left(\left\{\int_0^t R(t-s)\int_0^s a(s,\tau)f(\tau, y_\tau + v_{n\tau}, y(\tau) + v_n(\tau))d\tau ds\right\}\right) + \varepsilon \\
&\leq 4\sup_{t\in J} \int_0^t \alpha\left(\left\{R(t-s)\int_0^s a(s,\tau)f(\tau, y_\tau + v_{n\tau}, y(\tau) + v_n(\tau))d\tau ds\right\}\right) + \varepsilon \\
&\leq 8\sup_{t\in J} \int_0^t \int_0^s \alpha(\{R(t-s)a(s,\tau)f(\tau, y_\tau + v_{n\tau}, y(\tau) + v_n(\tau))d\tau ds\}) + \varepsilon \\
&\leq 8\ a\sup_{t\in J} \int_0^t \int_0^s \alpha(\{R(t-s)f(\tau, y_\tau + v_{n\tau}, y(\tau) + v_n(\tau))d\tau ds\}) + \varepsilon \\
&\leq 8\ a\ \sup_{t\in J} \int_0^t \int_0^s \eta_t(s,\tau)\left[\alpha(v_n(\tau)) + \sup_{-\infty<\theta\leq 0} \alpha(v_n(\theta+\tau))\right] d\tau ds + \varepsilon \\
&\leq 8\ a\ \sup_{t\in J} \int_0^t \int_0^s \eta_t(s,\tau)\left[\alpha(v_n) + \sup_{0<\mu\leq\tau} \alpha(v_n(\mu))\right] d\tau ds + \varepsilon \\
&\leq 16\ a\ \alpha(v_n)\sup_{t\in J} \int_0^t \int_0^s \eta_t(s,\tau)d\tau ds + \varepsilon \\
&\leq 16\ a\ \eta^* \alpha(V) + \varepsilon.
\end{aligned}$$

For any $t \in [t_i, s_i), i = 1, 2, \ldots, N$, we get

$$\begin{aligned}
\alpha(PV) &= \alpha\left(h_i(t, \overline{x}_{\rho(t,\overline{x}_t)}, \overline{x}(t))\right) \\
&\leq L\left(\alpha(v_n(t)) + \sup_{-\infty<\theta\leq 0} \alpha(v_n(\theta+t))\right) \\
&\leq L\left(\alpha(v_n) + \sup_{0<\mu\leq\tau} \alpha(v_n(\mu))\right) \\
&\leq 2L\alpha(v_n) \\
&\leq 2L\alpha(V).
\end{aligned}$$

In the same way, for any $t \in (s_i, t_{i+1}], i = 1, 2, \ldots, N$, we obtain

$$\begin{aligned}
\alpha(PV) &\leq 2\alpha(\{Pv_n\}) + \varepsilon \\
&= 2\sup_{t\in J} \alpha(\{Pv_n(t)\}) + \varepsilon \\
&= 2\sup_{t\in J} \alpha\left(Q(t-s_i)h_i(s_i, \overline{x}_{\rho(s_i,\overline{x}_{s_i})}, \overline{x}(s_i))\right) \\
&+ 2\sup_{t\in J} \alpha\left(\left\{\int_{s_i}^{t} R(t-s)\int_0^s a(s,\tau)f(\tau, y_\tau + v_{n\tau}, y(\tau) + v_n(\tau))d\tau ds\right\}\right) + \varepsilon \\
&\leq 2ML\alpha(v_n) \\
&+ 4\sup_{t\in J} \int_{s_i}^{t} \alpha\left(\left\{R(t-s)\int_0^s a(s,\tau)f(\tau, y_\tau + v_{n\tau}, y(\tau) + v_n(\tau))d\tau ds\right\}\right) + \varepsilon \\
&\leq 2ML\alpha(v_n) \\
&+ 8\sup_{t\in J} \int_{s_i}^{t}\int_0^s \alpha(\{R(t-s)a(s,\tau)f(\tau, y_\tau + v_{n\tau}, y(\tau) + v_n(\tau))d\tau ds\}) + \varepsilon \\
&\leq 2ML\alpha(v_n) + 8\ a\sup_{t\in J} \int_{s_i}^{t}\int_0^s \alpha(\{R(t-s)f(\tau, y_\tau + v_{n\tau}, y(\tau) + v_n(\tau))d\tau ds\}) + \varepsilon \\
&\leq 2ML\alpha(v_n) + 8\ a\sup_{t\in J} \int_{s_i}^{t}\int_0^s \eta_t(s,\tau)\left[\alpha(v_n(\tau)) + \sup_{-\infty<\theta\leq 0} \alpha(v_n(\theta+\tau))\right] d\tau ds + \varepsilon \\
&\leq 2ML\alpha(v_n) + 8\ a\ \sup_{t\in J} \int_{s_i}^{t}\int_0^s \eta_t(s,\tau)\left[\alpha(v_n) + \sup_{0<\mu\leq\tau} \alpha(v_n(\mu))\right] d\tau ds + \varepsilon \\
&\leq 2ML\alpha(v_n) + 16\ a\ \alpha(v_n)\sup_{t\in J} \int_0^t\int_0^s \eta_t(s,\tau)d\tau ds + \varepsilon \\
&\leq 2ML\alpha(V) + 16\ a\ \eta^*\alpha(V) + \varepsilon \\
&\leq (2ML + 16\ a\ \eta^*)\,\alpha(V) + \varepsilon.
\end{aligned}$$

Therefore, in view of Lemma 2.5, we have

$$\alpha(V) \leq \alpha(PV) \leq (2ML + 16\ a\ \eta^*)\,\alpha(V) + \varepsilon,$$

since ε is arbitrary we obtain that

$$\alpha(V) \leq (2ML + 16\ a\ \eta^*)\,\alpha(V).$$

This means that

$$\alpha(V)\,(1 - (2ML + 16\ a\ \eta^*)) \leq 0.$$

By (3.3) it follows that $\alpha(V) = 0$. In view of the Ascoli-Arzelà theorem, V is relatively compact in B_r. Applying now Theorem 2.10, we conclude that P has a fixed point which is a solution of the problem (1.1).

The second result is established using the Darbo's fixed point theorem.

Theorem 3.8. *Assume that* (H1)−(H5) *are satisfied, then the problem (1.1) has at least one mild solution.*

Proof. In what follows we show that the operator $P : Y \to Y$ is a strict set contraction. We know that $P : Y \to Y$ is bounded and continuous, we need to prove that there exists a constant $0 \le \nu < 1$ such that $\alpha(PV) \le \nu\alpha(V)$ for $V \subset B_r$.

Using the same method as the proof of Theorem 3.7, for $t \in [0, T]$, we have

$$\alpha\,(PV) \quad \le \quad (2ML + 16\; a\; \eta^*)\,\alpha(V) + \varepsilon,$$

since ε is arbitrary we obtain that

$$\alpha\,(PV) \le \nu\alpha(V).$$

Hence P is a set contraction. According to Theorem 2.11 the operator P has at least one fixed point which is obviously a mild solution of the problem (2.4). This completes the proof.

4 An Example

We consider the fractional integro-differential equations with state-dependent delay and non-instantaneous impulses of the form

$$\begin{aligned}
&\frac{\partial^q_t}{\partial t^q}\nu(t,\zeta) + \frac{\partial^2}{\partial\zeta^2}\nu(t,\zeta) = \int_0^t (t-s)^2 \int_{-\infty}^s \gamma(\tau - s)\nu(\tau - \rho_1(s)\rho_2(|\nu(s,\zeta)|),\zeta)d\tau ds \\
&\qquad\qquad + \int_0^t (t-s)^2 \cos|\nu(s,\zeta)|ds,\;\; (t,x) \in \mathbb{N} \in \cup_{i=1}^n [s_i, t_{i+1}] \times [0,\pi], \\
&\nu(t,0) = \nu(t,\pi) = 0, \qquad t \in [0,T], \\
&\nu(\tau,\zeta) = \nu_0(\theta,\zeta), \qquad \theta \in (-\infty, 0], x \in [0,\pi] \\
&\nu(t,\zeta) = H_i(t, \nu(t - \rho_1(t)\rho_2(|\nu(t,\zeta)|),\zeta),\zeta),\;\; (t,x) \in (t_i, s_i] \times [0,\pi],\;\; i = 1,2,\ldots,N,
\end{aligned} \tag{4.1}$$

where $0 < q < 1, 0 = t_0 = s_0 < t_1 \le s_1 \le t_2 < \ldots < t_{N-1} \le s_N \le t_N \le t_{N+1} = T$ are prefixed real numbers and the functions $\gamma : \mathbb{R} \to \mathbb{R}, \rho_i : [0, +\infty) \to [0, +\infty),\; i = 1, 2$ are continuous functions.

Let $X = L^2([0,\pi])$ and define the operator $A : D(A) \subset X \to X$ by $A\omega = \omega''$ with domain

$$D(A) = \{\omega \in E : \omega, \omega' \text{ are absolutely continuous, } \omega'' \in E, \omega(0) = \omega(\pi) = 0\}.$$

Then

$$A\omega = \sum_{n=1}^{\infty} n^2(\omega, \omega_n)\omega_n, \quad \omega \in D(A),$$

where $\omega_n(x) = \sqrt{\frac{2}{\pi}}\sin(nx), n \in \mathbb{N}$ is the orthogonal set of eigenvectors of A. It is well known that A is the infinitesimal generator of an analytic semigroup $\{S(t)\}_{t\geq 0}$ in X and is given by

$$S(t)\omega = \sum_{n=1}^{\infty} e^{-n^2 t}(\omega, \omega_n)\omega_n, \quad \forall \omega \in X, \text{ and every } t > 0.$$

From these expressions, it follows that $\{S(t)\}_{t\geq 0}$ is a uniformly bounded compact semigroup on X. For the phase space, we choose $\mathcal{B} = C_0 \times L^2(g, X)$, see Example 2.9 for details.

Set

$$\begin{aligned}
x(t)(\zeta) &= \nu(t,\zeta), \\
\phi(\theta)(\zeta) &= \nu_0(\theta,\zeta), \\
a(t,s) &= (t-s)^2 \\
f(t,\varphi,x(t))(\zeta) &= \int_{-\infty}^{0} \gamma(t)\varphi(t,\zeta)ds + \cos|x(t)(\zeta)|, \\
h_i(t,\varphi,x(t))(\zeta) &= H_i(t,\nu(t-\rho_1(t)\rho_2(|x(t)|),\zeta),\zeta) \\
\rho(t,\varphi) &= t-\rho_1(t)\rho_2(|\varphi(0)|).
\end{aligned}$$

Under the above conditions, we can represent the problem (4.1) by the abstract problem (1.1).

Proposition 4.1. *Let $\varphi \in \mathcal{B}$ be such that (H_φ) holds, and let $t \to \varphi_t$ be continuous on $\mathcal{R}(\rho^-)$. Then there exists a mild solution of (4.1).*

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Positive periodic solutions of functional discrete systems with a parameter

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ABSTRACT

The existence of multiple positive periodic solutions of the system of difference equations with a parameter

$$x(n+1) = A(n, x(n))x(n) + \lambda f(n, x_n),$$

is studied. In particular, we use the eigenvalue problems of completely continuous operators to obtain our results. We apply our results to a well-known model in population dynamics.

RESUMEN

Estudiamos la existencia de soluciones periódicas múltiples del siguiente sistema de ecuaciones diferenciales con un parámetro

$$x(n+1) = A(n, x(n))x(n) + \lambda f(n, x_n).$$

En particular, usamos los problemas de valores propios de operadores completamente continuos para obtener nuestros resultados. Aplicamos nuestros resultados a modelos de dinámica poblacional bien conocidos.

Keywords and Phrases: Functional difference system, Positive periodic solution, Eigenvalue, Population model

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1 Introduction

Let $\mathbb{R}$ denote the real numbers, $\mathbb{Z}$ the integers, $\mathbb{Z}_-$ the negative integers, $\mathbb{R}^k_+ = \{(x_1, x_2, ..., x_k)^T \in \mathbb{R}^k : \ x_j \geq 0, \ j = 1, 2, ..., k\}$, $\mathbb{R}^+ = \{x \in \mathbb{R} : \ x > 0\}$, and $\mathbb{Z}^+$ the nonnegative integers. Also, let BC denote the normed vector space of bounded functions $\phi : \mathbb{Z} \to \mathbb{R}^k$, with the norm $||\phi|| = \sum_{j=1}^{k} \max_{n\in[0,\omega-1]} |\phi_j(n)|$, where $\phi = (\phi_1, \phi_2, ..., \phi_k)^T$ and $[0, \omega - 1] = \{0, 1, ..., \omega - 1\}$. Particularly for each $x = (x_1, x_2, ..., x_k)^T \in \mathbb{R}^k$, we define the norm $|x|_0 = \sum_{j=1}^{k} |x_j|$. Also, denote by $BC^k_+ = \{\phi \in BC : \ \phi(n) \in \mathbb{R}^k_+$ for $n \in \mathbb{Z}\}$.

In [12], Raffoul used a Krasnoselskii's fixed point theorem in cones to prove the existence of positive periodic solutions of the scaler difference equation with parameter

$$x(n + 1) = a(n)x(n) + \lambda h(n)f(x(n - \tau(n))).$$

Also, in [10], Zhu and Li generalized the work in [12] by proving that the system of difference equations with parameter

$$x(n + 1) = A(n)x(n) + \lambda h(n)f(x(n - \tau(n)))$$

where $A(n) = \text{diag}[a_1(n), a_2(n), ..., a_m(n)]$ and $h(n) = \text{diag}[h_1(n), h_2(n), ..., h_m(n)]$ has positive periodic solutions. Motivated by the above considerations we investigate the existence of multiple positive periodic solutions of the nonautonomous system of difference equations

$$x(n + 1) = A(n, x(n))x(n) + \lambda f(n, x_n), \tag{1.1}$$

where, $\lambda > 0$ is a parameter, $A(n, x(n)) = \text{diag}[a_1(n, x(n)), ..., a_k(n, x(n))]$, $a_j(n+\omega, .) = a_j(n, .)$, $f(n, x) : \mathbb{Z} \times BC \to \mathbb{R}^k$ is continuous in x and $f(n, x)$ is ω-periodic in n and x, whenever x is ω-periodic, $\omega \geq 1$ is an integer. If $x \in BC$, then $x_n \in BC$ for any $n \in \mathbb{Z}$ is defined by $x_n(\theta) = x(n+\theta)$ for $\theta \in \mathbb{Z}$. Throughout this paper, we denote the product of $y(n)$ from $n = a$ to $n = b$ by $\prod_{n=a}^{b} y(n)$ with the understanding that $\prod_{n=a}^{b} y(n) = 1$ for all $a > b$. Also, for two $m \times n$ matrices A and B, $A \geq B$ ($A < B$) means that the inequality is satisfied entrywisely. In particular, A is said to be a nonnegative matrix if $A \geq 0$.

Definition 3.1. [4] Let X be a Banach space and P a closed, nonempty subset of X. P is a (convex) cone if

(i) $x, y \in P$ and $\alpha, \beta \in \mathbb{R}_+$ imply $\alpha x + \beta y \in P$.

(ii) $x \in P$ and $-x \in P$ imply $x = 0$.

Definition 3.2. [4] Let X be a Banach space and $D \subset X$, $0 \in D$. The operator $L : D \to X$ is such that $L0 = 0$. $x_\lambda \neq 0$ is said to be an eigenvector of the eigenvalue λ of L if $Lx_\lambda = \lambda x_\lambda$.

Lemma 3.1. [4] Suppose D is an open subset of an infinite-dimensional real Banach space X, $0 \in D$, and P is a cone of X. If the operator $\Gamma : P \cap \overline{D} \to P$ is completely continuous with $\Gamma 0 = 0$ and satisfies $\inf_{x \in P \cap \partial D} \|\Gamma x\| > 0$, then Γ has an eigenvector on $P \cap \partial D$ associated with a positive eigenvalue. That is, there exist $x_0 \in P \cap \partial D$ and $\mu_0 > 0$ such that $\Gamma x_0 = \mu_0 x_0$.

In this paper we make the following assumptions.

(H1) $0 < a_j(n) < 1, \ j = 1, 2, ...k$, and $n \in [0, \ \omega - 1]$.

(H2) There exist $B(n) = \text{diag}[b_1(n), b_2(n), ..., b_k(n)]$ and $C(n) = \text{diag}[c_1(n), c_2(n), ..., c_k(n)]$ where $b_j, c_j : \mathbb{Z} \to \mathbb{R}_+$ are ω-periodic with $0 < b_j, c_j < 1$, such that

$$B(n) \le A(n, \varphi(n)) \le C(n)$$

for all $(n, \varphi) \in \mathbb{Z} \times BC_+^k$.

(H3) $f(n, 0) = 0$ for all $n \in \mathbb{Z}$.

(H4) $f(n, \varphi_n) \le 0$ for all $(n, \varphi) \in \mathbb{Z} \times BC_+^k$.

(H5) For any $L > 0$ and $\epsilon > 0$, there exists $\delta > 0$ such that $[\phi, \psi \in BC_+^k, \ \|\phi\| \le L, \ \|\psi\| \le L, \ \|\phi - \psi\| < \delta, \ 0 \le s \le \omega]$ imply

$$|f(s, \phi_s) - f(s, \psi_s)| < \epsilon.$$

To study system (1.1) we let $\mathcal{X} = \{x : \mathbb{Z} \to \mathbb{R}^k, \ x(n + \omega) = x(n)\}$, then it is clear that $\mathcal{X} \subset BC$, endowed with the norm $\|x\| = \sum_{j=1}^{k} |x_j|_0$, where $|x_j|_0 = \max_{n \in [0, \omega - 1]} |x_j(n)|$.

For the next lemma we consider

$$x_j(n + 1) = a_j(n, x(n))x_j(n) + f_j(n, x_n), j = 1, 2, ..., k. \tag{1.2}$$

The proof of the next lemma can be easily deduced from [12] and hence we omit it.

Lemma 3.2. Suppose that (H1) hold. If $x(n) \in \mathcal{X}$ then $x_j(n)$ is a solution of equation (1.2) if and only if

$$x_j(n) = \sum_{u=n}^{n+T-1} G_j^x(n, u) f_j(n, x_n), \ j = 1, 2, ..., k, \tag{1.3}$$

where

$$G_j^x(n, u) = \frac{\prod_{s=u+1}^{n+T-1} a_j(s, x(s))}{1 - \prod_{s=n}^{n+T-1} a_j(s, x(s))}, \quad u \in [n, n + T - 1], \ j = 1, 2, ..., k. \tag{1.4}$$

Let

$$\sigma = \min_{1\le j\le k} \frac{\left(\prod_{s=0}^{\omega-1} b_j(s)\right)\left[1-\prod_{s=0}^{\omega-1} c_j(s)\right]}{\left(\prod_{s=0}^{\omega-1} c_j(s)\right)\left[1-\prod_{s=0}^{\omega-1} b_j(s)\right]} \tag{1.5}$$

It can easily be obtained from (H2) that $\sigma < 1$. We next define two cones in $\mathcal{X}$ as follows:

$$P_1 = \left\{y \in \mathcal{X} : y_j(n) \ge \sigma |y_j|_0, n \in \mathbb{Z} \quad \text{and } j = 1, ..., k\right\},$$

and

$$P_2 = \left\{y \in \mathcal{X} : y(n) \ge 0,\ n \in \mathbb{Z}\right\}.$$

Define an operator T on $\mathcal{X}$ by $T : \mathcal{X} \to \mathcal{X}$ by

$$(Tx) = (T_1x, T_2x, ..., T_kx)^T. \tag{1.6}$$

where

$$(T_jx)(n) = \sum_{u=n}^{n+\omega-1} G_j^x(n,u) f_j(u, x_u),\ j = 1, ..., k.$$

It is not very difficult to see that $G_j^x(n+\omega, u+\omega) = G_j^x(n,u)$. Also, it can easily be verified that $x^*(n) = (x_1^*(n), ..., x_k^*(n)) \ge 0$ is a positive ω-periodic solution of system (1.1) associated with λ^* if and only if $x^* \in P_2$ is an eigenvector of the operator T associated with the eigenvalue $\frac{1}{\lambda^*} > 0$, that is $Tx^* = \frac{1}{\lambda^*}x^*$.

Lemma 3.2. Suppose that (H1) and (H2) hold. Then the mapping T maps P_1 into P_1, i.e., $TP_1 \subset P_1$.

Proof. In view of (H1) and (H2), we have that, for $j = 1, 2, ..., k$, and $0 \le u \le \omega - 1$,

$$\frac{\prod_{s=0}^{\omega-1} b_j(s)}{1-\prod_{s=0}^{\omega-1} b_j(s)} \le G_j^x(n,u) \le \frac{\prod_{s=0}^{\omega-1} c_j(s)}{1-\prod_{s=0}^{\omega-1} c_j(s)} \tag{1.7}$$

$$
\begin{aligned}
|(T_j x)(n)| &\leq \sum_{u=n}^{n+\omega-1} \frac{\prod_{s=0}^{\omega-1} c_j(s)}{1-\prod_{s=0}^{\omega-1} c_j(s)} |f_j(u, x_u)| \\
&\leq \frac{\prod_{s=0}^{\omega-1} c_j(s)}{1-\prod_{s=0}^{\omega-1} c_j(s)} \sum_{u=0}^{\omega-1} |f_j(u, x_u)|
\end{aligned}
$$

It follows that

$$
|(T_j x)|_0 \leq \frac{\prod_{s=0}^{\omega-1} c_j(s)}{1-\prod_{s=0}^{\omega-1} c_j(s)} \sum_{u=0}^{\omega-1} |f_j(u, x_u)|
$$

or

$$
\sum_{u=0}^{\omega-1} |f_j(u, x_u)| \geq \frac{1-\prod_{s=0}^{\omega-1} c_j(s)}{\prod_{s=0}^{\omega-1} c_j(s)} |(T_j x)|_0.
$$

Therefore,

$$
\begin{aligned}
(T_j x)(n) &\geq \frac{\prod_{s=0}^{\omega-1} b_j(s)}{1-\prod_{s=0}^{\omega-1} b_j(s)} \sum_{u=0}^{\omega-1} |f_j(u, x_u)| \\
&\geq \frac{\left(\prod_{s=0}^{\omega-1} b_j(s)\right)\left[1-\prod_{s=0}^{\omega-1} c_j(s)\right]}{\left(\prod_{s=0}^{\omega-1} c_j(s)\right)\left[1-\prod_{s=0}^{\omega-1} b_j(s)\right]} |(T_j x)|_o \\
&\geq \sigma |(T_j x)|_o,
\end{aligned}
$$

which means that $Tx \in P_1$. This completes the proof.

Lemma 3.3. Suppose (H5) hold. Then the operator $T : P_2 \to \mathcal{X}$ is completely continuous.

Proof. In view of (H5) and the assumption that $f(n, x)$ is continuous in x, we have that the operator T is continuous. We will show that T is compact.
Let $U \subseteq P_2$ be any bounded set. Then, by the (H5), there exists a constant $M > 0$ such that

$$
|f_j(n, x_n)| \leq M, \text{ for } (n, x) \in [0, \omega - 1] \times U, \ j = 1, 2, ..., k.
$$

Thus we have,

$$
|(T_j x)| \leq \frac{\prod_{s=0}^{\omega-1} c_j(s)}{1-\prod_{s=0}^{\omega-1} c_j(s)} M\omega.
$$

It follows that,

$$\begin{aligned} \|(Tx)\| &= \sum_{j=1}^{k} |T_j x|_0 \\ &\leq M\omega \sum_{j=1}^{k} \frac{\prod_{s=0}^{\omega-1} c_j(s)}{1-\prod_{s=0}^{\omega-1} c_j(s)} \\ &\leq Mk\omega\gamma, \end{aligned}$$

where

$$\gamma = \max{}_{1\leq j\leq k} \frac{\prod_{s=0}^{\omega-1} c_j(s)}{1-\prod_{s=0}^{\omega-1} c_j(s)}.$$

Next, we show that T maps bounded subsets into compact sets. Let $J > 0$ be given, and define $\rho = \{\varphi \in P_2 :\| \varphi \|\leq J\}$ and $Q = \{(T\varphi)(n) : \varphi \in \rho\}$, then ρ is a subset of $\mathbb{R}^{\omega k}$ which is closed and bounded thus compact. As T is continuous in φ it maps compact sets into compact sets. Therefore $Q = T(\rho)$ is compact.
This completes the proof of lemma 3.3.

2 Main Results

In this section we state and prove our main results. For our main results we let

$$f_0 = \lim_{\phi\in P_1,\, \|\phi\|\to 0} \frac{\sum_{u=0}^{\omega-1} |f(u, x_u)|}{\|\phi\|}, \text{ and } f_\infty = \lim_{\phi\in P_1,\, \|\phi\|\to \infty} \frac{\sum_{u=0}^{\omega-1} |f(u, x_u)|}{\|\phi\|}.$$

Also, define, for r a positive number, Ω_r, by

$$\Omega_r = \{x \in \mathcal{X} \ : \ \|x\| < r\}.$$

Theorem 4.1 Suppose that (H1)-(H5) hold and $0 < f_\infty < \infty$. Then there exist positive constants R_0, λ_1, and λ_2 with $\lambda_1 < \lambda_2$ such that, for any $r > R_0$, system (1.1) has a positive ω-periodic solution $x^r(n)$ associated with some $\lambda_r \in [\lambda_1, \lambda_2]$ and $\|x^r\| = r$.

Proof. Since $0 < f_\infty < +\infty$, there exist $\epsilon_2 > \epsilon_1 > 0$ and $R_0 > 0$ such that

$$\epsilon_1\|\phi\| < \sum_{u=0}^{\omega-1} |f(u, \phi_u)| < \epsilon_2\|\phi\| \text{ for } \|\phi\| \geq R_0,\ \phi \in P_1.$$

Suppose $r > R_0$, then Ω_r is a bounded open subset of $\mathcal{X}$ and $0 \in \Omega_r$. For $x \in P_1 \cap \partial\Omega_r$, we have

$$\begin{aligned}
\|Tx\| &= \sum_{j=1}^{k} \max_{n\in[0,\omega-1]} |(T_j x)(n)| \\
&\geq \sum_{j=1}^{k} |(T_j x)(n)| \\
&= \sum_{j=1}^{k} \sum_{u=0}^{\omega-1} G_j^x(n,u) f_j(u, x_u) \\
&\geq \sum_{j=1}^{k} \frac{\prod_{s=0}^{\omega-1} b_j(s)}{1-\prod_{s=0}^{\omega-1} b_j(s)} \sum_{u=0}^{\omega-1} f_j(u, x_u) \\
&\geq \min_{1\leq j\leq k} \frac{\prod_{s=0}^{\omega-1} b_j(s)}{1-\prod_{s=0}^{\omega-1} b_j(s)} \sum_{u=0}^{\omega-1} \sum_{j=1}^{k} |f_j(u, x_u)| \\
&\geq \min_{1\leq j\leq k} \frac{\prod_{s=0}^{\omega-1} b_j(s)}{1-\prod_{s=0}^{\omega-1} b_j(s)} \epsilon_1 r > 0.
\end{aligned}$$

It follows that

$$\inf_{x\in P_1\cap\partial\Omega_r} \|Tx\| \geq \min_{1\leq j\leq k} \left\{ \frac{\prod_{s=0}^{\omega-1} b_j(s)}{1-\prod_{s=0}^{\omega-1} b_j(s)} \right\} \epsilon_1 r > 0.$$

Since, T is completely continuous with $T(0) = 0$, it follows from Lemma 3.1 that the operator T has an eigenvector $x^r \in P_1$ associated with the eigenvalue $\mu_r > 0$ such that $\|x^r\| = r$. Set $\lambda_r = \frac{1}{\mu_r}$. Then x^r is a positive ω-periodic solution of system (1.1).

We next determine λ_1 and λ_2 as follows. From

$$\begin{aligned}
(x^r)_j(n) &= \lambda_r \sum_{u=n}^{n+\omega-1} G_j^{x^r}(n,u) f_j(u, x_u^r) \\
&\leq \lambda_r \sum_{u=0}^{\omega-1} \frac{\prod_{s=0}^{\omega-1} c_j(s)}{1-\prod_{s=0}^{\omega-1} c_j(s)} |f_j(u, x_u^r)| \\
&\leq \lambda_r \frac{\prod_{s=0}^{\omega-1} c_j(s)}{1-\prod_{s=0}^{\omega-1} c_j(s)} \sum_{u=0}^{\omega-1} |f_j(u, x_u^r)| \\
&\leq \lambda_r \frac{\prod_{s=0}^{\omega-1} c_j(s)}{1-\prod_{s=0}^{\omega-1} c_j(s)} \epsilon_2 r, \ j = 1, 2, \ldots, k,
\end{aligned}$$

and $\|x^r\| = r$ we can get

$$\lambda_r \geq \frac{1}{\epsilon_2 \sum_{j=1}^{k} \frac{\prod_{s=0}^{\omega-1} c_j(s)}{1-\prod_{s=0}^{\omega-1} c_j(s)}} =: \lambda_1$$

On the other hand,

$$(x^r)_j(n) \quad \geq \quad \lambda_r \frac{\prod_{s=0}^{\omega-1} b_j(s)}{1-\prod_{s=0}^{\omega-1} b_j(s)} \sum_{u=0}^{\omega-1} |f_j(u, x_u^r)|, \; j = 1, ..., k.$$

It follows from

$$\begin{aligned} \|x^r\| = r \quad &\geq \quad \lambda_r \min_{1\leq j\leq k} \left\{ \frac{\prod_{s=0}^{\omega-1} b_j(s)}{1-\prod_{s=0}^{\omega-1} b_j(s)} \right\} \sum_{u=0}^{\omega-1} |f(u, x_u^r)| \\ &\geq \quad \lambda_r \min_{1\leq j\leq k} \left\{ \frac{\prod_{s=0}^{\omega-1} b_j(s)}{1-\prod_{s=0}^{\omega-1} b_j(s)} \right\} \epsilon_1 r \end{aligned}$$

that

$$\lambda_r \quad \leq \quad \lambda_r \max_{1\leq j\leq k} \left\{ \frac{1-\prod_{s=0}^{\omega-1} b_j(s)}{\epsilon_1 \prod_{s=0}^{\omega-1} b_j(s)} \right\} := \lambda_2.$$

Therefore, $\lambda_r \in [\lambda_1, \lambda_2]$ and this completes the proof.

Theorem 4.2. Suppose that (H1)-(H5) hold and $0 < f_0 < \infty$. Then there exist positive constants $r_0 > 0$, $\tilde{\lambda_1}$ and $\tilde{\lambda_2}$ with $\tilde{\lambda_1} < \tilde{\lambda_2}$ such that, for any $0 < r < r_0$, system (1.1) has a positive ω-periodic solution $\tilde{x^r}(n)$ associated with some $\tilde{\lambda_r} \in [\tilde{\lambda_1}, \tilde{\lambda_2}]$ and $\|\tilde{x^r}\| = r$.

Proof. Since $0 < f_0 < \infty$, there exist $0 < l_1 < l_2$ and $r_0 > 0$ such that

$$l_1\|\phi\| < \sum_{u=0}^{\omega-1} |f(u, \phi_u)| < l_2\|\phi\| \text{ for } 0 < \|\phi\| < r_0, \; \phi \in P_1.$$

For $r \in (0, r_0)$, Ω_r is a bounded subset of $\mathcal{X}$ and $0 \in \Omega_r$. Moreover, for $x \in P_1 \cap \partial\Omega_r$,

$$\begin{aligned} \|Tx\| \quad &\geq \quad \sum_{j=1}^{k} |(T_j x)(n)| \\ &= \quad \sum_{j=1}^{k} \sum_{u=n}^{n+\omega-1} G_j^x(n, u) f_j(u, x_u) \\ &\geq \quad \min_{1\leq j\leq k} \left\{ \frac{\prod_{s=0}^{\omega-1} b_j(s)}{1-\prod_{s=0}^{\omega-1} b_j(s)} \right\} l_1 r > 0. \end{aligned}$$

This implies that $\inf_{x\in P_1\cap\partial\omega_r} \|Tx\| > 0$. The remaining part of the proof is similar to that of Theorem 4.1 and so we omit it. This completes the proof.

Using arguments similar to that of Theorem 4.1 and Theorem 4.2, the following results can be established respectively.

Theorem 4.3. Suppose that (H1)-(H5) hold and $f_\infty = \infty$. Then there exist positive constants $\breve{R}_0$ and $\breve{\lambda}$ such that, for any $r > \breve{R}_0$, system (1.1) has a positive ω-periodic solution $\breve{x}^r(n)$ associated with some $\breve{\lambda}_r \le \breve{\lambda}$ and $\|\breve{x}^r\| = r$.

Theorem 4.4. Suppose that (H1)-(H5) hold and $f_0 = \infty$. Then there exist positive constants $\bar{r}_0$ and $\bar{\lambda}$ such that, for any $0 < r < \bar{r_0}$, system (1.1) has a positive ω-periodic solution $\bar{x}^r(n)$ associated with some $\bar{\lambda}_r \le \bar{\lambda}$ and $\|\bar{x}^r\| = r$.

3 An application

In this section, we apply our results from the previous section to the Volterra discrete system

$$\begin{aligned} x_j(n+1) &= x_j(n)\Big[a_j(n) - \lambda \sum_{i=1}^{k}\Big(b_{ji}(n)x_i(n) + \sum_{s=-\infty}^{n} C_{ji}(n,s)g_{ji}(x_i(s))\Big)\Big], \\ & \quad j = 1, 2, ..., k, \end{aligned} \tag{3.1}$$

where $x_j(n)$ is the population of the jth species, $a_j, b_{ji} : \mathbb{Z} \to \mathbb{R}_+$ are ω-periodic and $C_{ji}(n,s) \ge 0$ and $C_{ji}(n+\omega, s+\omega) = C_{ji}(n,s)$ for all $(n,s) \in \mathbb{Z}^2$; $g_{ji} : \mathbb{R}_+ \to \mathbb{R}_+,\ i, j = 1, ..., k.$

Theorem 5.1. Suppose that $\max_{n\in\mathbb{Z}} \sum_{s=-\infty}^{n} |C_{ji}(n,s)| < +\infty$. Then there exist positive constants R_0 and λ_0 such that, for any $r > R_0$, system (3.1) has a positive ω-periodic solution $x^r(n)$ associated with $\lambda_r \le \lambda_0$ and $\|x^r\| = r$.

Proof. Note that $A(n, x(n)) = \text{diag}[a_1(n), a_2(n), ..., a_k(n)]$ and $f = (f_1, f_2, ..., f_k)$ where

$$f_j(n, x_n) = -x_j(n) \sum_{i=1}^{k}\Big(b_{ji}(n)x_i(n) + \sum_{s=-\infty}^{n} C_{ji}(n,s)g_{ji}(x_i(s))\Big)$$

for $j = 1, 2, ..., k$ and (H1)-(H5) are satisfied.

For $x \in P_1$ and $j = 1, ..., k$ we have

$$\begin{aligned}
\sum_{u=0}^{\omega-1} |f_j(u, x_u)| &= \sum_{i=1}^{k} \sum_{u=o}^{\omega-1} x_j(u)\Big(x_i(u)b_{ji}(u) + \sum_{s=-\infty}^{u} C_{ji}(u,s)g_{ji}(x_i(s))\Big) \\
&\geq \sum_{i=1}^{k} \sum_{u=o}^{\omega-1} x_j(u)x_i(u)b_{ji}(u) \\
&\geq \sum_{u=o}^{\omega-1} x_j^2(u)b_{jj}(u) \\
&\geq \sigma^2 |x_j|_0^2 \sum_{u=o}^{\omega-1} b_{jj}(u).
\end{aligned}$$

Thus,

$$\begin{aligned}
\sum_{u=0}^{\omega-1} |f(u, x_u)| &= \sum_{j=1}^{k} \sum_{u=0}^{\omega-1} |f_j(u, x_u)| \\
&\geq \sum_{j=1}^{k} \sigma^2 |x_j|_0^2 \sum_{u=o}^{\omega-1} b_{jj}(u) \\
&\geq \sigma^2 \min_{1\leq j\leq k} \sum_{u=o}^{\omega-1} b_{jj}(u) \sum_{j=1}^{k} |x_j|_0^2 \\
&\geq \frac{\sigma^2}{k} \|x\|^2 \min_{1\leq j\leq k} \sum_{u=o}^{\omega-1} b_{jj}(u).
\end{aligned}$$

It follows that

$$\frac{\sum_{u=0}^{\omega-1} |f(u, x_u)|}{\|x\|} \to \text{ as } \|x\| \to \infty.$$

The conclusion follows directly from Theorem 4.3 and this completes the proof.

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