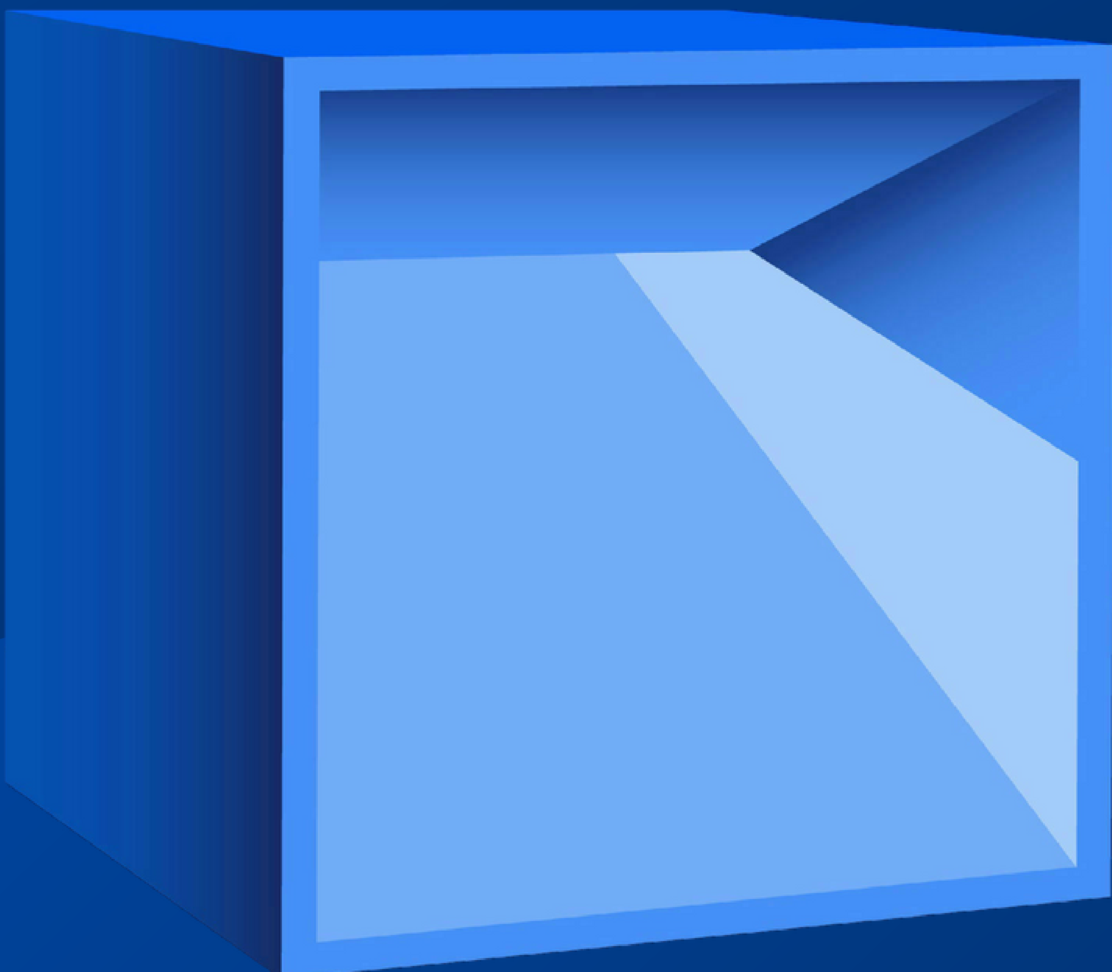
ISSN 0719-0646
ONLINE VERSION

UNIVERSIDAD
DE LA FRONTERA

VOLUME 23 · ISSUE 1
2021

Cubo
A Mathematical Journal



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CUBO
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CUBO, A Mathematical Journal, is a scientific journal founded in 1985, and published by the Department of Mathematics and Statistics of the Universidad de La Frontera, Temuco, Chile. CUBO appears in three issues per year and is indexed in DOAJ, Latindex, MathSciNet, MIAR, REDIB, SciELO-Chile, Scopus and zbMATH. The journal publishes original results of research papers, preferably not more than 20 pages, which contain substantial results in all areas of pure and applied mathematics.

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Universidad de La Frontera

Volume 23/Nº1 – APRIL 2021

SUMMARY

Cubo A Mathematical Journal

CUBO, A Mathematical Journal
Vol. 23, no. 01, pp. 01–20, April 2021
DOI: 10.4067/S0719-06462021000100001

Tan-G class of trigonometric distributions and its applications

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ABSTRACT

In this paper, we introduce a new general class of trigonometric distributions based on the tangent function, called the Tan-G class. A mathematical procedure of the Tan-G class is carried out, including expansions for the probability density function, moments, central moments and Rényi entropy. The estimates are acquired in a non-closed form by the maximum likelihood estimation method. Then, an emphasis is put on a particular member of this class defined with the Burr XII distribution as baseline, called the Tan-BXII distribution. The inferential properties of the Tan-BXII model are investigated. Finally, the Tan-BXII model is applied to a practical data set, illustrating the interest of the Tan-G class for the practitioner.

RESUMEN

En este artículo, introducimos una nueva clase general de distribuciones trigonométricas basada en la función tangente, llamada la clase Tan-G. Se lleva a cabo un procedimiento matemático para la clase Tan-G, incluyendo expansiones para la función de densidad de probabilidad, momentos, momentos centrales y entropía de Rényi. Las estimaciones se obtienen en forma no-cerrada para el método de estimación de máxima verosimilitud. Luego, se pone énfasis en un miembro particular de esta clase definido con la distribución Burr XII como línea de base, llamada la distribución Tan-BXII. Se investigan las propiedades inferenciales del modelo Tan-BXII. Finalmente, el modelo Tan-BXII es aplicado para un conjunto de datos prácticos, ilustrando el interés de la clase Tan-G para el practicante.

Keywords and Phrases: Trigonometric class of distributions, Tangent function, Burr XII distribution, Maximum likelihood estimation, Entropy.

2020 AMS Mathematics Subject Classification: 60E05, 62E15, 62F10.

Accepted: 23 December, 2020
Received: 12 February, 2020

1 Introduction

The recent years of research on probabilistic distributions have been marked by the rise of general classes of trigonometric distributions, more or less sophisticated. Modern statistical developments can be found in, e.g., [10], [16], [18], [19], [11], [4] and [8]. In particular, among the most fundamental of them, [18] introduced the Sin-G class defined by the cumulative distribution function (cdf) given by

$$H_G^{(1)}(x) = \sin\left(\frac{\pi}{2}G(x)\right), \quad x \in \mathbb{R},$$

where $G(x)$ denotes a baseline cdf of a continuous distribution and [19] proposed the Cos-G class defined by the cdf given by

$$H_G^{(2)}(x) = 1 - \cos\left(\frac{\pi}{2}G(x)\right), \quad x \in \mathbb{R}.$$

One can notice that the eventual parameter(s) of these classes is (are) (the one) (those) of $G(x)$ only, and that the following elementary equation hold: $[H_G^{(1)}(x)]^2 + [1 - H_G^{(2)}(x)]^2 = 1$, i.e., $H_G^{(2)}(x) = 1 - \sqrt{1 - [H_G^{(1)}(x)]^2}$ (showing that $H_G^{(2)}(x)$ belongs to the so-called Kum-G class with the parameters $1/2$ and 2 and the baseline cdf $H_G^{(1)}(x)$, see [5]). In addition to their simplicity, both of these two trigonometric classes benefit from the smooth periodic oscillations of the involved trigonometric functions to attain new levels of flexibility in statistical modeling. In [18] and [19], this fact is illustrated by means of several practical data sets, with winning results in comparison to useful model competitors. In this study, following the spirit of [18] and [19], we introduce a new and simple general class of trigonometric distributions having the feature to be centered around the tangent function. For the purpose of this paper, we call it the Tan-G class. It is defined by the following cdf:

$$H_G(x) = \tan\left(\frac{\pi}{4}G(x)\right), \quad x \in \mathbb{R}. \tag{1.1}$$

Several existing constructions give this cdf, beginning by the integral techniques developed by [2]; we have $H_G(x) = \int_0^{(\pi/4)G(x)} \sec^2(t)dt$, where $\sec(t) = 1/\cos(t)$. After some algebra, one can also notice that $H_G(x)$ can be expressed in terms of the cdfs $H_G^{(1)}(x)$ and $H_G^{(2)}(x)$ as

$$H_G(x) = \frac{\sqrt{1 - [1 - H_G^{(2)}(x)]^2}}{2 - H_G^{(2)}(x)}, \quad H_G(x) = \frac{H_G^{(1)}(x)}{1 + \sqrt{1 - [H_G^{(1)}(x)]^2}}.$$

From these expressions, we immediately get the following stochastic ordering: $H_G(x) \leq H_G^{(1)}(x)$, attesting that $H_G(x)$ can provide different statistical models to those of $H_G^{(1)}(x)$. In full generality, the main qualities of the Tan-G class are to be simple: there is no additional parameter and the related functions are very tractable, and its ability to create flexible statistical models, well-adapted to fit with precision several kinds of data sets, beyond those related to the Sin-G or Cos-G class.

All these aspects are developed in this paper according to the following plan. In Section 2, the main theoretical features of the Tan-G class are presented. Section 3 is devoted to a special member of the class defined with the Burr XII distribution as baseline. Concluding remarks are given in Section 4.

2 Main theoretical features of the Tan-G class

A theoretical treatment of the Tan-G class is performed in this section, investigating the related distributional functions, asymptotic and critical points, useful expansion, moments and central moments, expansion for the general coefficient, entropy and the mathematics of the maximum likelihood estimation.

2.1 Distributional functions

We recall that the Tan-G class of distributions is defined by the cdf given by (1.1). Upon differentiation, the corresponding pdf is given by

$$h_G(x) = \frac{\pi}{4} g(x) \sec^2\left(\frac{\pi}{4} G(x)\right), \quad x \in \mathbb{R}, \tag{2.1}$$

where $g(x)$ denotes the pdf corresponding to $G(x)$. The hazard function (hf) of the Tan-G class is given by

$$R_G(x) = \frac{\frac{\pi}{4} g(x) \sec^2\left(\frac{\pi}{4} G(x)\right)}{1 - \tan\left(\frac{\pi}{4} G(x)\right)}, \quad x \in \mathbb{R}. \tag{2.2}$$

The curvatures properties of $h_G(x)$ and $R_G(x)$ are crucial to define an appropriate statistical model for a given data set. Further elements on these curvature properties will be presented in the subsection below. Another important function is the quantile function (qf) given by

$$Q(u) = H_G^{-1}(u) = G^{-1}\left[\frac{4}{\pi} \arctan(u)\right], \quad u \in (0, 1).$$

That is, the median of the Tan-G class is given by

$$M = Q(0.5) \approx G^{-1}(0.5903345).$$

Other properties of the Tan-G class can be studied through this qf. For instance, the main steps to generate random numbers from the Tan-G class via the qf are described in Table 1.

Table 1: Generated numbers from the Tan-G class by the use of the qf

Algorithm
1. Generate n values from $u \sim U(0,1)$
2. Specify $G^{-1}(x)$
3. Obtain an outcome of X with cdf (1.1) by $X = Q(u)$

2.2 Asymptotic and critical points

Let us now investigate the asymptotic and critical points for $h_G(x)$ and $R_G(x)$. Owing to (2.1) and (2.2), when $G(x) \to 0$, we have

$$H_G(x) \sim \frac{\pi}{4} G(x), \quad h_G(x) \sim \frac{\pi}{4} g(x), \quad R_G(x) \sim \frac{\pi}{4} g(x).$$

Also, when $G(x) \to 1$, we have

$$H_G(x) \sim 1 - \frac{\pi}{2}(1 - G(x)), \quad h_G(x) \sim \frac{\pi}{2} g(x), \quad R_G(x) \sim \frac{g(x)}{1 - G(x)}.$$

If x_* denotes a critical point for $h_G(x)$, then it satisfies the following equation: $\{\ln[h_G(x)]\}' \mid_{x=x_*} = 0$, i.e.,

$$g(x)' \mid_{x=x_*} + \frac{\pi}{2} g(x_*)^2 \tan\left(\frac{\pi}{4} G(x_*)\right) = 0.$$

With similar arguments, if x_{**} denotes a critical point for $R_G(x)$, then it satisfies the following equation: $\{\ln[R_G(x)]\}' \mid_{x=x_{**}} = 0$, i.e.,

$$\left[g(x)' \mid_{x=x_{**}} + \frac{\pi}{2} g(x_{**})^2 \tan\left(\frac{\pi}{4} G(x_{**})\right)\right] \left[1 - \tan\left(\frac{\pi}{4} G(x_{**})\right)\right] + \frac{\pi}{4} g(x_{**})^2 \sec^2\left(\frac{\pi}{4} G(x_{**})\right) = 0.$$

None of these non-linear equations has solution(s) with closed form. That is, for a specific $G(x)$, we can determine x_* and x_{**} numerically by the use of any scientific software as R, Matlab, Mathematica...

2.3 Useful expansion

The following result presents an useful expansion of the pdf of the Tan-G class involving functions of the exponentiated-G class (see [7]).

Theorem 2.1. *The pdf of the Tan-G class given by* (2.1) *can be expressed as a linear combination of pdfs of the exponentiated-G class as*

$$h_G(x) = \sum_{k=1}^{+\infty} \omega_k g_{(2k-1)}(x),$$

where

$$\omega_k = \left(\frac{\pi}{4}\right)^{2k-1} \frac{B_{2k}(-4)^k(1-4^k)}{(2k)!}, \tag{2.3}$$

B_{2k} is the so-called $2k$th Bernoulli number and $g_{(2k-1)}(x) = (2k-1)g(x)G^{2k-2}(x)$ is the pdf of the exponentiated-G class with parameter $2k-1$.

Proof. Using the Taylor series for the tangent function, since $(\pi/4)G(x) \in (0, \pi/2)$, we have

$$\tan\left(\frac{\pi}{4}G(x)\right) = \sum_{k=1}^{+\infty} \frac{B_{2k}(-4)^k(1-4^k)}{(2k)!} \left(\frac{\pi}{4}G(x)\right)^{2k-1}.$$

Thus, we obtain the following expansion for $H_G(x)$:

$$H_G(x) = \sum_{k=1}^{+\infty} \left(\frac{\pi}{4}\right)^{2k-1} \frac{B_{2k}(-4)^k(1-4^k)}{(2k)!} G^{2k-1}(x)\cdot$$

The desired expansion for $h_G(x)$ is deduced by differentiation. This ends the proof of Theorem 2.1. □

2.4 Moments and central moments

An expansion for the moment of order m of the Tan-G class is studied in the following result.

Theorem 2.2. *Let μ_m be the moment of order m of the Tan-G class and $\mu_m^{(2k-1)}$ be the moment of order m of the exponentiated-G class with parameter $2k-1$. Then, we have*

$$\mu_m = \sum_{k=1}^{+\infty} \omega_k \mu_m^{(2k-1)},$$

where ω_k is given by (2.3).

Proof. The moment of order m of the Tan-G class is defined by

$$\mu_m = \int_{-\infty}^{+\infty} x^m dH_G(x).$$

It follows from Theorem 2.1 that

$$\mu_m = \int_{-\infty}^{+\infty} x^m \sum_{k=1}^{+\infty} \omega_k g_{(2k-1)}(x)dx = \sum_{k=1}^{+\infty} \omega_k \int_{-\infty}^{+\infty} x^m g_{(2k-1)}(x)dx = \sum_{k=1}^{+\infty} \omega_k \mu_m^{(2k-1)}.$$

This ends the proof of Theorem 2.2. □

The mean is given by $\mu = \mu_1$.

Remark 2.3. *By applying the change of variable $u = G(x)$, we can express $\mu_m^{(2k-1)}$ as*

$$\mu_m^{(2k-1)} = (2k-1)\int_{-\infty}^{+\infty} x^m g(x)G^{2k-2}(x)dx = (2k-1)\int_0^1 \left[G^{-1}(u)\right]^m u^{2k-2}du.$$

Similarly, we can obtain an expansion of the central moments of order m by using Theorem 2.2.

Corollary 2.4. *Let μ'_m be the central moment of order m of the Tan-G class and $\mu_m^{(2k-1)}$ be the moment of order m of the exponentiated-G class with parameter $2k-1$. Then, we have*

$$\mu'_m = \sum_{k=1}^{+\infty}\sum_{r=0}^{m} \gamma_{k,m,r}\mu_{m-r}^{(2k-1)},$$

where

$$\gamma_{k,m,r} = \omega_k \binom{m}{r}(-1)^r\mu^r$$

and ω_k is defined by (2.3).

Proof. The central moment of order m of the Tan-G class is defined by

$$\mu'_m = \int_{-\infty}^{+\infty}(x-\mu)^m dH_G(x).$$

By using the binomial theorem and Theorem 2.2, we have

$$\begin{aligned}\mu'_m &= \sum_{r=0}^{m}\binom{m}{r}(-1)^r\mu^r\int_{-\infty}^{+\infty}x^{m-r}dH_G(x) = \sum_{r=0}^{m}\binom{m}{r}(-1)^r\mu^r\mu_{m-r}\\ &= \sum_{r=0}^{m}\binom{m}{r}(-1)^r\mu^r\sum_{k=1}^{+\infty}\omega_k\mu_{m-r}^{(2k-1)} = \sum_{k=1}^{+\infty}\sum_{r=0}^{m}\gamma_{k,m,r}\mu_{m-r}^{(2k-1)}.\end{aligned}$$

The proof of Corollary 2.4 is ended. □

By considering $m = 2$, the variance is given by

$$\sigma^2 = \mu'_2 = \sum_{k=1}^{+\infty}\sum_{r=0}^{2}\gamma_{k,2,r}\mu_{2-r}^{(2k-1)}.$$

By using similar summation techniques, one can set expansions of the incomplete moments, the moment generating function and the characteristic function, among others.

2.5 Expansion to the general coefficient

The general coefficient of the Tan-G class is defined by

$$C_m = \frac{\mu'_m}{\sigma^m}.$$

By applying Corollary 2.4, it can be written as

$$C_m = \frac{\sum_{k=1}^{+\infty}\sum_{r=0}^{m}\gamma_{k,m,r}\mu_{m-r}^{(2k-1)}}{\left[\sum_{k=1}^{+\infty}\sum_{r=0}^{2}\gamma_{k,2,r}\mu_{2-r}^{(2k-1)}\right]^{\frac{m}{2}}}.$$

So, the asymmetry and kurtosis of the Tan-G class can be respectively expressed by

$$C_3 = \frac{\sum_{k=1}^{+\infty}\sum_{r=0}^{3}\gamma_{k,3,r}\mu_{3-r}^{(2k-1)}}{\left[\sum_{k=1}^{+\infty}\sum_{r=0}^{2}\gamma_{k,2,r}\mu_{2-r}^{(2k-1)}\right]^{\frac{3}{2}}}, \quad C_4 = \frac{\sum_{k=1}^{+\infty}\sum_{r=0}^{4}\gamma_{k,4,r}\mu_{4-r}^{(2k-1)}}{\left[\sum_{k=1}^{+\infty}\sum_{r=0}^{2}\gamma_{k,2,r}\mu_{2-r}^{(2k-1)}\right]^{2}}.$$

2.6 Entropy

Entropy measures the uncertainty; the greater the entropy, the higher the disorder and the less likely it will be to observe a phenomenon; the lower the entropy, the lower its disorder and the higher the probability of observing a particular event. Among the most useful entropy, there is the Rényi entropy introduced by [13]. In the context of the Tan-G class, it is defined by

$$\mathfrak{L}_G(\gamma) = \frac{1}{1-\gamma}\ln\left[\int_{-\infty}^{+\infty} h_G^{\gamma}(x)\right]dx,$$

where $\gamma > 0$ with $\gamma \neq 1$ and

$$h_G^{\gamma}(x) = \left(\frac{\pi}{4}\right)^{\gamma} g^{\gamma}(x)\sec^{2\gamma}\left(\frac{\pi}{4}G(x)\right).$$

Let us now consider the function $W(s) = \sec^{2\gamma}[(\pi/4)s]$, $s \in (0,1)$. By applying the Taylor series formula to $W(s)$ at a fixed point $s_0 \in (0,1)$ (say $s_0 = 0.5$), we get

$$\sec^{2\gamma}\left[\frac{\pi}{4}s\right] = \sum_{k=0}^{+\infty} a_k(s-s_0)^k = \sum_{k=0}^{+\infty}\sum_{r=0}^{k}\binom{k}{r}a_k s^r(-1)^{k-r}s_0^{k-r},$$

where $a_k = W^{(k)}(s)\mid_{s=s_0}/k!$. We are now able to derive an expansion of the Rényi entropy of the Tan-G class. After some algebra, we obtain

$$\mathfrak{L}_G(\gamma) = \frac{1}{1-\gamma}\left\{\gamma\ln\left(\frac{\pi}{4}\right) + \ln\left[\sum_{k=0}^{+\infty}\sum_{r=0}^{k} a_k s^r(-1)^{k-r}s_0^{k-r}I_r\right]\right\}, \tag{2.4}$$

where

$$I_r = \int_{-\infty}^{+\infty} G^r(x)g^{\gamma}(x)dx.$$

Even if it has no closed form, the integral I_r can be computed numerically. The Shannon entropy, pioneered by [15], is given by

$$\mathfrak{S}_G = -\int_{-\infty}^{+\infty}\ln[h_G(x)]h_G(x)dx.$$

It can deduced from $\mathfrak{L}_G(\gamma)$ via the relation $\lim_{\gamma\to 1}\mathfrak{L}_G(\gamma) = \mathfrak{S}_G$.

2.7 Maximum likelihood estimation and scores

Here, we consider the estimation of the parameters of the Tan-G class by the method of maximum likelihood. Let $\tilde{x} = (x_1, \ldots, x_n)^\top$ be a random sample observations from the Tan-G class with vector parameter $\tilde{\theta} = (\theta_1, \ldots, \theta_p)$ (thus, p is the number of parameters of the distribution). Then, the log-likelihood (LL) function for the Tan-G class is given by

$$\ell(\tilde{\theta}) = n \ln\left(\frac{\pi}{4}\right) + \sum_{i=1}^{n} \ln\left(g(x_i|\tilde{\theta})\right) + 2\sum_{i=1}^{n} \ln\left[\sec\left(\frac{\pi}{4}G(x_i|\tilde{\theta})\right)\right].$$

The maximum likelihood estimators (MLEs) are obtained by maximizing this function according to $\tilde{\theta}$. In this regards, if $G(x|\tilde{\theta})$ is differentiable according to $\tilde{\theta}$, one can consider the jth score given by

$$U(\theta_j) = \frac{\partial \ell(\tilde{\theta})}{\partial \theta_j} = \sum_{i=1}^{n} \frac{1}{g(x_i|\tilde{\theta})} \frac{\partial g(x_i|\tilde{\theta})}{\partial \theta_j} + \frac{\pi}{2}\sum_{i=1}^{n} \tan\left(\frac{\pi}{4}G(x_i|\tilde{\theta})\right) \frac{\partial G(x_i|\tilde{\theta})}{\partial \theta_j}$$

and consider the following equations: $U(\theta_1) = 0, \ldots, U(\theta_p) = 0$. Thus, the MLEs are defined as the simultaneous solutions of these equations.

3 The Tan-BXII distribution

We now focus on a special distribution of the Tan-G class, called the Tan-BXII distribution.

3.1 Definition

Tan-BXII distribution is defined by the cdf given by (1.1) with the cdf $G(x)$ of the Burr XII distribution, i.e., $G(x) = 1 - \left[1 + \left(\frac{x}{s}\right)^c\right]^{-\kappa}$, $x, s, c, \kappa > 0$. Hence, the cdf of the Tan-BXII distribution is given by

$$H_G(x) = \tan\left\{\frac{\pi}{4}\left(1 - \left[1 + \left(\frac{x}{s}\right)^c\right]^{-\kappa}\right)\right\}, \quad x > 0.$$

The corresponding pdf is given by

$$h_G(x) = \frac{\pi}{4}\left\{x^{c-1}c\kappa s^{-c}\left[1 + \left(\frac{x}{s}\right)^c\right]^{-\kappa-1}\right\}\sec^2\left\{\frac{\pi}{4}\left(1 - \left[1 + \left(\frac{x}{s}\right)^c\right]^{-\kappa}\right)\right\}, \quad x > 0.$$

Finally, the corresponding hf is given by

$$R_G(x) = \frac{\frac{\pi}{4}\left\{x^{c-1}c\kappa s^{-c}\left[1 + \left(\frac{x}{s}\right)^c\right]^{-\kappa-1}\right\}\sec^2\left\{\frac{\pi}{4}\left(1 - \left[1 + \left(\frac{x}{s}\right)^c\right]^{-\kappa}\right)\right\}}{1 - \tan\left\{\frac{\pi}{4}\left(1 - \left[1 + \left(\frac{x}{s}\right)^c\right]^{-\kappa}\right)\right\}}, \quad x > 0.$$

It is expected that the hf is unimodal or decreasing, as it can be seen in Figures 3 and 4, respectively, but an analytic verification of this fact using all three parameters is an unnecessarily complicated computation. One can check for given parameters that it is indeed the case using computing software.

3.2 Shape characteristics of probability density and hazard functions

The asymptotic and critical points for $h_G(x)$ and $R_G(x)$ can be obtained in non-closed form by applying Subsection 2.2. Also, some possible shapes of $h_G(x)$ for some parameter values are displayed in Figure 1. Some plots of $H_G(x)$ are given in Figure 2.

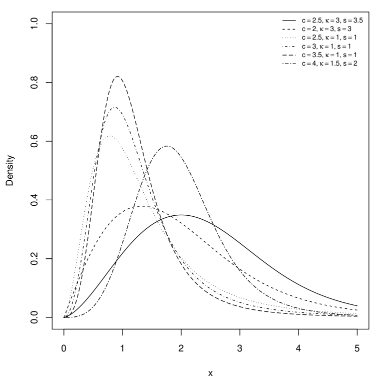


Figure 1: Plots of the pdf of the Tan-BXII distribution

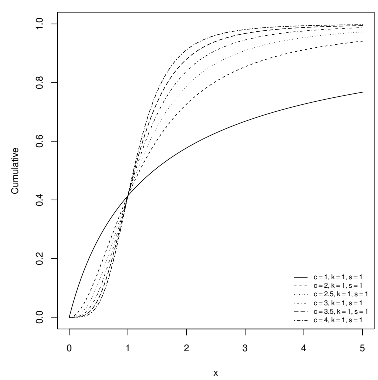


Figure 2: Plots of the cdf of the Tan-BXII distribution

Figures 3 and 4 present plots of $R_G(x)$ for some parameter values. We observe that the hf can be unimodal or only be decreasing.

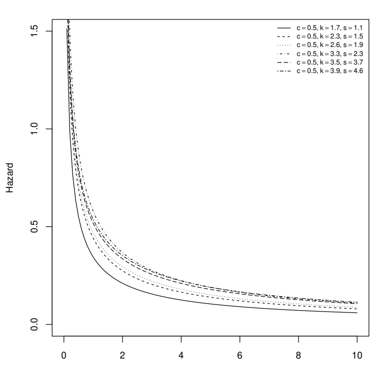


Figure 3: Plots of decreasing hf of the Tan-BXII distribution.

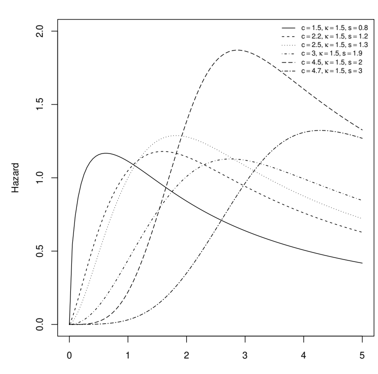


Figure 4: Plots of unimodal hf of the Tan-BXII distribution.

3.3 Expansion of the probability density function

Here, we use the general results proved for the Tan-G class of distributions to reveal properties for the Tan-BXII distribution. An useful expansion of the pdf is presented below.

Theorem 3.1. *The pdf of the Tan-G class can be expanded as a mixture of pdfs of the Burr XII distribution, i.e.,*

$$h_G(x) = \sum_{k=1}^{+\infty} \sum_{j=0}^{2k-2} \omega_{j,k} g_{BurrXII}(x; s, c, \kappa(j+1)),$$

where

$$\omega_{j,k} = \omega_k (2k-1) \binom{2k-2}{j} (-1)^j \frac{1}{j+1}, \tag{3.1}$$

ω_k *is given by* (2.3) *and* $g_{BurrXII}(x; s, c, \kappa(j+1))$ *is the pdf of the Burr XII distribution with parameters* s, c *and* $\kappa(j+1)$, *i.e.,* $g_{BurrXII}(x; s, c, \kappa(j+1)) = x^{c-1} c\kappa(j+1) s^{-c} \left[1 + (x/s)^c\right]^{-\kappa(j+1)-1}$, $x > 0$.

Proof. Owing to Theorem 2.1, we can write

$$h_G(x) = \sum_{k=1}^{+\infty} \omega_k g_{(2k-1)}(x),$$

where ω_k is given by (2.3) and

$$\begin{aligned} g_{(2k-1)}(x) &= (2k-1) g(x) G^{2k-2}(x) \\ &= (2k-1) x^{c-1} c\kappa s^{-c} \left[1 + \left(\frac{x}{s}\right)^c\right]^{-\kappa-1} \left\{1 - \left[1 + \left(\frac{x}{s}\right)^c\right]^{-\kappa}\right\}^{2k-2}. \end{aligned}$$

The standard binomial theorem gives

$$\begin{aligned} g_{(2k-1)}(x) &= (2k-1) x^{c-1} c\kappa s^{-c} \sum_{j=0}^{2k-2} \binom{2k-2}{j} (-1)^j \left[1 + \left(\frac{x}{s}\right)^c\right]^{-\kappa(j+1)-1} \\ &= (2k-1) \sum_{j=0}^{2k-2} \binom{2k-2}{j} (-1)^j \frac{1}{j+1} g_{BurrXII}(x; s, c, \kappa(j+1)). \end{aligned}$$

The proof ends by putting the above equalities together. □

3.4 Moments and central moments

By using identical manipulations to those used in Theorem 2.2, we introduce the moment expansion of the Tan-BXII distribution in the following result.

Theorem 3.2. *First of all, the moment of order m of the Tan-BXII distribution exists if and only if $c\kappa > m$. In this case, the moment of order m of the Tan-BXII distribution is given by*

$$\mu_m = \sum_{k=1}^{+\infty} \sum_{j=0}^{2k-2} \omega_{j,k} s^m \kappa(j+1) B\left(\kappa(j+1) - mc^{-1}, 1 + mc^{-1}\right),$$

where $\omega_{j,k}$ is given by (3.1) and $B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}dt$, $a, b > 0$ (the standard beta function).

Proof. It follows from Theorem 3.1 that

$$\mu_m = \sum_{k=1}^{+\infty} \sum_{j=0}^{2k-2} \omega_{j,k} J_{j,k,m},$$

where

$$J_{j,k,m} = \int_0^{+\infty} x^m g_{BurrXII}(x; s, c, \kappa(j+1)) dx = \int_0^{+\infty} x^m x^{c-1} c\kappa(j+1) s^{-c} \left[1 + \left(\frac{x}{s}\right)^c\right]^{-\kappa(j+1)-1} dx.$$

By applying the changes of variables $u = \left(\frac{x}{s}\right)^c$ and $\nu = (1+u)^{-1}$, in turn, we get

$$\begin{aligned} J_{j,k,m} &= s^m \kappa(j+1) \int_0^{+\infty} u^{\frac{m}{c}} (1+u)^{-\kappa(j+1)-1} du \\ &= s^m \kappa(j+1) \int_0^1 \nu^{\kappa(j+1) - \frac{m}{c} - 1} (1-\nu)^{\frac{m}{c}} d\nu \\ &= s^m \kappa(j+1) B\left(\kappa(j+1) - mc^{-1}, 1 + mc^{-1}\right). \end{aligned}$$

By combining the above equalities together, we end the proof of Theorem 3.2. □

The mean is given by $\mu = \mu_1$.

Remark 3.3. *By adopting the notations introduced in Section 2, following the lines of the proof of Theorem 3.2, one can show that*

$$\mu_m^{(2k-1)} = (2k-1) s^m \kappa \sum_{j=0}^{2k-2} \binom{2k-2}{j} (-1)^j B\left(\kappa(j+1) - mc^{-1}, 1 + mc^{-1}\right).$$

Similarly to Corollary 2.4, the central moment of order m of the Tan-BXII distribution is given

$$\mu'_m = \sum_{r=0}^{m} \binom{m}{r} (-1)^r \mu^r \mu_{m-r} = \sum_{k=1}^{+\infty} \sum_{j=0}^{2k-2} \sum_{r=0}^{m} \rho_{j,k,m,r} B\left(\kappa(j+1) - (m-r)c^{-1}, 1 + (m-r)c^{-1}\right),$$

where

$$\rho_{j,k,m,r} = \omega_{j,k} s^{m-r} \kappa(j+1) \binom{m}{r} (-1)^r \mu^r.$$

By considering $m = 2$, we get the following expansion for variance of the distribution:

$$\sigma^2 = \mu'_2 = \sum_{k=1}^{+\infty} \sum_{j=0}^{2k-2} \sum_{r=0}^{2} \rho_{j,k,2,r} B\left(\kappa(j+1) - (2-r)c^{-1}, 1 + (2-r)c^{-1}\right).$$

3.5 Expansion to the general coefficient

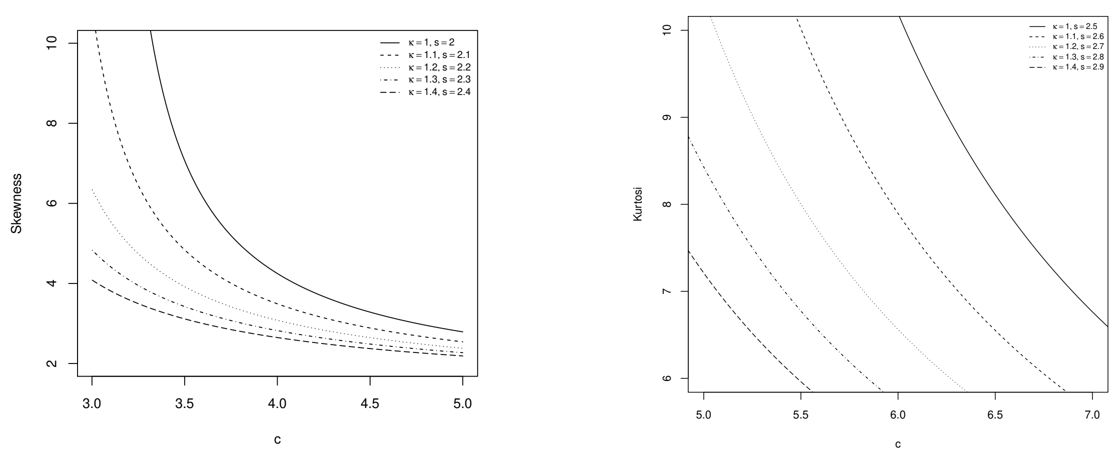
The general coefficient of the Tan-BXII distribution can be expressed as

$$C_m = \frac{\mu'_m}{\sigma^m} = \frac{\sum_{k=1}^{+\infty} \sum_{j=0}^{2k-2} \sum_{r=0}^{m} \rho_{j,k,m,r} B\left(\kappa(j+1) - (m-r)c^{-1}, 1 + (m-r)c^{-1}\right)}{\left\{\sum_{k=1}^{+\infty} \sum_{j=0}^{2k-2} \sum_{r=0}^{2} \rho_{j,k,2,r} B\left(\kappa(j+1) - (2-r)c^{-1}, 1 + (2-r)c^{-1}\right)\right\}^{m/2}}.$$

Thus, the asymmetry and kurtosis can be expressed by taking $m = 3$ and $m = 4$, respectively, which is the object of the next part.

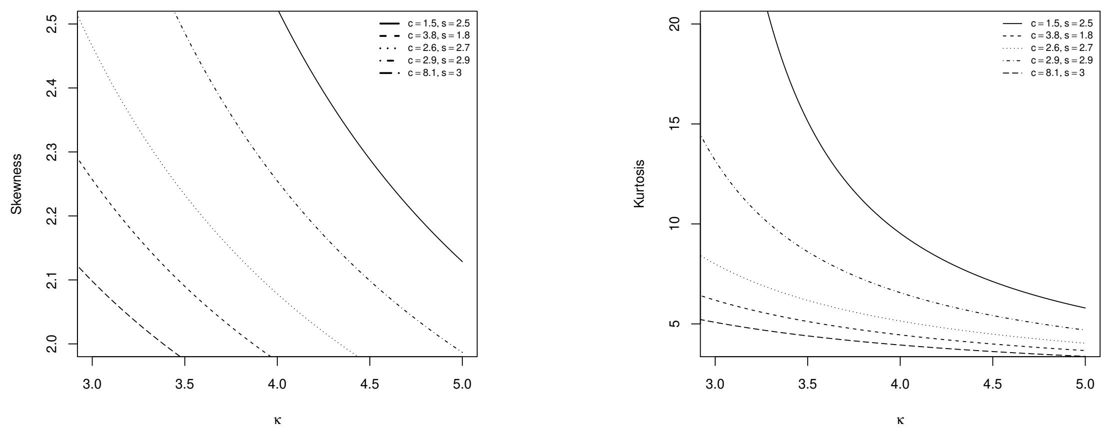
3.6 Figures of asymmetry and kurtosis

In Figures 5, 6 and 7, we present the asymmetry and kurtosis graphs for the Tan-BXII distribution. It is possible to observe that this new distribution has a great flexibility on these aspects, showing varying values, small and large.



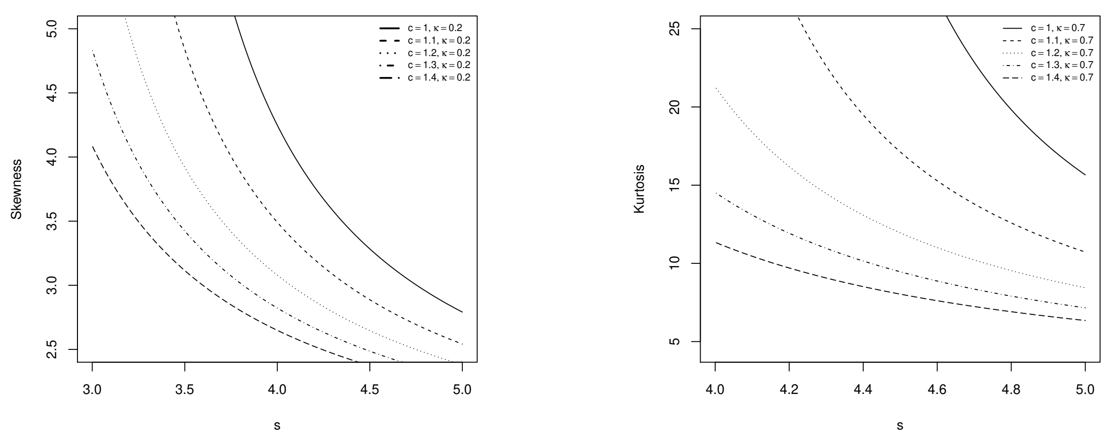
(a) (b)

Figure 5: Plots of the skewness and kurtosis coefficients of the Tan-BXII distribution as a function of c for selected values of κ and s



(a) (b)

Figure 6: Plots of the skewness and kurtosis coefficients of the Tan-BXII distribution as a function of κ for selected values of c and s



(a) (b)

Figure 7: Plots of the skewness and kurtosis coefficients of the Tan-BXII distribution as a function of s for selected values of c and κ

3.7 Entropy

By applying (2.4), the Rényi entropy is given by

$$\mathfrak{L}_G(\gamma) = \frac{1}{1-\gamma}\left\{\gamma \ln\left(\frac{\pi}{4}\right) + \ln\left[\sum_{k=0}^{+\infty}\sum_{r=0}^{k} a_k s^r (-1)^{k-r} s_0^{k-r} I_r\right]\right\},$$

where $\gamma > 0$ with $\gamma \neq 1$ and, after some algebra,

$$\begin{aligned} I_r &= \int_{-\infty}^{+\infty} G^r(x) g^\gamma(x) dx \\ &= \sum_{j=0}^{r}\binom{r}{j}(-1)^j \kappa^\gamma s^{-(\gamma-1)} c^{\gamma-1} B(\kappa(j+\gamma)+(\gamma-1)c^{-1}, (\gamma-1)(c-1)c^{-1}+1), \end{aligned}$$

assuming that $\kappa\gamma + (\gamma-1)c^{-1} > 0$ and $(\gamma-1)(c-1)c^{-1}+1 > 0$.

Figure 8 displays this Rényi entropy for some values of the parameters.

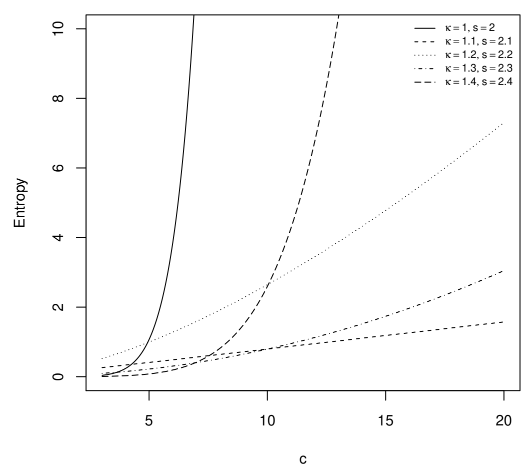


Figure 8: Plots of the Rényi entropy of the Tan-BXII distribution as a function of c for selected values of κ and s

3.8 Maximum likelihood estimation

Here, we provide the mathematical background related to the MLEs of the Tan-BXII model parameters, i.e., c, κ and s. Let $\mathbf{x} = \{x_1, \ldots, x_n\}^\top$ be n independent random variables from the Tan-BXII distribution. Then, the log-likelihood function is given by

$$\begin{aligned} L = {} & n\ln\left(\frac{\pi}{4}\right) + n\ln(c) + n\ln(\kappa) - nc\ln(s) + (c-1)\sum_{i=1}^{n}\ln(x_i) \\ & - (\kappa+1)\sum_{i=1}^{n}\ln\left[1+\left(\frac{x_i}{s}\right)^c\right] + 2\sum_{i=1}^{n}\ln\left[\sec\left\{\frac{\pi}{4}\left(1-\left[1+\left(\frac{x_i}{s}\right)^c\right]^{-\kappa}\right)\right\}\right]. \end{aligned}$$

The scores are presented below:

$$U_c = \frac{n}{c} - n\ln(s) + \sum_{i=1}^{n} \ln(x_i) - (\kappa+1)\sum_{i=1}^{n} \frac{x_i^c \ln\left(\frac{x_i}{s}\right)}{s^c + x_i^c}$$
$$+ \frac{\pi}{2}\kappa \sum_{i=1}^{n} \left(\frac{x_i}{s}\right)^c \ln\left(\frac{x_i}{s}\right)\left[1+\left(\frac{x_i}{s}\right)^c\right]^{-\kappa-1} \tan\left\{\frac{\pi}{4}\left(1-\left[1+\left(\frac{x_i}{s}\right)^c\right]^{-\kappa}\right)\right\},$$

$$U_\kappa = \frac{n}{\kappa} - \sum_{i=1}^{n} \ln\left[1+\left(\frac{x_i}{s}\right)^c\right]$$
$$+ \frac{\pi}{2}\sum_{i=1}^{n}\left[1+\left(\frac{x_i}{s}\right)^c\right]^{-\kappa} \ln\left[1+\left(\frac{x_i}{s}\right)^c\right] \tan\left\{\frac{\pi}{4}\left(1-\left[1+\left(\frac{x_i}{s}\right)^c\right]^{-\kappa}\right)\right\}$$

and

$$U_s = -\frac{nc}{s} + c(\kappa+1)s^{-1}\sum_{i=1}^{n} \frac{x_i^c}{s^c + x_i^c}$$
$$- \frac{\pi}{2}c\kappa s^{-(c+1)} \sum_{i=1}^{n} x_i^c \left[1+\left(\frac{x_i}{s}\right)^c\right]^{-\kappa-1} \tan\left\{\frac{\pi}{4}\left(1-\left[1+\left(\frac{x_i}{s}\right)^c\right]^{-\kappa}\right)\right\}.$$

The MLEs of c, κ and s are defined by the simultaneous solutions of the following non-linear equations: $U_c = 0$, $U_\kappa = 0$ and $U_s = 0$ according to c, κ and s. Under some standard regularity conditions, the well-known theory on MLE can be applied, ensuring nice asymptotic properties (see [3]).

3.9 Simulation

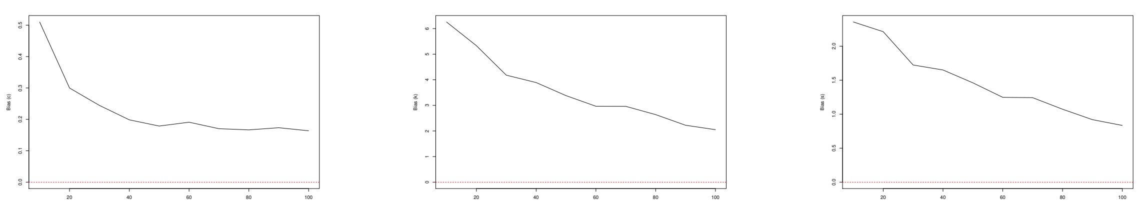
Using the `TanB` R package [17], we perform a simulation study using several random samples of the Tan-BXII distribution. For each sample, we calculate the MLEs using native R language's `optim` implementation. Biases, and Mean Square Errors (MSEs) are also calculated using the MLEs obtained.

For this simulation, we use samples with sizes $10, 20, 30, \ldots, 100$ and 1000 replicas for the parameter's configuration: $c = 1$, $\kappa = 1.4$ and $s = 0.15$. Figures 9a, 9b and 9c show the bias for c, κ and s, respectively, in this simulation and we can see it decreasing over the sample sizes. Figures 10a, 10b and 10c show the MSE for the same parameters and also decreases over the sample sizes.

Table 2 summarizes the simulation, given the means of MLEs, biases and MSEs of the samples with sizes of 10, 20, 30, 50 and 100. We can see in the table that all the parameters are overestimated by the maximum likelihood method. The biases and MSEs decrease over the sample sizes as we see in Figures 9a, 9b, 9c, 10a, 10b and 10c.

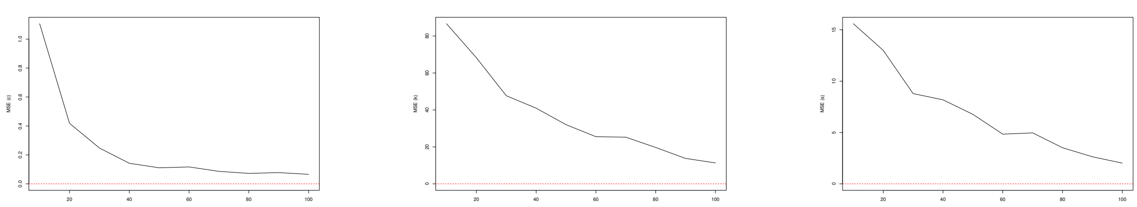
Table 2: MLEs, Biases and MSEs for $c = 1$, $\kappa = 1.4$, $s = 0.15$ using 1000 replicas

Sample size(n)	Parameters	MLEs	Biases	MSEs
	c	1.5102	0.5102	1.1065
10	κ	7.6587	6.2587	86.6797
	s	2.5062	2.3562	15.5951
	c	1.2998	0.2998	0.4181
20	κ	6.7327	5.3327	68.2502
	s	2.3631	2.2131	12.9993
	c	1.2444	0.2444	0.2478
30	κ	5.5806	4.1806	47.7063
	s	1.8732	1.7232	8.7874
	c	1.1787	0.1787	0.111
50	κ	4.7807	3.3807	32.0412
	s	1.6109	1.4609	6.7689
	c	1.1636	0.1636	0.066
100	κ	3.4506	2.0506	11.3414
	s	0.9844	0.8344	2.0205



(a) Plots of bias(c) (b) Plots of bias(κ) (c) Plots of bias(s)

Figure 9: Plots of the biases for the simulated experiment related to the Tan-BurXII model parameters



(a) Plots of MSE(c) (b) Plots of MSE(κ) (c) Plots of MSE(s)

Figure 10: Plots of the MSEs for the simulated experiment related to the Tan-BurXII model parameters

3.10 Application

Now, we apply the Tan-BXII model to fit a practical data set and compare it with three other models, namely Kum-BXII, BurrXII and Kum-W models. These data are on the Aircraft windshield failures (thousands of hours) reported in Murthy [12] (see Table 3). A brief statistical description of these data can be found in Table 4. Table 5 shows the MLEs of the parameters of the Tan-BXII, Kum-BXII, BurrXII and Kum-W models with error in parentheses, as well as the related Akaike Information Criterion (AIC), Corrected Akaike Information Criterion (CAIC), Bayesian Information Criterion (BIC), Cramér-von Mises (W^*) and Anderson-Darling (A^*) statistics. We refer to [1], [6] and the book of [9] for precise definitions and use of these fundamental statistical tools.

Table 3: Data on aircraft windshield failures (thousands of hours)

0.040	1.866	2.385	3.443	0.301	1.876	2.481	3.467	0.309	1.899	2.610
3.478	0.557	1.911	2.625	3.578	0.943	1.912	2.632	3.595	1.070	1.914
2.646	3.699	1.124	1.981	2.661	3.779	1.248	2.010	2.688	3.924	1.281
2.038	2.823	4.035	1.281	2.085	2.890	4.121	1.303	2.089	2.902	4.167
1.432	2.097	2.934	4.240	1.480	2.135	2.962	4.255	1.505	2.154	2.964
4.278	1.506	2.190	3.000	4.305	1.568	2.194	3.103	4.376	1.615	2.223
3.114	4.449	1.619	2.224	3.117	4.485	1.652	2.229	3.166	4.570	1.652
2.300	3.344	4.602	1.757	2.324	3.376	4.663				

Table 4: Descriptive statistics of the considered data

Min.	Q_1	Median	Mean	Q_3	Max.	Var.
0.040	1.839	2.354	2.557	3.393	4.663	1.252

Table 5: MLEs of the parameters of the Tan-BXII, Kum-BXII, Kum-W and BurrXII models, with errors in parentheses, and AIC, BIC, CAIC, W^* and A^* statistics

Models	Estimates					AIC	BIC	CAIC	W^*	A^*
Tan-BXII(c, κ, s)	2.27	186.02	26.00	—	—	267.76	275.09	268.06	0.06	0.58
	(0.20)	(659.52)	(41.42)	—	—					
Kum-BXII(a, b, c, d, k)	0.28	1.96	7.17	4.54	5.82	267.95	280.17	268.71	0.08	0.64
	(0.11)	(1.36)	(2.38)	(5.07)	(1.46)					
Kum-W(a, b, c, β)	0.38	8.53	5.78	0.13	—	268.82	278.59	269.32	0.06	0.56
	(0.04)	(6.89)	(0.06)	(0.04)	—					
BXII(a, c, k)	2.48	11.31	7.47	—	—	270.24	277.57	270.54	0.06	0.63
	(0.23)	(8.05)	(2.57)	—	—					

It follows from Table 5 that, when compared to other ones, the Tan-BXII model is the best. We illustrate this claim by showing the fits of the estimated pdfs and cdfs in Figures 11 and 12, respectively. Thus, we conclude that the Tan-BXII distribution is quite flexible in the modeling of the proposed data.

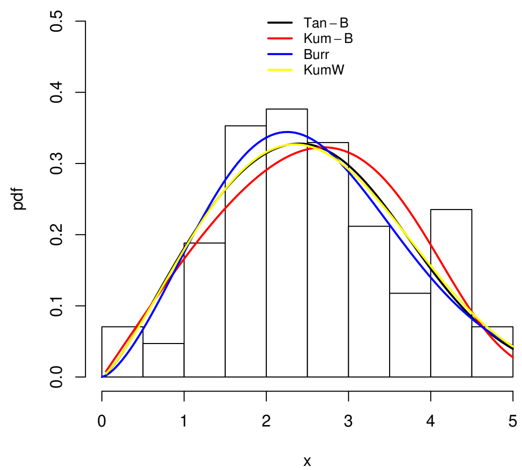


Figure 11: Some fitted pdfs of the data

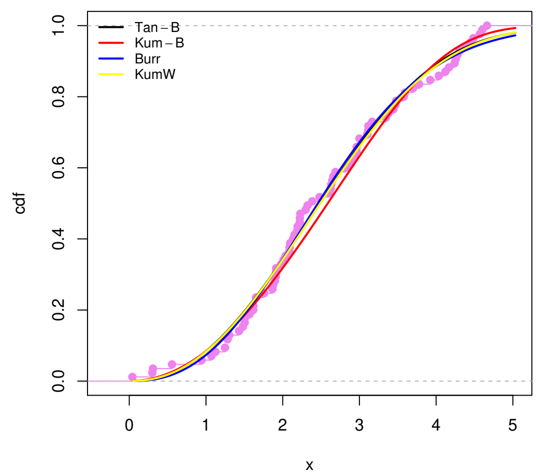


Figure 12: Some fitted cdfs of the data

4 Concluding remarks

In this paper, we introduced and discussed a new class of trigonometric distributions, called the Tan-G class, with a focus on a new lifetime trigonometric distribution of the class, called the Tan-BXII distribution. We obtain probability density function, cumulative distribution function, hazard function and various moments. The entropy is also calculated. A complete part is devoted to the estimation of the model parameters via the maximum likelihood method. We put the light on the applicability of the new related models by considering a practical data set. Even though our class of distributions does not optimally fit the data presented, it still proves to be a powerful tool for statistical analysis. We will apply this distribution to other data sets to show its full power and it will be reported elsewhere.

Acknowledgments

We would like to thank the reviewer and the associated editor for constructive comments on the article, improving it on several important aspects.

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Cubo
A Mathematical Journal

CUBO, A Mathematical Journal
Vol. 23, no. 01, pp. 21–62, April 2021
DOI: 10.4067/S0719-06462021000100021

Anisotropic problem with non-local boundary conditions and measure data

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ABSTRACT

We study a nonlinear anisotropic elliptic problem with non-local boundary conditions and measure data. We prove an existence and uniqueness result of entropy solution.

RESUMEN

Estudiamos un problema elíptico nolineal anisotrópico con condiciones de borde no-locales y data de medida. Probamos un resultado de existencia y unicidad de la solución de entropía.

Keywords and Phrases: Entropy solution, non-local boundary conditions, Leray-Lions operator, bounded Radon diffuse measure, Marcinkiewicz spaces.

2020 AMS Mathematics Subject Classification: 35J05, 35J25, 35J60, 35J66.

Accepted: 06 January, 2021
Received: 13 November, 2019

1 Introduction and assumptions

Let Ω be a bounded domain in $\mathbb{R}^N$ $(N \geq 3)$ such that $\partial\Omega$ is Lipschitz and $\partial\Omega = \Gamma_D \cup \Gamma_{Ne}$ with $\Gamma_D \cap \Gamma_{Ne} = \emptyset$. Our aim is to study the following problem.

$$P(\rho,\mu,d)\begin{cases} -\displaystyle\sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i\left(x, \frac{\partial}{\partial x_i} u\right) + |u|^{p_M(x)-2}u = \mu & \text{in } \Omega \\ u = 0 & \text{on } \Gamma_D \\ \left.\begin{array}{l} \rho(u) + \displaystyle\sum_{i=1}^{N} \int_{\Gamma_{Ne}} a_i\left(x, \frac{\partial}{\partial x_i} u\right)\eta_i = d \\ u \equiv constant \end{array}\right\} & \text{on } \Gamma_{Ne}, \end{cases} \tag{1.1}$$

where the right-hand side μ is a bounded Radon diffuse measure (that is μ does not charge the sets of zero $p_m(.)$-capacity), $\rho : \mathbb{R} \to \mathbb{R}$ a surjective, continuous and non-decreasing function, with $\rho(0) = 0$, $d \in \mathbb{R}$ and $\eta_i,\ i \in \{1, ..., N\}$ are the components of the outer normal unit vector.
For any $\Omega \subset \mathbb{R}^N$, we set

$$C_+(\bar{\Omega}) = \{h \in C(\bar{\Omega}) : \inf_{x\in\Omega} h(x) > 1\} \tag{1.2}$$

and we denote

$$h^+ = \sup_{x\in\Omega} h(x), \qquad h^- = \inf_{x\in\Omega} h(x). \tag{1.3}$$

For the exponents, $\vec{p}(.) : \bar{\Omega} \to \mathbb{R}^N$, $\vec{p}(.) = (p_1(.), ..., p_N(.))$ with $p_i \in C_+(\bar{\Omega})$ for every $i \in \{1, ..., N\}$ and for all $x \in \bar{\Omega}$. We put $p_M(x) = \max\{p_1(x), ..., p_N(x)\}$ and $p_m(x) = \min\{p_1(x), ..., p_N(x)\}$.
We assume that for $i = 1, ..., N$, the function $a_i : \Omega \times \mathbb{R} \to \mathbb{R}$ is Carathéodory and satisfies the following conditions.

- (H_1): $a_i(x, \xi)$ is the continuous derivative with respect to ξ of the mapping $A_i = A_i(x, \xi)$, that is, $a_i(x, \xi) = \dfrac{\partial}{\partial \xi} A_i(x, \xi)$ such that the following equality holds.

$$A_i(x, 0) = 0, \tag{1.4}$$

  for almost every $x \in \Omega$.

- (H_2) : There exists a positive constant C_1 such that

$$|a_i(x, \xi)| \leq C_1(j_i(x) + |\xi|^{p_i(x)-1}), \tag{1.5}$$

  for almost every $x \in \Omega$ and for every $\xi \in \mathbb{R}$, where j_i is a non-negative function in $L^{p_i'(.)}(\Omega)$, with $\dfrac{1}{p_i(x)} + \dfrac{1}{p_i'(x)} = 1$.

- (H_3) : there exists a positive constant C_2 such that

$$(a_i(x,\xi) - a_i(x,\eta)).(\xi - \eta) \geq \begin{cases} C_2|\xi - \eta|^{p_i(x)} & \text{if } |\xi - \eta| \geq 1, \\ C_2|\xi - \eta|^{p_i^-} & \text{if } |\xi - \eta| < 1, \end{cases} \tag{1.6}$$

for almost every $x \in \Omega$ and for every ξ, $\eta \in \mathbb{R}$, with $\xi \neq \eta$.

- (H_4) : For almost every $x \in \Omega$ and for every $\xi \in \mathbb{R}$,

$$|\xi|^{p_i(x)} \leq a_i(x,\xi).\xi \leq p_i(x)A_i(x,\xi). \tag{1.7}$$

- (H_5) : The variable exponents $p_i(.) : \bar{\Omega} \to [2, N)$ are continuous functions for all $i = 1, ..., N$ such that

$$\frac{\bar{p}(N-1)}{N(\bar{p}-1)} < p_i^- < \frac{\bar{p}(N-1)}{N-\bar{p}}, \ \sum_{i=1}^{N} \frac{1}{p_i^-} > 1 \text{ and } \frac{p_i^+ - p_i^- - 1}{p_i^-} < \frac{\bar{p}-N}{\bar{p}(N-1)}, \tag{1.8}$$

where $\dfrac{1}{\bar{p}} = \dfrac{1}{N}\displaystyle\sum_{i=1}^{N}\frac{1}{p_i^-}$.

As examples under assumptions (H_1) -(H_5), we can give the following.

(1) Set $A_i(x,\xi) = (\frac{1}{p_i(x)})|\xi|^{p_i(x)}$ and $a_i(x,\xi) = |\xi|^{p_i(x)-2}\xi$, where $2 \leq p_i(x) < N$.

(2) $A_i(x,\xi) = (\frac{1}{p_i(x)})((1+|\xi|^2)^{\frac{p_i(x)}{2}} - 1)$ and $a_i(x,\xi) = (1+|\xi|^2)^{\frac{p_i(x)-2}{2}}\xi$, where $2 \leq p_i(x) < N$.

We put for all $x \in \partial\Omega$,

$$p^{\partial}(x) = \begin{cases} \dfrac{(N-1)p(x)}{N-p(x)} & \text{if } p(x) < N, \\ \infty & \text{if } p(x) \geq N. \end{cases}$$

We introduce the numbers

$$q = \frac{N(\bar{p}-1)}{N-1}, \ q^* = \frac{Nq}{N-q} = \frac{N(\bar{p}-1)}{N-\bar{p}}. \tag{1.9}$$

We denote by $\mathcal{M}_b(\Omega)$ the space of bounded Radon measure in Ω, equipped with its standard norm $\|.\|_{\mathcal{M}_b(\Omega)}$. Note that, if u belongs to $\mathcal{M}_b(\Omega)$, then $|\mu|(\Omega)$ (the total variation of μ) is a bounded positive measure on Ω.
Given $\mu \in \mathcal{M}_b(\Omega)$, we say that μ is diffuse with respect to the capacity $W_0^{1,p(.)}(\Omega)$ ($p(.)$-capacity for short) if $\mu(A) = 0$, for every set A such that $\mathrm{Cap}_{p(.)}(A,\Omega) = 0$.
For every $A \subset \Omega$, we denote

$$S_{p(.)}(A) = \{u \in W_0^{1,p(.)}(\Omega) \cap C_0(\Omega) : u = 1 \text{ on } A, u \geq 0 \text{ on } \Omega\}.$$

The $p(.)$-capacity of every subset A with respect to Ω is defined by

$$\mathrm{Cap}_{p(.)}(A,\Omega)=\inf_{u\in S_{p(.)}(A)}\{\int_\Omega |\nabla u|^{p(x)}dx\}.$$

In the case $S_{p(.)}(A)=\emptyset$, we set $\mathrm{Cap}_{p(.)}(A,\Omega)=\infty$.
The set of bounded Radon diffuse measure in the variable exponent setting is denoted by $\mathcal{M}_b^{p(.)}(\Omega)$. We use the following result of decomposition of bounded Radon diffuse measure proved by Nyanquini et *al.* (see [31]).

Theorem 1.1. *Let $p(.):\bar{\Omega}\to(1,\infty)$ be a continuous function and $\mu\in\mathcal{M}_b(\Omega)$. Then $\mu\in\mathcal{M}_b^{p(.)}(\Omega)$ if and only if $\mu\in L^1(\Omega)+W^{-1,p'(.)}(\Omega)$.*

Remark 1.2. *Since $\mu\in\mathcal{M}_b^{p_m(.)}(\Omega)$, the Theorem 1.1 implies that there exist $f\in L^1(\Omega)$ and $F\in(L^{p'_m(.)}(\Omega))^N$ such that*

$$\mu=f-divF, \tag{1.10}$$

where $\dfrac{1}{p_m(x)}+\dfrac{1}{p'_m(x)}=1$, $\forall x\in\Omega$.

The study of nonlinear elliptic equations involving the p-Laplace operator is based on the theory of standard Sobolev spaces $W^{m,p}(\Omega)$ in order to find weak solutions. For the nonhomogeneous $p(.)$-Laplace operators, the natural setting for this approach is the use of the variable exponent Lebesgue and Sobolev spaces $L^{p(.)}(\Omega)$ and $W^{m,p(.)}(\Omega)$.
Variable exponent Lebesgue spaces appeared in the literature for the first time in a article by Orlicz in 1931. In the 1950's, this study was carred on by Nakano who made the first systematic study of spaces with variable exponent (called modular spaces). Nakano explicitly mentioned variable exponent Lebesgue spaces as an example of more general spaces he considered (see [30], p. 284). Later, the polish mathematicians investigated the modular function spaces (see [29]). Note also that H. Hudzik [18] investigated the variable exponent Sobolev spaces. Variable exponent Lebesgue spaces on the real line have been independently developed by Russian researchers, notably Sharapudinov [40] and Tsenov [42]. The next major step in the investigation of variable exponent Lebesgue and Sobolev spaces was the comprehensive paper by O. Kovacik and J. Rakosnik in the early 90's [23]. This paper established many of basic properties of Lebesgue and Sobolev spaces with variables exponent. Variable Sobolev spaces have been used in the last decades to model various phenomena. In [9], Chen, Levine and Rao proposed a framework for image restoration based on a Laplacian variable exponent. Another application which uses nonhomogeneous Laplace operators is related to the modelling of electrorheological fluids see [38]. The first major discovery in electrorheological fluids was due to Winslow in 1949 (cf. [43]). These fluids have the interesting property that their viscosity depends on the electric field in the fluid. They can raise the viscosity by as much as five orders of magnitude. This phenomenon is known as the Winslow effect. For some technical applications, we refer the readers to the work by Pfeiffer et *al* [33]. Electrorheological fluids have been used in robotics and space technology. The experimental research has been done mainly in

the USA, for instance in NASA laboratories. For more information on properties, modelling and the application of variable exponent spaces to these fluids, we refer to Diening [11], Rajagopal and Ruzicka [35], and Ruzicka [36]. In this paper, the operator involved in (1.1) is more general than the $p(.)$-Laplace operator. Thus, the variable exponent Sobolev space $W^{1,p(.)}(\Omega)$ is not adequate to study nonlinear problems of this type. This leads us to seek entropy solutions for problems (1.1) in a more general variable exponent Sobolev space which was introduced for the first time by Mihaïlescu et *al.* [28], see also [34, 26, 27].

The need for such theory comes naturally every time we want to consider materials with inhomogeneities that have different behavior on different space directions. Non-local boundary value problems of various kinds for partial differential equations are of great interest by now in several fields of application. In a typical non-local problem, the partial differential equation (resp. boundary conditions) for an unknown function u at any point in a domain Ω involves not only the local behavior of u in a neighborhood of that point but also the non-local behavior of u elsewhere in Ω. For example, at any point in Ω the partial differential equation and/or the boundary conditions may contains integrals of the unknown u over parts of Ω, values of u elsewhere in D or, generally speaking, some non-local operator on u. Beside the mathematical interest of nonlocal conditions, it seems that this type of boundary condition appears in petroleum engineering model for well modeling in a $3D$ stratified petroleum reservoir with arbitrary geometry (see [12] and [15]). A lot of papers (see [34], [24], [25], [2], [19], [1]) on problems like (1.1) considered cases of generally boundary value condition. In [6], Bonzi et al. studied the following problems.

$$\begin{cases} -\displaystyle\sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i\left(x, \frac{\partial}{\partial x_i} u\right) + |u|^{p_M(x)-2}u = f & \text{in } \Omega \\ \displaystyle\sum_{i=1}^{N} a_i\left(x, \frac{\partial}{\partial x_i} u\right)\eta_i = -|u|^{r(x)-2}u & \text{on } \partial\Omega, \end{cases} \tag{1.11}$$

which correspond to the Robin type boundary condition. The authors used minimization techniques used in [8] to prove the existence and uniqueness of entropy solution. By the same techniques, Koné and *al.* proved the existence and uniqueness of entropy solution for the following problem.

$$\begin{cases} -\displaystyle\sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i\left(x, \frac{\partial}{\partial x_i} u\right) + |u|^{p_M(x)-2}u = f & \text{in } \Omega \\ \displaystyle\sum_{i=1}^{N} a_i\left(x, \frac{\partial}{\partial x_i} u\right)\eta_i + \lambda u = g & \text{on } \partial\Omega, \end{cases} \tag{1.12}$$

which correspond to the Fourier type boundary condition.

In a recent paper we studied a nonlinear elliptic anisotropic problem involving non- local conditions. We also considered variable exponent and general maximal monotone graph datum at the boundary and proved existence and uniqueness of weak solution to the following problem.

$$S(\rho,\mu,d)\begin{cases} -\displaystyle\sum_{i=1}^{N}\frac{\partial}{\partial x_i}a_i\left(x,\frac{\partial}{\partial x_i}u\right)+|u|^{p_M(x)-2}u=f & \text{in } \Omega\\ u=0 & \text{on } \Gamma_D\\ \left.\begin{array}{l}\rho(u)+\displaystyle\sum_{i=1}^{N}\int_{\Gamma_{Ne}}a_i\left(x,\frac{\partial}{\partial x_i}u\right)\eta_i\ni d\\ u\equiv constant\end{array}\right\} & \text{on } \Gamma_{Ne},\end{cases}$$

where the right-hand side $f\in L^{\infty}(\Omega)$ and ρ a maximal monotone graph on $\mathbb{R}$ such that $D(\rho)=Im(\rho)=\mathbb{R}$ and $0\in\rho(0)$, $d\in\mathbb{R}$, by using the technique of monotone operators in Banach spaces (see [21]) and approximation methods. There are two difficulties associated with the study of problem $P(\rho,\mu,d)$. The first is to give a sense to the partial derivative of u which appear in the term $a_i\left(x,\frac{\partial}{\partial x_i}u\right)$. As μ is a measure (even if μ is a integrable function), then we cannot take the partial derivative of u in the usual distribution sense. The idea consists in considering troncatures of the solution u (see [5]). The second difficulty appears with the question of uniqueness of solutons. We obtain existence and uniqueness of a special class of solutions of problem $P(\rho,\mu,d)$ that satisfy an extra condition that we call the entropy condition (see formula (2.9)). An alternative notion of solution which can leads to existence and uniqueness of solution to problem $P(\rho,\mu,d)$ is the notion of renormalized solution. But in this work, we consider the notion of entropy solution.

The paper is organized as follows. Section 2 is devoted to mathematical preliminaries including, among other things, a brief discussion on variable exponent Lebesgue, Sobolev, anisotropic and Marcinkiewicz spaces. In Section 3, we study an approximated problem and in Section 4, we prove by using the results of the Section 3, the existence and uniqueness of entropy solution of problem $P(\rho,\mu,d)$.

2 Preliminary

This part is related to anisotropic Lebesgue and Sobolev spaces with variable exponent and some of their properties.

Given a measurable function $p(.):\Omega\to[1,\infty)$. We define the Lebesgue space with variable exponent $L^{p(.)}(\Omega)$ as the set of all measurable functions $u:\Omega\to\mathbb{R}$ for which the convex modular

$$\rho_{p(.)}(u):=\int_{\Omega}|u|^{p(x)}dx$$

is finite.

If the exponent is bounded, i.e, if $p_+<\infty$, then the expression

$$|u|_{p(.)}:=\inf\left\{\lambda>0:\rho_{p(.)}(\frac{u}{\lambda})\le 1\right\}$$

defines a norm in $L^{p(.)}(\Omega)$, called the Luxembourg norm. The space $(L^{p(.)}(\Omega), |.|_{p(.)})$ is a separable Banach space. Then, $L^{p(.)}(\Omega)$ is uniformly convex, hence reflexive and its dual space is isomorphic to $L^{p'(.)}(\Omega)$, where $\dfrac{1}{p(x)} + \dfrac{1}{p'(x)} = 1$, for all $x \in \Omega$. We have the following properties (see [13]) on the modular $\rho_{p(.)}$.

If $u, u_n \in L^{p(.)}(\Omega)$ and $p_+ < \infty$, then

$$|u|_{p(.)} < 1 \Rightarrow |u|_{p(.)}^{p^+} \le \rho_{p(.)}(u) \le |u|_{p(.)}^{p^-}, \tag{2.1}$$

$$|u|_{p(.)} > 1 \Rightarrow |u|_{p(.)}^{p^-} \le \rho_{p(.)}(u) \le |u|_{p(.)}^{p^+}, \tag{2.2}$$

$$|u|_{p(.)} < 1 (= 1; > 1) \Rightarrow \rho_{p(.)}(u) < 1 (= 1; > 1), \tag{2.3}$$

and

$$|u_n|_{p(.)} \to 0 \ (|u_n|_{p(.)} \to \infty) \Leftrightarrow \rho_{p(.)}(u_n) \to 0 \ (\rho_{p(.)}(u_n) \to \infty). \tag{2.4}$$

If in addition, $(u_n)_{n\in\mathbb{N}} \subset L^{p(.)}(\Omega)$, then $\lim_{n\to\infty} |u_n - u|_{p(.)} = 0 \Leftrightarrow \lim_{n\to\infty} \rho_{p(.)}(u_n - u) = 0 \Leftrightarrow (u_n)_{n\in\mathbb{N}}$ converges to u in measure and $\lim_{n\to\infty} \rho_{p(.)}(u_n) = \rho_{p(.)}(u)$.

We introduce the definition of the isotropic Sobolev space with variable exponent,

$$W^{1,p(.)}(\Omega) := \left\{ u \in L^{p(.)}(\Omega) : |\nabla u| \in L^{p(.)}(\Omega) \right\},$$

which is a Banach space equipped with the norm

$$\|u\|_{1,p(.)} := |u|_{p(.)} + |\nabla u|_{p(.)}.$$

Now, we present the anisotropic Sobolev space with variable exponent which is used for the study of $P(\rho, \mu, d)$.

The anisotropic variable exponent Sobolev space $W^{1,\vec{p}(.)}(\Omega)$ is defined as follow.

$$W^{1,\vec{p}(.)}(\Omega) := \left\{ u \in L^{p_M(.)}(\Omega) : \frac{\partial u}{\partial x_i} \in L^{p_i(.)}(\Omega), \text{ for all } i \in \{1, ..., N\} \right\}.$$

Endowed with the norm

$$\|u\|_{\vec{p}(.)} := |u|_{p_M(.)} + \sum_{i=1}^{N} \left| \frac{\partial u}{\partial x_i} \right|_{p_i(.)},$$

the space $\left(W^{1,\vec{p}(.)}(\Omega), \|.\|_{\vec{p}(.)}\right)$ is a reflexive Banach space (see [14], Theorem 2.1 and Theorem 2.2). As consequence, we have the following.

Theorem 2.1. *(see [14]) Let $\Omega \subset \mathbb{R}^{\mathbb{N}}$ $(N \ge 3)$ be a bounded open set and for all $i \in \{1, ..., N\}$, $p_i \in L^\infty(\Omega)$, $p_i(x) \ge 1$ a.e. in Ω. Then, for any $r \in L^\infty(\Omega)$ with $r(x) \ge 1$ a.e. in Ω such that*

$$ess \inf_{x\in\Omega} (p_M(x) - r(x)) > 0,$$

we have the compact embedding

$$W^{1,\vec{p}(.)}(\Omega) \hookrightarrow L^{r(.)}(\Omega).$$

We also need the following trace theorem due to [7].

Theorem 2.2. *Let $\Omega \subset \mathbb{R}^{\mathbb{N}}$ $(N \geq 2)$ be a bounded open set with smooth boundary and let $\vec{p}(.) \in C(\bar{\Omega})$ satisfy the condition*

$$1 \leq r(x) < \min_{x \in \partial\Omega} \{p_1^{\partial}(x), ..., p_N^{\partial}(x)\}, \ \forall x \in \partial\Omega. \tag{2.5}$$

Then, there is a compact boundary trace embedding

$$W^{1,\vec{p}(.)}(\Omega) \hookrightarrow L^{r(.)}(\partial\Omega).$$

Let us introduce the following notation:

$$\vec{p}_- = (p_1^-, ..., p_N^-).$$

We will use in this paper, the Marcinkiewicz spaces $\mathcal{M}^q(\Omega)$ $(1 < q < \infty)$ with constant exponent. Note that the Marcinkiewicz spaces $\mathcal{M}^{q(.)}(\Omega)$ in the variable exponent setting was introduced for the first time by Sanchon and Urbano (see [37]).
Marcinkiewicz spaces $\mathcal{M}^q(\Omega)$ $(1 < q < \infty)$ contain all measurable function $h : \Omega \to \mathbb{R}$ for which the distribution function

$$\lambda_h(\gamma) := \text{meas}(\{x \in \Omega : |h(x)| > \gamma\}), \ \gamma \geq 0,$$

satisfies an estimate of the form $\lambda_h(\gamma) \leq C\gamma^{-q}$, for some finite constant $C > 0$.
The space $\mathcal{M}^q(\Omega)$ is a Banach space under the norm

$$\|h\|^*_{\mathcal{M}^q(\Omega)} = \sup_{t>0} t^{\frac{1}{q}} \left(\frac{1}{t} \int_0^t h^*(s) ds \right),$$

where h^* denotes the nonincreasing rearrangement of h.

$$h^*(t) := \inf \left\{ C : \lambda_h(\gamma) \leq C\gamma^{-q}, \ \forall \gamma > 0 \right\},$$

which is equivalent to the norm $\|h\|^*_{\mathcal{M}^q(\Omega)}$ (see [3]).
We need the following Lemma (see [4], Lemma A-2).

Lemma 2.3. *Let $1 \leq q < p < \infty$. Then, for every measurable function u on Ω,*

(i) $\dfrac{(p-1)^p}{p^{p+1}} \|u\|^p_{\mathcal{M}^p(\Omega)} \leq \sup_{\lambda > 0} \{\lambda^p meas[x \in \Omega : |u| > \lambda]\} \leq \|u\|^p_{\mathcal{M}^p(\Omega)}$.
Moreover,

(ii) $\displaystyle\int_K |u|^q dx \leq \frac{p}{p-q} (\frac{p}{q})^{\frac{q}{p}} \|u\|^q_{\mathcal{M}^p(\Omega)} (meas(K))^{\frac{p-q}{p}}$, *for every measurable subset $K \subset \Omega$.*

In particular, $\mathcal{M}^p(\Omega) \subset L^q_{loc}(\Omega)$, with continuous embedding and $u \in \mathcal{M}^p(\Omega)$ implies $|u|^q \in \mathcal{M}^{\frac{p}{q}}(\Omega)$.

The following result is due to Troisi (see [39]).

Theorem 2.4. *Let $p_1, ..., p_N \in [1, \infty)$, $\vec{p} = (p_1, ..., p_N)$; $g \in W^{1,\vec{p}}(\Omega)$, and let*

$$\begin{cases} q = \bar{p}^* & \text{if} \quad \bar{p}^* < N, \\ q \in [1, \infty) & \text{if} \quad \bar{p}^* \geq N; \end{cases} \tag{2.6}$$

where $p^* = \dfrac{N}{\sum_{i=1}^N \frac{1}{p_i} - 1}$, $\sum_{i=1}^N \frac{1}{p_i} > 1$ *and* $\bar{p}^* = \dfrac{N\bar{p}}{N - \bar{p}}$.
Then, there exists a constant $C > 0$ depending on N, $p_1, ..., p_N$ if $\bar{p} < N$ and also on q and meas(Ω) if $\bar{p} \geq N$ such that

$$\|g\|_{L^q(\Omega)} \leq c \prod_{i=1}^N \left[\|g\|_{L^{p_M}(\Omega)} + \|\frac{\partial g}{\partial x_i}\|_{L^{p_i}(\Omega)} \right]^{\frac{1}{N}}, \tag{2.7}$$

where $p_M = \max\{p_1, ..., p_N\}$ and $\frac{1}{\bar{p}} = \frac{1}{N}\sum_{i=1}^N \frac{1}{p_i}$. In particular, if $u \in W_0^{1,\vec{p}}(\Omega)$, we have

$$\|g\|_{L^q(\Omega)} \leq c \prod_{i=1}^N \left[\left\| \frac{\partial g}{\partial x_i} \right\|_{L^{p_i}(\Omega)} \right]^{\frac{1}{N}}. \tag{2.8}$$

In the sequel, we consider the following spaces.

$$W_D^{1,\vec{p}(.)}(\Omega) = \{\xi \in W^{1,\vec{p}(.)}(\Omega) \ : \ \xi = 0 \text{ on } \Gamma_D\}$$

and

$$W_{Ne}^{1,\vec{p}(.)}(\Omega) = \{\xi \in W_D^{1,\vec{p}(.)}(\Omega) \ : \ \xi \equiv \text{constant on } \Gamma_{Ne}\}.$$

$$\mathcal{T}_D^{1,\vec{p}(.)}(\Omega) = \{\xi \text{ measurable on } \Omega \text{ such that } \forall k > 0, \ T_k(\xi) \in \ W_D^{1,\vec{p}(.)}(\Omega)\}$$

and

$$\mathcal{T}_{Ne}^{1,\vec{p}(.)}(\Omega) = \{\xi \text{ measurable on } \Omega \text{ such that } \forall k > 0, \ T_k(\xi) \in \ W_{Ne}^{1,\vec{p}(.)}(\Omega)\},$$

where T_k is a truncation function defined by

$$T_k(s) = \begin{cases} k & \text{if } s > k, \\ s & \text{if } |s| \leq k, \\ -k & \text{if } s < -k. \end{cases}$$

For any $v \in W_{Ne}^{1,\vec{p}(.)}(\Omega)$, we set $v_N = v_{Ne} := v|_{\Gamma_{Ne}}$.

Definition 2.5. *A measurable function $u : \Omega \to \mathbb{R}$ is an entropy solution of $P(\rho, \mu, d)$ if $u \in \mathcal{T}_{Ne}^{1,\vec{p}(.)}(\Omega)$ and for every $k > 0$,*

$$\begin{cases} \displaystyle\int_\Omega \left(\sum_{i=1}^N a_i \left(x, \frac{\partial}{\partial x_i} u \right) \frac{\partial}{\partial x_i} T_k(u - \xi) \right) dx + \int_\Omega |u|^{p_M(x)-2} u T_k(u - \xi) dx \leq \\ \displaystyle\int_\Omega T_k(u - \xi) d\mu + (d - \rho(u_{Ne})) T_k(u_{Ne} - \xi), \end{cases} \tag{2.9}$$

for all $\xi \in W_{Ne}^{1,\vec{p}(.)}(\Omega) \cap L^\infty(\Omega)$.

Our main result in this paper is the following theorem.

Theorem 2.6. *Assume* (H_1)-(H_5). *Then for any* $(\mu, d) \in \mathcal{M}_b^{p_m(.)}(\Omega) \times \mathbb{R}$, *the problem* $P(\rho, \mu, d)$ *admits a unique entropy solution* u.

3 The approximated problem corresponding to $P(\rho, \mu, d)$

We define a new bounded domain $\tilde{\Omega}$ in $\mathbb{R}^N$ as follow.
We fix $\theta > 0$ and we set $\tilde{\Omega} = \Omega \cup \{x \in \mathbb{R}^N / dist(x, \Gamma_{Ne}) < \theta\}$. Then, $\partial\tilde{\Omega} = \Gamma_D \cup \tilde{\Gamma}_{Ne}$ is Lipschitz with $\Gamma_D \cap \tilde{\Gamma}_{Ne} = \emptyset$.

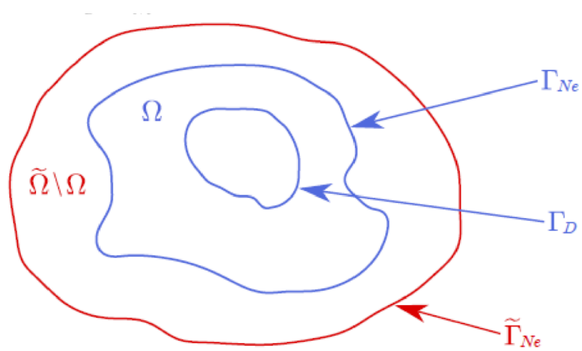

Figure 1: Domains representation

Let us consider $\tilde{a}_i(x, \xi)$ (to be defined later) Carathéodory and satisfying (1.4), (1.5), (1.6) and (1.7), for all $x \in \tilde{\Omega}$.
We also consider a function $\tilde{d}$ in $L^1(\tilde{\Gamma}_{Ne})$ such that

$$\int_{\tilde{\Gamma}_{Ne}} \tilde{d} d\sigma = d. \tag{3.1}$$

For any $\epsilon > 0$, we set $\mu_\epsilon = f_\epsilon - \mathrm{div}F$, where $f_\epsilon = T_{\frac{1}{\epsilon}}(f) \in L^\infty(\Omega)$. Note that $f_\epsilon \to f$ as $\epsilon \to 0$ in $L^1(\Omega)$ and $\|f_\epsilon\|_1 \leq \|f\|_1$.

We set $\tilde{\mu}_\epsilon = f_\epsilon \chi_\Omega - \mathrm{div}F\chi_\Omega$, $\tilde{d}_\epsilon = T_{\frac{1}{\epsilon}}(\tilde{d})$ and we consider the problem

$$P(\tilde{\rho}, \tilde{\mu}_\epsilon, \tilde{d}_\epsilon) \begin{cases} -\displaystyle\sum_{i=1}^N \frac{\partial}{\partial x_i} \tilde{a}_i(x, \frac{\partial}{\partial x_i} u_\epsilon) + |u_\epsilon|^{p_M(x)-2} u_\epsilon \chi_\Omega(x) = \tilde{\mu}_\epsilon & \text{in } \tilde{\Omega} \\ u_\epsilon = 0 & \text{on } \Gamma_D \\ \tilde{\rho}(u_\epsilon) + \displaystyle\sum_{i=1}^N \tilde{a}_i(x, \frac{\partial}{\partial x_i} u_\epsilon)\eta_i = \tilde{d}_\epsilon & \text{on } \tilde{\Gamma}_{Ne}, \end{cases} \tag{3.2}$$

where the function $\tilde{\rho}$ is defined as follow.

- $\tilde{\rho}(s) = \dfrac{1}{|\tilde{\Gamma}_{Ne}|}\rho(s)$, where $|\tilde{\Gamma}_{Ne}|$ denotes the Hausdorff measure of $\tilde{\Gamma}_{Ne}$.

We obviously have $\forall \epsilon > 0$, $\tilde{d}_\epsilon \in L^\infty(\tilde{\Gamma}_{Ne})$.
The following definition gives the notion of solution for the problem $P_\epsilon(\tilde{\rho}, \tilde{\mu}_\epsilon, \tilde{d}_\epsilon)$.

Definition 3.1. *A measurable function $u_\epsilon : \tilde{\Omega} \to \mathbb{R}$ is a solution to problem $P_\epsilon(\tilde{\rho}, \tilde{\mu}_\epsilon, \tilde{d}_\epsilon)$ if $u_\epsilon \in W_D^{1,\vec{p}(.)}(\tilde{\Omega})$ and*

$$\int_{\tilde{\Omega}} \sum_{i=1}^{N} \tilde{a}_i(x, \frac{\partial}{\partial x_i} u_\epsilon) \frac{\partial}{\partial x_i} \tilde{\xi} dx + \int_\Omega |u_\epsilon|^{p_M(x)-2} u_\epsilon \tilde{\xi} dx = \int_\Omega f_\epsilon \tilde{\xi} dx + \int_\Omega F.\nabla \tilde{\xi} + \int_{\tilde{\Gamma}_{Ne}} (\tilde{d}_\epsilon - \tilde{\rho}(u_\epsilon)) \tilde{\xi} d\sigma, \tag{3.3}$$

for any $\tilde{\xi} \in W_D^{1,\vec{p}(.)}(\tilde{\Omega}) \cap L^\infty(\Omega)$.

Theorem 3.2. *The problem $P_\epsilon(\tilde{\rho}, \tilde{\mu}_\epsilon, \tilde{d}_\epsilon)$ admits at least one solution in the sense of Definition 3.1.*

Step 1: Approximated problem we study an existence result to the following problem. For any $k > 0$ we consider

$$P_{\epsilon,k}(\tilde{\rho}, \tilde{\mu}_\epsilon, \tilde{d}_\epsilon) \begin{cases} -\displaystyle\sum_{i=1}^{N} \frac{\partial}{\partial x_i} \tilde{a}_i(x, \frac{\partial}{\partial x_i} u_{\epsilon,k}) + T_k(b(u_{\epsilon,k}))\chi_\Omega(x) = \tilde{\mu}_\epsilon & \text{in } \tilde{\Omega} \\ u_{\epsilon,k} = 0 & \text{on } \Gamma_D \\ T_k(\tilde{\rho}(u_{\epsilon,k})) + \displaystyle\sum_{i=1}^{N} \tilde{a}_i(x, \frac{\partial}{\partial x_i} u_{\epsilon,k})\eta_i = \tilde{d}_\epsilon & \text{on } \tilde{\Gamma}_{Ne}, \end{cases} \tag{3.4}$$

where $b(u) = |u|^{p_M(x)-2}u$.
We have to prove that $P_{\epsilon,k}(\tilde{\rho}, \tilde{\mu}_\epsilon, \tilde{d}_\epsilon)$ admits at least one solution in the following sense.

$$\begin{cases} u_{\epsilon,k} \in W_D^{1,\vec{p}(.)}(\tilde{\Omega}) \text{ and for all } \tilde{\xi} \in W_D^{1,\vec{p}(.)}(\tilde{\Omega}), \\ \displaystyle\int_{\tilde{\Omega}} \sum_{i=1}^{N} \tilde{a}_i(x, \frac{\partial}{\partial x_i} u_{\epsilon,k}) \frac{\partial}{\partial x_i} \tilde{\xi} dx + \int_\Omega T_k(b(u_{\epsilon,k}))\tilde{\xi} dx = \int_\Omega \tilde{\xi} d\mu_\epsilon + \int_{\tilde{\Gamma}_{Ne}} (\tilde{d}_\epsilon - T_k(\tilde{\rho}(u_{\epsilon,k})))\tilde{\xi} d\sigma. \end{cases} \tag{3.5}$$

For any $k > 0$, let us introduce the operator $\Lambda_k : W_D^{1,\vec{p}(.)}(\tilde{\Omega}) \to (W_D^{1,\vec{p}(.)}(\tilde{\Omega}))'$ such that for any $(u, v) \in W_D^{1,\vec{p}(.)}(\tilde{\Omega}) \times W_D^{1,\vec{p}(.)}(\tilde{\Omega})$,

$$\langle \Lambda_k(u), v \rangle = \int_{\tilde{\Omega}} \left(\sum_{i=1}^{N} \tilde{a}_i(x, \frac{\partial}{\partial x_i} u) \frac{\partial}{\partial x_i} v \right) dx + \int_\Omega T_k(b(u)) v dx + \int_{\tilde{\Gamma}_{Ne}} T_k(\tilde{\rho}(u)) v d\sigma. \tag{3.6}$$

We need to prove that for any $k > 0$, the operator Λ_k is bounded, coercive, of type M and therefore, surjective.

(i) Boundedness of Λ_k. Let $(u, v) \in F \times W_D^{1,\vec{p}(.)}(\tilde{\Omega})$ with F a bounded subset of $W_D^{1,\vec{p}(.)}(\tilde{\Omega})$.

We have

$$\begin{cases} |\langle \Lambda_k(u), v\rangle| \leq \sum_{i=1}^{N}\left(\int_{\tilde{\Omega}}\left|\tilde{a}_i(x,\frac{\partial}{\partial x_i}u)\right|\left|\frac{\partial}{\partial x_i}v\right|dx\right)+\int_{\tilde{\Omega}}|T_k(b(u))||v|dx+\int_{\tilde{\Gamma}_{Ne}}|T_k(\tilde{\rho}(u))||v|d\sigma \\ = I_1+I_2+I_3, \end{cases}$$

where we denote by I_1, I_2 and I_3 the three terms on the right hand side of the first inequality. By (H_2) and the Hölder type inequality, we have

$$\begin{cases} I_1 \leq C_1\sum_{i=1}^{N}\left(\int_{\tilde{\Omega}}|j_i(x)|\left|\frac{\partial}{\partial x_i}v\right|dx+\int_{\tilde{\Omega}}\left|\frac{\partial}{\partial x_i}u\right|^{p_i(x)-1}\left|\frac{\partial}{\partial x_i}v\right|dx\right) \\ \leq C_1\sum_{i=1}^{N}\left(\frac{1}{p_i'^{-}}+\frac{1}{p_i^{-}}\right)|j_i|_{p_i'(.)}\left|\frac{\partial}{\partial x_i}v\right|_{p_i(.)}+\sum_{i=1}^{N}\left(\frac{1}{p_i'^{-}}+\frac{1}{p_i^{-}}\right)\left|\left|\frac{\partial}{\partial x_i}u\right|^{p_i(x)-1}\right|_{p_i'(.)}\left|\frac{\partial}{\partial x_i}v\right|_{p_i(.)}. \end{cases}$$

As $u\in F$, $\forall\, i\in\{1,...,N\}$, there exists a constant $M>0$ such that

$$\sum_{i=1}^{N}\left|\left|\frac{\partial}{\partial x_i}u\right|^{p_i(x)-1}\right|_{p_i'(.)}<M;$$

so

$$\left|\left|\frac{\partial}{\partial x_i}u\right|^{p_i(x)-1}\right|_{p_i'(.)}<M,\ \forall\, i\in\{1,...,N\}.$$

Let $C_4=\max_{i=1,...,N}\left\{\left|\left|\frac{\partial}{\partial x_i}u\right|^{p_i(x)-1}\right|_{p_i'(.)}\right\}$.

As $j_i\in L^{p_i'(.)}(\tilde{\Omega})$, we have

$$I_1\leq C_5(C_1,p_i^-,(p_i')^-,C_3(j_i))\sum_{i=1}^{N}\left|\frac{\partial}{\partial x_i}v\right|_{p_i(.)}+C_6(C_1,p_i^-,(p_i')^-,C_4)\sum_{i=1}^{N}\left|\frac{\partial}{\partial x_i}v\right|_{p_i(.)}.$$

It is easy to see that

$$I_2\leq k\int_{\tilde{\Omega}}|v|dx.$$

Using Theorem 2.1, we have

$$\|v\|_{L^1(\tilde{\Omega})}\leq C_7\|v\|_{W_D^{1,\vec{p}(.)}(\tilde{\Omega})}.$$

So,

$$I_2\leq kC_7\|v\|_{W_D^{1,\vec{p}(.)}(\tilde{\Omega})}.$$

Similarly, by using Theorem 2.2, we have

$$I_3\leq kC_8\|v\|_{W_D^{1,\vec{p}(.)}(\tilde{\Omega})}\Box$$

Therefore, Λ_k maps bounded subsets of $W_D^{1,\vec{p}(.)}(\tilde{\Omega})$ into bounded subsets of $(W_D^{1,\vec{p}(.)}(\tilde{\Omega}))'$. Thus, Λ_k is bounded on $W_D^{1,\vec{p}(.)}(\tilde{\Omega})$.

(ii) Coerciveness of Λ_k. We have to show that for any $k > 0$, $\dfrac{\langle \Lambda_k(u), u\rangle}{\|u\|_{W_D^{1,\vec{p}(.)}(\tilde{\Omega})}} \to \infty$ as $\|u\|_{W_D^{1,\vec{p}(.)}(\tilde{\Omega})} \to \infty$.

For any $u \in W_D^{1,\vec{p}(.)}(\tilde{\Omega})$, we have

$$\langle \Lambda_k(u), u\rangle = \langle \Lambda(u), u\rangle + \int_\Omega T_k(b(u))u dx + \int_{\tilde{\Gamma}_{Ne}} T_k(\tilde{\rho}(u))u d\sigma, \tag{3.7}$$

where $\langle \Lambda(u), u\rangle = \displaystyle\sum_{i=1}^{N}\left(\int_{\tilde{\Omega}} \tilde{a}_i(x, \frac{\partial}{\partial x_i}u)\frac{\partial}{\partial x_i}u dx\right)$.

The last two terms on the right-hand side of (3.7) are non-negative by the monotonicity of T_k, b and $\tilde{\rho}$. We can assert that

$$\begin{cases} \langle \Lambda_k(u), u\rangle \geq \langle \Lambda(u), u\rangle \\ \geq \dfrac{1}{N^{p_m^- - 1}}\|u\|^{p_m^-}_{W_D^{1,\vec{p}(.)}(\tilde{\Omega})} - N. \end{cases}$$

Indeed, since $\displaystyle\int_{\tilde{\Omega}} |T_k(b(u))||u| dx + \int_{\tilde{\Gamma}_{Ne}} |T_k(\tilde{\rho}(u))||u| d\sigma \geq 0$, for all $u \in W_D^{1,\vec{p}(.)}(\tilde{\Omega})$, we have

$$\langle \Lambda_k(u), u\rangle \geq \langle \Lambda(u), u\rangle.$$

So,

$$\langle \Lambda_k(u), u\rangle \;\geq\; \sum_{i=1}^{N}\left(\int_{\tilde{\Omega}} \tilde{a}_i(x, \frac{\partial}{\partial x_i}u)\frac{\partial}{\partial x_i}u dx\right) \geq \sum_{i=1}^{N}\left(\int_{\tilde{\Omega}} \left|\frac{\partial}{\partial x_i}u\right|^{p_i(x)} dx\right).$$

We make the following notations:

$$\mathcal{I} = \left\{ i \in \{1, ..., N\} : \left|\frac{\partial}{\partial x_i}u\right|_{p_i(.)} \leq 1 \right\} \text{ and } \mathcal{J} = \left\{ i \in \{1, ..., N\} : \left|\frac{\partial}{\partial x_i}u\right|_{p_i(.)} > 1 \right\}.$$

We have

$$\begin{aligned}
\langle \Lambda_k(u), u\rangle \;&\geq\; \sum_{i\in\mathcal{I}}\left(\int_{\tilde{\Omega}} \left|\frac{\partial}{\partial x_i}u\right|^{p_i(x)} dx\right) + \sum_{i\in\mathcal{J}}\left(\int_{\tilde{\Omega}} \left|\frac{\partial}{\partial x_i}u\right|^{p_i(x)} dx\right) \\
&\geq\; \sum_{i\in\mathcal{I}}\left(\left|\frac{\partial}{\partial x_i}u\right|^{p_i^+}_{p_i(.)}\right) + \sum_{i\in\mathcal{J}}\left(\left|\frac{\partial}{\partial x_i}u\right|^{p_i^-}_{p_i(.)}\right) \\
&\geq\; \sum_{i\in\mathcal{J}}\left(\left|\frac{\partial}{\partial x_i}u\right|^{p_i^-}_{p_i(.)}\right) \\
&\geq\; \sum_{i\in\mathcal{J}}\left(\left|\frac{\partial}{\partial x_i}u\right|^{p_m^-}_{p_i(.)}\right) \\
&\geq\; \sum_{i=1}^{N}\left(\left|\frac{\partial}{\partial x_i}u\right|^{p_m^-}_{p_i(.)}\right) - \sum_{i\in\mathcal{I}}\left(\left|\frac{\partial}{\partial x_i}u\right|^{p_m^-}_{p_i(.)}\right) \\
&\geq\; \sum_{i=1}^{N}\left(\left|\frac{\partial}{\partial x_i}u\right|^{p_m^-}_{p_i(.)}\right) - N.
\end{aligned}$$

We now use Jensen's inequality on the convex function $Z: \mathbb{R}^+ \to \mathbb{R}^+$, $Z(t) = t^{p_m^-}$, $p_m^- > 1$ to get

$$\begin{cases} \langle \Lambda_k(u), u \rangle \geq \langle \Lambda(u), u \rangle \\ \geq \dfrac{1}{N^{p_m^- - 1}} \|u\|^{p_m^-}_{W_D^{1,\vec{p}(.)}(\tilde{\Omega})} - N. \end{cases}$$

Hence, Λ_k is coercive (as $p_m^- > 1$).

(iii) The operator Λ_k is of type M.

Lemma 3.3. *(cf [41]) Let $\mathcal{A}$ and $\mathcal{B}$ be two operators. If $\mathcal{A}$ is of type M and $\mathcal{B}$ is monotone and weakly continuous, then $\mathcal{A} + \mathcal{B}$ is of type M.*

Now , we set $\langle \mathcal{A}u, v \rangle := \langle \Lambda(u), v \rangle$ and $\langle \mathcal{B}_k u, v \rangle := \int_\Omega T_k(b(u))v dx + \int_{\tilde{\Gamma}_{Ne}} T_k(\tilde{\rho}(u))v d\sigma$.
Then, for every $k > 0$, we have $\Lambda_k = \mathcal{A} + \mathcal{B}_k$. We now have to show that for every $k > 0$, $\mathcal{B}_k$ is monotone and weakly continuous, because it is well-known that $\mathcal{A}$ is of type M. For the monotonicity of $\mathcal{B}_k$, we have to show that

$$\langle \mathcal{B}_k u - \mathcal{B}_k v, u - v \rangle \geq 0 \text{ for all } (u, v) \in W_D^{1,\vec{p}(.)}(\tilde{\Omega}) \times W_D^{1,\vec{p}(.)}(\tilde{\Omega}).$$

We have

$$\begin{aligned} \langle \mathcal{B}_k u - \mathcal{B}_k v, u - v \rangle &= \int_\Omega (T_k(b(u)) - T_k(b(v)))(u - v) dx \\ &+ \int_{\tilde{\Gamma}_{Ne}} (T_k(\tilde{\rho}(u)) - T_k(\tilde{\rho}(v)))(u - v) d\sigma. \end{aligned}$$

From the monotonicity of b, $\tilde{\rho}$ and the map T_k, we conclude that

$$\langle \mathcal{B}_k u - \mathcal{B}_k v, u - v \rangle \geq 0. \tag{3.8}$$

We need now to prove that for each $k > 0$ the operator $\mathcal{B}_k$ is weakly continuous, that is, for all sequences $(u_n)_{n \in \mathbb{N}} \subset W_D^{1,\vec{p}(.)}(\tilde{\Omega})$ such that $u_n \rightharpoonup u$ in $W_D^{1,\vec{p}(.)}(\tilde{\Omega})$, we have $\mathcal{B}_k u_n \rightharpoonup \mathcal{B}_k u$ as $n \to \infty$. For all $\phi \in W_D^{1,\vec{p}(.)}(\tilde{\Omega})$, we have

$$\langle \mathcal{B}_k u_n, \phi \rangle := \int_\Omega T_k(b(u_n))\phi dx + \int_{\tilde{\Gamma}_{Ne}} T_k(\tilde{\rho}(u_n))\phi d\sigma. \tag{3.9}$$

Passing to the limit in (3.9) as n goes to ∞ and using the Lebesgue dominated convergence theorem, since $u_n \rightharpoonup u$ in $W_D^{1,\vec{p}(.)}(\tilde{\Omega})$; up to a subsequence, we have $u_n \to u$ in $L^1(\tilde{\Omega})$ and a.e. in $\tilde{\Omega}$. As $|T_k(b(u_n))\phi| \leq k|\phi|$ and $\phi \in W_D^{1,\vec{p}(.)}(\tilde{\Omega}) \hookrightarrow L^1(\tilde{\Omega})$, for the first term on the right-hand side of (3.9), we obtain

$$\lim_{n \to \infty} \int_\Omega T_k(b(u_n))\phi dx = \int_\Omega T_k(b(u))\phi dx. \tag{3.10}$$

Furthermore, since $u_n \rightharpoonup u$ in $W_D^{1,\vec{p}(.)}(\tilde{\Omega})$; up to a subsequence, we have $u_n \to u$ *in* $L^1(\partial\tilde{\Omega})$ and a.e. on $\partial\tilde{\Omega}$. As $|T_k(\tilde{\rho}(u_n))\phi| \leq k|\phi|$ and $\phi \in W_D^{1,\vec{p}(.)}(\tilde{\Omega}) \hookrightarrow L^1(\partial\tilde{\Omega})$, we deduce by the Lebesgue dominated convergence theorem that

$$\lim_{n \to \infty} \int_{\tilde{\Gamma}_{Ne}} T_k(\tilde{\rho}(u_n))\phi dx = \int_{\tilde{\Gamma}_{Ne}} T_k(\tilde{\rho}(u))\phi dx. \tag{3.11}$$

From (3.10) and (3.11) we conclude that for every $k>0$, $\mathcal{B}_k(u_n)\to\mathcal{B}_k(u)$ as $n\to\infty$.

The operator $\mathcal{A}$ is type M and as $\mathcal{B}_k$ is monotone and weakly continuous, thanks to Lemma 3.3, we conclude that the operator Λ_k is of type M. Then for any $L\in(W_D^{1,\vec{p}(.)}(\tilde{\Omega}))'$, there exists $u_{\epsilon,k}\in W_D^{1,\vec{p}(.)}(\tilde{\Omega})$, such that $\Lambda_k(u_{\epsilon,k})=L$.

We now consider $L\in(W_D^{1,\vec{p}(.)}(\tilde{\Omega}))'$ defined by $L(v)=\int_\Omega v\,d\mu_\epsilon+\int_{\tilde{\Gamma}_{Ne}}\tilde{d}_\epsilon v\,d\sigma$, for $v\in W_D^{1,\vec{p}(.)}(\tilde{\Omega})$ and we obtain (3.5)□

Step 2: A priori estimates

Lemma 3.4. *Let $u_{\epsilon,k}$ a solution of $P_{\epsilon,k}(\tilde{\rho},\tilde{\mu}_\epsilon,\tilde{d}_\epsilon)$. Then*

$$\begin{cases}|\tilde{\rho}(u_{\epsilon,k})|\le k_1:=\max\{\|\tilde{d}_\epsilon\|_\infty,(\tilde{\rho}_\epsilon\circ b^{-1})(\|\mu_\epsilon\|_\infty)\}\ a.e.\ on\ \tilde{\Gamma}_{Ne},\\ |b(u_{\epsilon,k})|\le k_2:=\max\{|\mu_\epsilon\|_\infty;(b\circ\rho_0^{-1})(|\tilde{\Gamma}_{Ne}|\|\tilde{d}_\epsilon\|_\infty)\}\ a.e.\ in\ \Omega.\end{cases}\tag{3.12}$$

Proof. For any $\tau>0$, let us introduce the function $H_\tau:\mathbb{R}\to\mathbb{R}$ by

$$H_\tau(s)=\begin{cases}0 & \text{if } s<0,\\ \dfrac{s}{\tau} & \text{if } 0\le s\le\tau,\\ 1 & \text{if } s>\tau.\end{cases}$$

In (3.5) we set $\tilde{\xi}=H_\tau(u_{\epsilon,k}-M)$, where $M>0$ is to be fixed later. We get

$$\begin{cases}\displaystyle\int_{\tilde{\Omega}}\sum_{i=1}^N\tilde{a}_i(x,\frac{\partial}{\partial x_i}u_{\epsilon,k})\frac{\partial}{\partial x_i}H_\tau(u_{\epsilon,k}-M)dx+\int_\Omega T_k(b(u_{\epsilon,k}))H_\tau(u_{\epsilon,k}-M)dx=\\ \displaystyle\int_\Omega H_\tau(u_{\epsilon,k}-M)d\mu_\epsilon+\int_{\tilde{\Gamma}_{Ne}}(\tilde{d}_\epsilon-T_k(\tilde{\rho}(u_\epsilon,k)))H_\tau(u_{\epsilon,k}-M)d\sigma.\end{cases}\tag{3.13}$$

The first term in (3.13) is non-negative. Indeed,

$$\int_{\tilde{\Omega}}\sum_{i=1}^N\tilde{a}_i(x,\frac{\partial}{\partial x_i}u_{\epsilon,k})\frac{\partial}{\partial x_i}H_\tau(u_{\epsilon,k}-M)dx=\frac{1}{\tau}\int_{\{0\le u_{\epsilon,k}-M\le\tau\}}\sum_{i=1}^N\tilde{a}_i(x,\frac{\partial}{\partial x_i}u_{\epsilon,k})\frac{\partial}{\partial x_i}u_{\epsilon,k}dx\ge0.$$

From (3.13) we obtain

$$\int_\Omega T_k(b(u_{\epsilon,k}))H_\tau(u_{\epsilon,k}-M)dx\le\int_\Omega H_\tau(u_{\epsilon,k}-M)d\mu_\epsilon+\int_{\tilde{\Gamma}_{Ne}}(\tilde{d}_\epsilon-T_k(\tilde{\rho}(u_\epsilon,k)))H_\tau(u_{\epsilon,k}-M)d\sigma.$$

Then, one has

$$\begin{cases}\displaystyle\int_\Omega(T_kb(u_{\epsilon,k})-T_k(b(M)))H_\tau(u_{\epsilon,k}-M)dx+\int_{\tilde{\Gamma}_{Ne}}(T_k(\tilde{\rho}(u_\epsilon,k))-T_k(\tilde{\rho}(M)))H_\tau(u_{\epsilon,k}-M)dx\le\\ \displaystyle\int_\Omega(\mu_\epsilon-T_k(b(M)))H_\tau(u_{\epsilon,k}-M)dx+\int_{\tilde{\Gamma}_{Ne}}(\tilde{d}_\epsilon-T_k(\tilde{\rho}(M)))H_\tau(u_{\epsilon,k}-M)d\sigma.\end{cases}$$

Letting τ go to 0 in the inequality above, we get

$$\begin{cases}\displaystyle\int_\Omega(T_k(b(u_{\epsilon,k}))-T_k(b(M)))^+dx+\int_{\tilde{\Gamma}_{Ne}}(T_k(\tilde{\rho}(u_{\epsilon,k}))-T_k(\tilde{\rho}(M)))^+d\sigma\le\\ \displaystyle\int_\Omega(\mu_\epsilon-T_k(b(M)))sign_0^+(u_k-M)dx+\int_{\tilde{\Gamma}_{Ne}}(\tilde{d}_\epsilon-T_k(\tilde{\rho}(M)))sign_0^+(u_{\epsilon,k}-M)d\sigma.\end{cases}$$

As $Im(b) = Im(\rho) = \mathbb{R}$, we can fix $M = M_0 = \max\{b^{-1}(\|\mu_\epsilon\|_\infty), \rho_0^{-1}(|\tilde{\Gamma}_{Ne}|\|\tilde{d}_\epsilon\|_\infty)\}$. From the above inequality we obtain

$$\begin{cases} \displaystyle\int_\Omega (T_k(b(u_{\epsilon,k})) - T_k(b(M_0)))^+ dx + \int_{\tilde{\Gamma}_{Ne}} (T_k(\tilde{\rho}(u_{\epsilon,k}) - T_k(\tilde{\rho}(M_0)))^+ d\sigma \leq \\ \displaystyle\int_\Omega (\mu_\epsilon - T_k(\|\mu_\epsilon\|_\infty)) sign_0^+(u_{\epsilon,k} - M_0) dx + \int_{\tilde{\Gamma}_{Ne}} (\tilde{d} - T_k(\|\tilde{d}_\epsilon\|_\infty)) sign_0^+(u_{\epsilon,k} - M_0) d\sigma. \end{cases}$$

For $k > k_0 := \max\{\|\mu_\epsilon\|, \|\tilde{d}_\epsilon\|_\infty\}$, it follows that

$$\int_\Omega (T_k(b(u_{\epsilon,k})) - T_k(b(M_0)))^+ dx + \int_{\tilde{\Gamma}_{Ne}} (T_k(\tilde{\rho}(u_{\epsilon,k})) - T_k(\tilde{\rho}(M_0)))^+ d\sigma \leq 0. \tag{3.14}$$

From (3.14), we deduce that

$$\begin{cases} T_k(\tilde{\rho}(u_{\epsilon,k})) \leq T_k(\tilde{\rho}(M_0)) \text{ a.e. on } \tilde{\Gamma}_{Ne}, \\ T_k(b(u_{\epsilon,k})) \leq T_k(b(M_0)) \text{ a.e. in } \Omega. \end{cases} \tag{3.15}$$

From (3.15), we deduce that for every $k > k_1 := \max\{\|\tilde{d}_\epsilon\|_\infty, \|\mu_\epsilon\|_\infty,\ b(M_0),\ \tilde{\rho}(M_0)\}$,

$$\tilde{\rho}(u_{\epsilon,k}) \leq \tilde{\rho}(M_0) \text{ a.e. on } \tilde{\Gamma}_{Ne}$$

and

$$b(u_{\epsilon,k}) \leq b(M_0) \text{ a.e. in } \Omega.$$

Note that with the choice of M_0 and the fact that $D(\rho) = D(b) = \mathbb{R}$, for every $k > k_1 := \max\{\|\tilde{d}_\epsilon\|_\infty, \|\mu_\epsilon\|_\infty,\ b(M_0),\ \tilde{\rho}(M_0)\}$, we have

$$\begin{cases} b(u_{\epsilon,k}) \leq \max\{\|\mu_\epsilon\|_\infty, b \circ \rho_0^{-1}(|\tilde{\Gamma}_{Ne}|\|\tilde{d}_\epsilon\|_\infty)\ \} \text{ a.e. in } \Omega, \\ \tilde{\rho}(u_{\epsilon,k}) \leq \max\{\|\tilde{d}_\epsilon\|_\infty, (\tilde{\rho} \circ b^{-1})(\|\mu_\epsilon\|_\infty)\} \text{ a.e. on } \tilde{\Gamma}_{Ne}. \end{cases} \tag{3.16}$$

We need to show that for any k large enough,

$$\begin{cases} b(u_{\epsilon,k}) \geq -\max\{\|\mu_\epsilon\|_\infty, b \circ \rho_0^{-1}(|\tilde{\Gamma}_{Ne}|\|\tilde{d}_\epsilon\|_\infty)\} \text{ a.e. in } \Omega, \\ \tilde{\rho}(u_{\epsilon,k}) \geq -\max\{\|\tilde{d}_\epsilon\|_\infty, (\tilde{\rho} \circ b^{-1})(\|\mu_\epsilon\|_\infty)\} \text{ a.e. on } \tilde{\Gamma}_{Ne}. \end{cases} \tag{3.17}$$

It is easy to see that if $(u_{\epsilon,k})$ is a solution of $P_{\epsilon,k}(\tilde{\rho}, \tilde{\mu}_\epsilon, \tilde{d}_\epsilon)$, then $(-u_{\epsilon,k})$ is a solution of

$$P_{\epsilon,k}(\hat{\rho}, \hat{\mu}_\epsilon, \hat{d}_\epsilon) \begin{cases} -\displaystyle\sum_{i=1}^N \frac{\partial}{\partial x_i} \hat{a}_i(x, \frac{\partial}{\partial x_i} u_{\epsilon,k}) + T_k(\hat{b}(u_{\epsilon,k}))\chi_\Omega(x) = \hat{\mu}_\epsilon & \text{in } \tilde{\Omega} \\ u_{\epsilon,k} = 0 & \text{on } \Gamma_D \\ T_k(\hat{\rho}(u_{\epsilon,k})) + \displaystyle\sum_{i=1}^N \hat{a}_i(x, \frac{\partial}{\partial x_i} u_{\epsilon,k}) \eta_i = \hat{d}_\epsilon & \text{on } \tilde{\Gamma}_{Ne}, \end{cases}$$

where $\hat{a}_i(x,\xi) = -\tilde{a}_i(x, -\xi)$, $\hat{\rho}(s) = -\tilde{\rho}(-s)$, $\hat{b}(s) = -b(-s)$, $\hat{\mu}_\epsilon = -\tilde{\mu}_\epsilon$ and $\hat{d} = -\tilde{d}_\epsilon$. Then for every $k > k_2 := \max\{\|\tilde{d}_\epsilon\|_\infty, \|\mu_\epsilon\|_\infty,\ -b(-M_0),\ -\tilde{\rho}(-M_0)\}$, we have

$$\begin{cases} -b(u_{\epsilon,k}) \leq \max\{\|\mu_\epsilon\|_\infty, b \circ \rho_0^{-1}(|\tilde{\Gamma}_{Ne}|\|\tilde{d}_\epsilon\|_\infty)\} \text{ a.e. in } \Omega, \\ -\tilde{\rho}(u_{\epsilon,k}) \leq \max\{\|\tilde{d}_\epsilon\|_\infty, (\tilde{\rho} \circ b^{-1})(\|\mu_\epsilon\|_\infty)\} \text{ a.e. on } \tilde{\Gamma}_{Ne}, \end{cases}$$

which implies (3.17).

From (3.16) and (3.17), we deduce (3.12). □

Step 3. Convergence Since $u_{\epsilon,k}$ is a solution of $P_{\epsilon,k}(\tilde\rho,\tilde\mu_\epsilon,\tilde d_\epsilon)$, thanks to Lemma 3.4 and the fact that Ω is bounded, we have $\tilde\rho(u_{\epsilon,k})\in L^1(\tilde\Gamma_{Ne})$ and $b(u_{\epsilon,k})\in L^1(\Omega)$. For $k=1+\max(k_1,k_2)$ fixed, by Lemma 3.4, one sees that problem $P_\epsilon(\tilde\rho,\tilde\mu_\epsilon,\tilde d_\epsilon)$ admits at least one solution u_ϵ □

Remark 3.5. *Using the relation* (3.12) *and the fact that the functions b and ρ are non-decreasing, it follows that for k large enough, the solution of the problem $P(\tilde\rho,\tilde\mu_\epsilon,\tilde d_\epsilon)$ belongs to $L^\infty(\Omega)\cap L^\infty(\tilde\Gamma_{Ne})$ and $|u_\epsilon|\le c(b,k_1)$ a.e. in Ω and $|u_\epsilon|\le c(\rho,k_2)$ a.e. on $\tilde\Gamma_{Ne}$.*

Now, we set $\tilde a_i(x,\xi)=a_i(x,\xi)\chi_\Omega(x)+\dfrac{1}{\epsilon^{p_i(x)}}|\xi|^{p_i(x)-2}\xi\chi_{\tilde\Omega\setminus\Omega}(x)$ for all $(x,\xi)\in\tilde\Omega\times\mathbb{R}^N$ and we consider the following problem. $P_\epsilon(\tilde\rho,\tilde\mu_\epsilon,\tilde d_\epsilon)$

$$\begin{cases} -\displaystyle\sum_{i=1}^N\frac{\partial}{\partial x_i}\left(a_i\left(x,\frac{\partial}{\partial x_i}u_\epsilon\right)\chi_\Omega(x)+\frac{1}{\epsilon^{p_i(x)}}\left|\frac{\partial}{\partial x_i}u_\epsilon\right|^{p_i(x)-2}\frac{\partial}{\partial x_i}u_\epsilon\chi_{\tilde\Omega\setminus\Omega}(x)\right)+ & \\ |u_\epsilon|^{p_M(x)-2}u_\epsilon\chi_\Omega=\tilde\mu_\epsilon & \text{in } \tilde\Omega\\ u_\epsilon=0 & \text{on } \Gamma_D\\ \tilde\rho(u_\epsilon)+\displaystyle\sum_{i=1}^N\tilde a_i(x,\frac{\partial}{\partial x_i}u_\epsilon)\eta_i=\tilde d_\epsilon & \text{on } \tilde\Gamma_{Ne}.\end{cases}\tag{3.18}$$

Thanks to Theorem 3.2, $P_\epsilon(\tilde\rho,\tilde\mu_\epsilon,\tilde d_\epsilon)$ has at least one solution. So, there exists at least one measurable function $u_\epsilon:\tilde\Omega\to\mathbb{R}$ such that

$$\begin{cases}\displaystyle\sum_{i=1}^N\int_\Omega a_i\left(x,\frac{\partial}{\partial x_i}u_\epsilon\right)\frac{\partial}{\partial x_i}\tilde\xi dx+\sum_{i=1}^N\int_{\tilde\Omega\setminus\Omega}\left(\frac{1}{\epsilon^{p_i(x)}}|\frac{\partial}{\partial x_i}u_\epsilon|^{p_i(x)-2}\frac{\partial}{\partial x_i}u_\epsilon.\frac{\partial}{\partial x_i}\tilde\xi\right)dx\\ +\displaystyle\int_\Omega|u_\epsilon|^{p_M(x)-2}u_\epsilon\tilde\xi dx=\int_\Omega\tilde\xi d\mu_\epsilon+\int_{\tilde\Gamma_{Ne}}(\tilde d_\epsilon-\tilde\rho(u_\epsilon)\tilde\xi d\sigma,\end{cases}\tag{3.19}$$

where $u_\epsilon\in W_D^{1,\vec p(.)}(\tilde\Omega)$ and $\tilde\xi\in W_D^{1,\vec p(.)}(\tilde\Omega)\cap L^\infty(\Omega)$.

Moreover $u_\epsilon\in L^\infty(\Omega)\cap L^\infty(\tilde\Gamma_{Ne})$.

Our aim is to prove that these approximated solutions u_ϵ tend, as ϵ goes to 0, to a measurable function u which is an entropy solution of the problem $P(\tilde\rho,\tilde\mu,\tilde d)$. To start with, we establish some a priori estimates.

Proposition 3.6. *Let u_ϵ be a solution of the problem $P_\epsilon(\tilde\rho,\tilde\mu_\epsilon,\tilde d_\epsilon)$. Then, the following statements hold.*

(i) $\forall k>0$,

$$\sum_{i=1}^N\int_\Omega\left|\frac{\partial}{\partial x_i}T_k(u_\epsilon)\right|^{p_i(x)}dx+\sum_{i=1}^N\int_{\tilde\Omega\setminus\Omega}\left(\frac{1}{\epsilon}\left|\frac{\partial}{\partial x_i}T_k(u_\epsilon)\right|\right)^{p_i(x)}dx\le k(\|\tilde d\|_{L^1(\tilde\Gamma_{Ne})}+|\mu|(\Omega));$$

(ii)
$$\int_\Omega |u_\epsilon|^{p_M(x)-1}dx + \int_{\tilde{\Gamma}_{Ne}} |\tilde{\rho}(u_\epsilon)|dx \leq (\|\tilde{d}\|_{L^1(\tilde{\Gamma}_{Ne})} + |\mu|(\Omega));$$

(iii) $\forall k > 0$,
$$\sum_{i=1}^N \int_{\tilde{\Omega}} \left|\frac{\partial}{\partial x_i} T_k(u_\epsilon)\right|^{p_i(x)} dx \leq k(\|\tilde{d}\|_{L^1(\tilde{\Gamma}_{Ne})} + |\mu|(\Omega)).$$

Proof. For any $k > 0$, we set $\tilde{\xi} = T_k(u_\epsilon)$ in (3.19), to get

$$\begin{cases} \displaystyle\sum_{i=1}^N \int_\Omega \left(a_i\left(x, \frac{\partial}{\partial x_i}u_\epsilon\right)\frac{\partial}{\partial x_i}T_k(u_\epsilon)\right)dx + \sum_{i=1}^N \int_{\tilde{\Omega}\setminus\Omega}\left(\frac{1}{\epsilon^{p_i(x)}}\left|\frac{\partial}{\partial x_i}u_\epsilon\right|^{p_i(x)-2}\frac{\partial}{\partial x_i}u_\epsilon\frac{\partial}{\partial x_i}T_k(u_\epsilon)\right)dx \\ \displaystyle\int_\Omega |u_\epsilon|^{p_M(x)-2}u_\epsilon T_k(u_\epsilon)dx = \int_\Omega T_k(u_\epsilon)d\mu_\epsilon + \int_{\tilde{\Gamma}_{Ne}}(\tilde{d}_\epsilon - \tilde{\rho}(u_\epsilon))T_k(u_\epsilon)d\sigma. \end{cases} \tag{3.20}$$

(i) Obviously, we have
$$\sum_{i=1}^N \int_{\tilde{\Omega}\setminus\Omega}\left(\frac{1}{\epsilon^{p_i(x)}}\left|\frac{\partial}{\partial x_i}u_\epsilon\right|^{p_i(x)-2}\frac{\partial}{\partial x_i}u_\epsilon\frac{\partial}{\partial x_i}T_k(u_\epsilon)\right)dx = \sum_{i=1}^N \int_{\tilde{\Omega}\setminus\Omega}\left(\frac{1}{\epsilon^{p_i(x)}}\left|\frac{\partial}{\partial x_i}T_k(u_\epsilon)\right|^{p_i(x)}\right)dx \geq 0,$$
$\displaystyle\int_{\tilde{\Gamma}_{Ne}} \tilde{\rho}(u_\epsilon)T_k(u_\epsilon)d\sigma \geq 0$ and $\displaystyle\int_\Omega |u_\epsilon|^{p_M(x)-2}u_\epsilon T_k(u_\epsilon)dx \geq 0$.
Moreover,
$$\begin{cases} \displaystyle\int_\Omega T_k(u_\epsilon)d\mu_\epsilon + \int_{\tilde{\Gamma}_{Ne}}\tilde{d}_\epsilon T_k(u_\epsilon)d\sigma & \displaystyle\leq k\int_\Omega d\mu_\epsilon + k\int_{\tilde{\Gamma}_{Ne}}|\tilde{d}_\epsilon|d\sigma \\ & \displaystyle\leq k\left(|\mu|(\Omega) + \int_{\tilde{\Gamma}_{Ne}}|\tilde{d}|d\sigma\right). \end{cases} \tag{3.21}$$
Using the inequalities above and (1.7), it follows that
$$\sum_{i=1}^N \int_\Omega \left|\frac{\partial T_k(u_\epsilon)}{\partial x_i}\right|^{p_i(x)} dx \leq k\left(|\mu|(\Omega) + \int_{\tilde{\Gamma}_{Ne}}|\tilde{d}|d\sigma\right). \tag{3.22}$$

As $\displaystyle\sum_{i=1}^N \int_\Omega \left(a_i\left(x, \frac{\partial}{\partial x_i}u_\epsilon\right)\frac{\partial}{\partial x_i}T_k(u_\epsilon)\right)dx \geq 0$, $\displaystyle\int_{\tilde{\Gamma}_{Ne}}\tilde{\rho}(u_\epsilon)T_k(u_\epsilon)d\sigma \geq 0$ and
$\displaystyle\int_\Omega |u_\epsilon|^{p_M(x)-2}u_\epsilon T_k(u_\epsilon)dx \geq 0$, therefore, we get from (3.20),
$$\sum_{i=1}^N \int_{\tilde{\Omega}\setminus\Omega}\left(\frac{1}{\epsilon^{p_i(x)}}|\frac{\partial}{\partial x_i}T_k(u_\epsilon)|^{p_i(x)}\right)dx \leq k\left(|\mu|(\Omega) + \int_{\tilde{\Gamma}_{Ne}}|\tilde{d}|d\sigma\right) \tag{3.23}$$
Adding (3.22) and (3.23), we obtain *(i)*.

(ii) The first two terms in (3.20) are non-negative and using (3.21), we have from (3.20) the following
$$\int_{\tilde{\Gamma}_{Ne}}\tilde{\rho}(u_\epsilon)T_k(u_\epsilon)d\sigma + \int_\Omega |u_\epsilon|^{p_M(x)-2}u_\epsilon T_k(u_\epsilon)dx \leq k\left(|\mu|(\Omega) + \int_{\tilde{\Gamma}_{Ne}}|\tilde{d}|d\sigma\right).$$
We divide the above inequality by $k > 0$ and let k go to zero, to get
$$\begin{aligned} \int_{\tilde{\Gamma}_{Ne}}\tilde{\rho}(u_\epsilon)sign(u_\epsilon)d\sigma + \int_\Omega |u_\epsilon|^{p_M(x)-2}u_\epsilon sign(u_\epsilon)dx &= \int_{\tilde{\Gamma}_{Ne}}|\tilde{\rho}(u_\epsilon)|d\sigma + \int_\Omega |u_\epsilon|^{p_M(x)-1}dx \\ &\leq \left(|\mu|(\Omega) + \int_{\tilde{\Gamma}_{Ne}}|\tilde{d}|d\sigma\right). \end{aligned}$$

(iii) For all $k > 0$, we have

$$\sum_{i=1}^{N} \int_{\tilde{\Omega}} \left| \frac{\partial}{\partial x_i} T_k(u_\epsilon) \right|^{p_i(x)} dx \leq \sum_{i=1}^{N} \int_{\Omega} \left| \frac{\partial}{\partial x_i} T_k(u_\epsilon) \right|^{p_i(x)} dx + \sum_{i=1}^{N} \int_{\tilde{\Omega}\setminus\Omega} \left| \frac{1}{\epsilon} \frac{\partial}{\partial x_i} T_k(u_\epsilon) \right|^{p_i(x)} dx,$$

for any $0 < \epsilon < 1$. According to (i), we deduce that

$$\sum_{i=1}^{N} \int_{\tilde{\Omega}} \left| \frac{\partial}{\partial x_i} T_k(u_\epsilon) \right|^{p_i(x)} dx \leq k \left(|\mu|(\Omega) + \int_{\tilde{\Gamma}_{Ne}} |\tilde{d}| d\sigma \right).$$

□

Lemma 3.7. *There is a positive constant D such that*

$$meas\{|u_\epsilon| > k\} \leq D^{p_m^-} \frac{(1+k)}{k^{p_m^- - 1}},\ \forall k > 0.$$

Proof. Let $k > 0$; by using Proposition 3.6-*(iii)*, we have

$$\begin{aligned}
\sum_{i=1}^{N} \int_{\tilde{\Omega}} \left| \frac{\partial T_k(u_\epsilon)}{\partial x_i} \right|^{p_m^-(x)} dx &\leq \sum_{i=1}^{N} \int_{\left\{ \left| \frac{\partial T_k(u_\epsilon)}{\partial x_i} \right| > 1 \right\}} \left| \frac{\partial T_k(u_\epsilon)}{\partial x_i} \right|^{p_m^-(x)} dx + Nmeas(\tilde{\Omega}) \\
&\leq \sum_{i=1}^{N} \int_{\tilde{\Omega}} \left| \frac{\partial T_k(u_\epsilon)}{\partial x_i} \right|^{p_i(x)} dx + Nmeas(\tilde{\Omega}) \\
&\leq k \left(|\mu|(\Omega) + \int_{\tilde{\Gamma}_{Ne}} |\tilde{d}| d\sigma \right) + Nmeas(\tilde{\Omega}) \\
&\leq C'(k+1),
\end{aligned}$$

with $C' = max \left(\left(|\mu|(\Omega) + \int_{\tilde{\Gamma}_{Ne}} |\tilde{d}| d\sigma \right); Nmeas(\tilde{\Omega}) \right)$.
We can write the above inequality as

$$\sum_{i=1}^{N} \left\| \frac{\partial T_k(u_\epsilon)}{\partial x_i} \right\|_{p_m^-}^{p_m^-} \leq C'(1+k) \text{ or } \|T_k(u_\epsilon)\|_{W_D^{1,p_m^-}(\tilde{\Omega})} \leq [C'(1+k)]^{\frac{1}{p_m^-}}.$$

By the Poincaré inequality in constant exponent, we obtain

$$\|T_k(u_\epsilon)\|_{L^{p_m^-}(\tilde{\Omega})} \leq D(1+k)^{\frac{1}{p_m^-}}.$$

The above inequality implies that

$$\int_{\tilde{\Omega}} |T_k(u_\epsilon)|^{p_m^-} dx \leq D^{p_m^-}(1+k),$$

from which we obtain

$$meas\{|u_\epsilon| > k\} \leq D^{p_m^-} \frac{(1+k)}{k^{p_m^-}},$$

since

$$\int_{\tilde{\Omega}} |T_k(u_\epsilon)|^{p_m^-} dx = \int_{\{|u_\epsilon| > k\}} |T_k(u_\epsilon)|^{p_m^-} dx + \int_{\{|u_\epsilon| \leq k\}} |T_k(u_\epsilon)|^{p_m^-} dx,$$

we get

$$\int_{\{|u_\epsilon|>k\}} |T_k(u_\epsilon)|^{p_m^-} dx \leq \int_{\tilde{\Omega}} |T_k(u_\epsilon)|^{p_m^-} dx$$

and

$$k^{p_m^-} meas\,\{|u_\epsilon| > k\} \leq \int_{\tilde{\Omega}} |T_k(u_\epsilon)|^{p_m^-} dx \leq D^{p_m^-}(1+k)$$

□

Lemma 3.8. *There is a positive constant C such that*

$$\sum_{i=1}^{N} \int_{\tilde{\Omega}} \left(\left| \frac{\partial}{\partial x_i} T_k(u_\epsilon) \right|^{p_i^-} \right) dx \leq C(k+1),\ \forall k > 0. \tag{3.24}$$

Proof. Let $k > 0$, we set $\Omega_1 = \left\{ |u| \leq k; \left| \frac{\partial}{\partial x_i} u_\epsilon \right| \leq 1 \right\}$ and $\Omega_2 = \left\{ |u| \leq k; \left| \frac{\partial}{\partial x_i} u_\epsilon \right| > 1 \right\}$; using Proposition 3.6-*(iii)*, we have

$$\begin{aligned}
\sum_{i=1}^{N} \int_{\tilde{\Omega}} \left(\left| \frac{\partial}{\partial x_i} T_k(u_\epsilon) \right|^{p_i^-} \right) dx &= \sum_{i=1}^{N} \int_{\Omega_1} \left(\left| \frac{\partial}{\partial x_i} T_k(u_\epsilon) \right|^{p_i^-} \right) dx + \sum_{i=1}^{N} \int_{\Omega_2} \left(\left| \frac{\partial}{\partial x_i} T_k(u_\epsilon) \right|^{p_i^-} \right) dx \\
&\leq N meas(\tilde{\Omega}) + \sum_{i=1}^{N} \int_{\tilde{\Omega}} \left(\left| \frac{\partial}{\partial x_i} T_k(u_\epsilon) \right|^{p_i(x)} \right) dx \\
&\leq N meas(\tilde{\Omega}) + k \left(|\mu|(\Omega) + \|\tilde{d}\|_{L^1(\tilde{\Gamma}_{Ne})} \right) \leq C(k+1),
\end{aligned}$$

with $C = \max \left\{ N meas(\tilde{\Omega}); \left(|\mu|(\Omega) + \|\tilde{d}\|_{L^1(\tilde{\Gamma}_{Ne})} \right) \right\}$. □

Lemma 3.9. *For all $k > 0$, there is two constants C_1 and C_2 such that*

(i) $\|u_\epsilon\|_{\mathcal{M}^{q^*}(\tilde{\Omega})} \leq C_1$;

(ii) $\left\| \frac{\partial}{\partial x_i} u_\epsilon \right\|_{\mathcal{M}^{p_i^- q/p}(\tilde{\Omega})} \leq C_2$.

Proof. *(i)* By Lemma 3.8, we have

$$\sum_{i=1}^{N} \int_{\tilde{\Omega}} \left| \frac{\partial}{\partial x_i} T_k(u_\epsilon) \right|^{p_i^-} dx \leq C(1+k),\ \forall k > 0 \text{ and } i = 1, ..., N.$$

• If $k > 1$, we have

$$\sum_{i=1}^{N} \int_{\tilde{\Omega}} \left| \frac{\partial}{\partial x_i} T_k(u_\epsilon) \right|^{p_i^-} dx \leq C' k,$$

which means $T_k(u_\epsilon) \in W^{1,(p_1^-,...,p_N^-)}(\tilde{\Omega})$. Using relation (2.8), we deduce that

$$\|T_k(u_\epsilon)\|_{L^{(\bar{p})^*}(\tilde{\Omega}} \leq C_1 \prod_{i=1}^{N} \left\| \frac{\partial}{\partial x_i} T_k(u_\epsilon) \right\|_{L^{p_i^-}(\tilde{\Omega})}^{\frac{1}{N}}.$$

So,

$$\begin{aligned}
\int_{\tilde{\Omega}} |T_k(u_\epsilon)|^{(\bar{p})^*} dx &\leq C \left[\prod_{i=1}^{N} \left(\int_{\tilde{\Omega}} \left| \frac{\partial}{\partial x_i} T_k(u_\epsilon) \right|^{p_i^-} dx \right)^{\dfrac{1}{Np_i^-}} \right]^{(\bar{p})^*} \\
&\leq C'' \left[\prod_{i=1}^{N} (k)^{\dfrac{1}{Np_i^-}} \right]^{(\bar{p})^*} \\
&\leq C'' \left[k^{\sum_{i=1}^{N} \dfrac{1}{Np_i^-}} \right]^{(\bar{p})^*} \\
&\leq C'' k^{\dfrac{(\bar{p})^*}{\bar{p}}}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\int_{\{|u_\epsilon|>k\}} |T_k(u_\epsilon)|^{(\bar{p})^*} dx &\leq \int_{\tilde{\Omega}} |T_k(u_\epsilon)|^{(\bar{p})^*} dx \\
&\leq C' k^{\dfrac{(\bar{p})^*}{\bar{p}}}
\end{aligned}$$

and so,

$$(k)^{(\bar{p})^*} meas\{x \in \tilde{\Omega} : |u_\epsilon| > k\} \leq C' k^{\dfrac{(\bar{p})^*}{\bar{p}}};$$

which means that

$$\lambda_{u_\epsilon}(k) \leq C' k^{(\bar{p})^*(\dfrac{1}{\bar{p}}-1)} = C' k^{-q^*}, \; \forall k \geq 1.$$

- If $0 < k < 1$, we have

$$\begin{aligned}
\lambda_{u_\epsilon}(k) &= meas\left\{ x \in \tilde{\Omega} : |u_\epsilon| > k \right\} \\
&\leq meas(\tilde{\Omega}) \\
&\leq meas(\tilde{\Omega}) k^{-q^*}.
\end{aligned}$$

So,

$$\lambda_{u_\epsilon}(k) \leq (C' + meas(\tilde{\Omega})) k^{-q^*} = C_1 k^{-q^*}.$$

Therefore,

$$\|u_\epsilon\|_{\mathcal{M}^{q^*}(\tilde{\Omega})} \leq C_1.$$

(ii) • Let $\alpha \geq 1$. For all $k \geq 1$, we have

$$\begin{array}{rcl}
\lambda_{\frac{\partial u_\epsilon}{\partial x_i}}(\alpha) & = & meas\left(\left\{\left|\frac{\partial u_\epsilon}{\partial x_i}\right| > \alpha\right\}\right) \\
& = & meas\left(\left\{\left|\frac{\partial u_\epsilon}{\partial x_i}\right| > \alpha; |u_\epsilon| \leq k\right\}\right) + meas\left(\left\{\left|\frac{\partial u_\epsilon}{\partial x_i}\right| > \alpha;; |u_\epsilon| > k\right\}\right) \\
& \leq & \int_{\left\{\left|\frac{\partial u_\epsilon}{\partial x_i}\right|>\alpha; |u_\epsilon|\leq k\right\}} dx + \lambda_{u_\epsilon}(k) \\
& \leq & \int_{\{|u_\epsilon|\leq k\}} \left(\frac{1}{\alpha}\left|\frac{\partial u_\epsilon}{\partial x_i}\right|\right)^{p_i^-} dx + \lambda_{u_\epsilon}(k) \\
& \leq & \alpha^{-p_i^-} C'k + Ck^{-q^*} \\
& \leq & B\left(\alpha^{-p_i^-} k + k^{-q^*}\right),
\end{array}$$

with $B = \max(C'; C)$.
Let $g : [1; \infty) \to \mathbb{R}$, $x \mapsto g(x) = \dfrac{x}{\alpha^{p_i^-}} + x^{-q^*}$.

We have $g'(x) = 0$ with $x = \left(q^* \alpha^{p^-}\right)^{\frac{1}{q^*+1}}$.

We set $k = \left(q^* \alpha^{p_i^-}\right)^{\frac{1}{q^*+1}} \geq 1$ in the above inequality to get,

$$\begin{array}{rcl}
\lambda_{\frac{\partial u_\epsilon}{\partial x_i}}(\alpha) & \leq & B\left[\alpha^{-p_i^-} \times \left(q^* \alpha^{p_i^-}\right)^{\frac{1}{q^*+1}} + \left(q^* \alpha^{p_i^-}\right)^{\frac{-q^*}{q^*+1}}\right] \\
& \leq & B\left[(q^*)^{\frac{1}{q^*+1}} \times \alpha^{-p_i^-\left(1-\frac{1}{q^*+1}\right)} + (q^*)^{\frac{-q^*}{q^*+1}} \times \alpha^{\frac{-p_i^- q^*}{q^*+1}}\right] \\
& \leq & B\left[(q^*)^{\frac{1}{q^*+1}} \times \alpha^{-p_i^-\left(\frac{q^*}{q^*+1}\right)} + (q^*)^{\frac{-q^*}{q^*+1}} \times \alpha^{\frac{-p_i^- q^*}{q^*+1}}\right] \\
& \leq & M\alpha^{-p_i^- \frac{q^*}{q^*+1}} \\
& \leq & M\alpha^{-p_i^- \frac{q}{\bar{p}}},
\end{array}$$

where $M = B \times \max\left((q^*)^{\frac{1}{q^*+1}}; (q^*)^{\frac{-q^*}{q^*+1}}\right)$ and as $q^* = \dfrac{N(\bar{p}-1)}{N-\bar{p}}$, $q = \dfrac{N(\bar{p}-1)}{N-1}$.

So,

$$\begin{aligned}\frac{q^*}{q^*+1} &= \frac{q^*(N-\bar{p})}{N(\bar{p}-1)+N-\bar{p}}\\ &= \frac{q^*(N-\bar{p})}{N\bar{p}-\bar{p}}\\ &= \frac{N(\bar{p}-1)}{(N-1)\bar{p}}\\ &= \frac{q}{\bar{p}}.\end{aligned}$$

• If $0 \leq \alpha < 1$, we have.

$$\begin{aligned}\lambda_{\frac{\partial u_\epsilon}{\partial x_i}}(\alpha) &= meas\left(\left\{x \in \tilde{\Omega} : \left|\frac{\partial u_\epsilon}{\partial x_i}\right| > \alpha\right\}\right)\\ &\leq meas(\tilde{\Omega})\alpha^{-p_i^-\frac{q}{\bar{p}}}.\end{aligned}$$

Therefore,

$$\lambda_{\frac{\partial u_\epsilon}{\partial x_i}}(\alpha) \leq \left(M + meas(\tilde{\Omega})\right)\alpha^{-p_i^-\frac{q}{\bar{p}}},\ \forall\, \alpha \geq 0.$$

So,

$$\left\|\frac{\partial u_\epsilon}{\partial x_i}\right\|_H \leq C_2,$$

where $H = \mathcal{M}(\tilde{\Omega})^{\frac{p_i^- q}{\bar{p}}}$ □

Proposition 3.10. *Let u_ϵ be a solution of the problem $P(\tilde{\rho}, \tilde{\mu}_\epsilon, \tilde{d}_\epsilon)$. Then,*

(i) $u_\epsilon \to u$ in measure, a.e. in Ω and a.e. on $\tilde{\Gamma}_N$;

(ii) For all $i = 1, ...N$, $\dfrac{\partial T_k(u_\epsilon)}{\partial x_i} \rightharpoonup \dfrac{\partial T_k(u)}{\partial x_i} = 0$ in $L^{p_i^-}(\tilde{\Omega} \setminus \Omega)$.

Proof. (i) By Proposition 3.6 (i), we deduce that $(T_k(u_\epsilon))_{\epsilon>0}$ is bounded in $W_D^{1,\vec{p}(.)}(\tilde{\Omega}) \hookrightarrow L^{p_m(.)}(\tilde{\Omega}) \hookrightarrow L^{p_m^-}(\tilde{\Omega})$ (with compact embedding). Therefore, up to a subsequence, we can assume that as $\epsilon \to 0$, $(T_k(u_\epsilon))_{\epsilon>0}$ converges strongly to some function σ_k in $L^{p_m^-}(\tilde{\Omega})$, $a.e.$ in $\tilde{\Omega}$ and $a.e.$ on $\tilde{\Gamma}_{Ne}$.

Let us see that the sequence $(u_\epsilon)_{\epsilon>0}$ is Cauchy in measure.

Indeed, let $s > 0$ and define:

$E_1 = [|u_{\epsilon_1}| > k]$, $E_2 = [|u_{\epsilon_2}| > k]$ and $E_3 = [|T_k(u_{\epsilon_1}) - T_k(u_{\epsilon_2})| > s]$,

where $k > 0$ is fixed. We note that

$$[|u_{\epsilon_1} - u_{\epsilon_2}| > s] \subset E_1 \cup E_2 \cup E_3;$$

hence,

$$meas([|u_{\epsilon_1} - u_{\epsilon_2}| > s]) \leq \sum_{i=1}^{3} meas(E_i). \tag{3.25}$$

Let $\theta > 0$, using Lemma 3.7, we choose $k = k(\theta)$ such that

$$meas(E_1) \leq \frac{\theta}{3} \text{ and } meas(E_2) \leq \frac{\theta}{3}. \tag{3.26}$$

Since $(T_k(u_\epsilon))_{\epsilon>0}$ converges strongly in $L^{p_m^-}(\tilde{\Omega})$, then, it is a Cauchy sequence in $L^{p_m^-}(\tilde{\Omega})$. Thus,

$$meas(E_3) \leq \frac{1}{s^{p_m^-}} \int_\Omega |T_k(u_{\epsilon_1}) - T_k(u_{\epsilon_2})|^{p_m^-} dx \leq \frac{\theta}{3}, \tag{3.27}$$

for all $\epsilon_1, \epsilon_2 \geq n_0(s, \theta)$. Finally, from (3.25), (3.26) and (3.27), we obtain

$$meas([|u_{\epsilon_1} - u_{\epsilon_2}| > s]) \leq \theta \text{ for all } \epsilon_1, \epsilon_2 \geq n_0(s, \theta); \tag{3.28}$$

which means that the sequence $(u_\epsilon)_{\epsilon>0}$ is Cauchy in measure, so $u_\epsilon \to u$ in measure and up to a subsequence, we have $u_\epsilon \to u$ *a.e.* in $\tilde{\Omega}$. Hence, $\sigma_k = T_k(u)$ *a.e.* in $\tilde{\Omega}$ and so, $u \in \mathcal{T}_D^{1,\vec{p}(.)}(\Omega)$.

(ii) According to the proof of (i), we have $T_k(u_\epsilon) \rightharpoonup T_k(u)$ in $W_D^{1,\vec{p}(.)}(\tilde{\Omega}) \hookrightarrow W_D^{1,\vec{p}_-}(\tilde{\Omega})$ which implies on one hand that for all $i = 1, ...N$, $\dfrac{\partial T_k(u_\epsilon)}{\partial x_i} \rightharpoonup \dfrac{\partial T_k(u)}{\partial x_i}$ in $L^{p_i(.)}(\tilde{\Omega})$ and on the other hand that for all $i = 1, ...N$, $\dfrac{\partial T_k(u_\epsilon)}{\partial x_i} \rightharpoonup \dfrac{\partial T_k(u)}{\partial x_i}$ in $L^{p_i(.)}(\tilde{\Omega})$ and then for all $i = 1, ...N$, $\dfrac{\partial T_k(u_\epsilon)}{\partial x_i} \rightharpoonup \dfrac{\partial T_k(u)}{\partial x_i}$ in $L^{p_i^-}(\tilde{\Omega} \setminus \Omega)$.

Let $i = 1, ..., N$, by Proposition 3.6-(i), we can assert that $\left(\dfrac{1}{\epsilon}\dfrac{\partial T_k(u_\epsilon)}{\partial x_i}\right)_{\epsilon>0}$ is bounded in $L^{p_i^-}(\tilde{\Omega} \setminus \Omega)$. Indeed, let $k > 0$, we set $\Omega^1 = \left\{x \in \tilde{\Omega} \setminus \Omega; |u(x)| \leq k; \left|\dfrac{\partial}{\partial x_i} u_\epsilon(x)\right| \leq \epsilon\right\}$ and $\Omega^2 = \left\{x \in \tilde{\Omega} \setminus \Omega; |u| \leq k; \left|\dfrac{\partial}{\partial x_i} u_\epsilon(x)\right| > \epsilon\right\}$; using Proposition 3.6-*(i)*, we have

$$\begin{aligned}
& \sum_{i=1}^{N} \int_{\tilde{\Omega}\setminus\Omega} \left(\frac{1}{\epsilon}\left|\frac{\partial T_k(u_\epsilon)}{\partial x_i}\right|^{p_i^-}\right) dx \\
= \; & \sum_{i=1}^{N} \int_{\Omega^1} \left(\frac{1}{\epsilon}\left|\frac{\partial T_k(u_\epsilon)}{\partial x_i}\right|^{p_i^-}\right) dx + \sum_{i=1}^{N} \int_{\Omega^2} \left(\frac{1}{\epsilon}\left|\frac{\partial}{\partial x_i} T_k(u_\epsilon)\right|^{p_i^-}\right) dx \\
\leq \; & N meas(\tilde{\Omega} \setminus \Omega) + \sum_{i=1}^{N} \int_{\tilde{\Omega}\setminus\Omega} \left(\frac{1}{\epsilon}\left|\frac{\partial}{\partial x_i} T_k(u_\epsilon)\right|^{p_i(x)}\right) dx \\
\leq \; & N meas(\tilde{\Omega} \setminus \Omega) + k\left(|\mu|(\Omega) + \|\tilde{d}\|_{L^1(\tilde{\Gamma}_{Ne})}\right) \leq C'(k+1),
\end{aligned}$$

with $C' = \max\left\{N meas(\tilde{\Omega} \setminus \Omega); \left(|\mu|(\Omega) + \|\tilde{d}\|_{L^1(\tilde{\Gamma}_{Ne})}\right)\right\}$. To end, we have

$$\int_{\tilde{\Omega}\setminus\Omega} \left(\frac{1}{\epsilon}\left|\frac{\partial T_k(u_\epsilon)}{\partial x_i}\right|^{p_i^-}\right) dx \leq \sum_{i=1}^{N} \int_{\tilde{\Omega}\setminus\Omega} \left(\frac{1}{\epsilon}\left|\frac{\partial T_k(u_\epsilon)}{\partial x_i}\right|^{p_i^-}\right) dx, \text{ for any} i = 1, \ldots, N.$$

Therefore, there exists $\Theta_k \in L^{p_i^-}(\tilde{\Omega} \setminus \Omega)$ such that

$$\frac{1}{\epsilon}\frac{\partial T_k(u_\epsilon)}{\partial x_i} \rightharpoonup \Theta_k \text{ in } L^{p_i^-}(\tilde{\Omega} \setminus \Omega) \text{ as } \epsilon \to 0.$$

For any $\psi \in L^{(p_i')^-}(\tilde{\Omega} \setminus \Omega)$, we have

$$\int_{\tilde{\Omega}\setminus\Omega} \frac{\partial T_k(u_\epsilon)}{\partial x_i}\psi dx = \int_{\tilde{\Omega}\setminus\Omega} \left(\frac{1}{\epsilon}\frac{\partial T_k(u_\epsilon)}{\partial x_i} - \Theta_k\right)(\epsilon\psi)dx + \epsilon\int_{\tilde{\Omega}\setminus\Omega} \Theta_k\psi dx. \tag{3.29}$$

As $(\epsilon\psi)_{\epsilon>0}$ converges strongly to zero in $L^{(p_i')^-}(\tilde{\Omega}\setminus\Omega)$, we pass to the limit as $\epsilon \to 0$ in (3.29), to get

$$\frac{\partial T_k(u_\epsilon)}{\partial x_i} \rightharpoonup 0 \text{ in } L^{p_i^-}(\tilde{\Omega} \setminus \Omega).$$

Hence, one has

$$\frac{\partial T_k(u_\epsilon)}{\partial x_i} \rightharpoonup \frac{\partial T_k(u)}{\partial x_i} = 0 \text{ in } L^{p_i^-}(\tilde{\Omega} \setminus \Omega),$$

for any $i = 1, ..., N$.

□

Lemma 3.11. *$b(u) \in L^1(\Omega)$ and $\tilde{\rho}(u) \in L^1(\tilde{\Gamma}_{Ne})$.*

Proof. Having in mind that by Proposition 3.6-*(ii)*,

$$\int_\Omega |b(u_\epsilon)|dx + \int_{\tilde{\Gamma}_{Ne}} |\tilde{\rho}(u_\epsilon)|d\sigma \le (|\mu|(\Omega) + \|\tilde{d}\|_{L^1(\tilde{\Gamma}_{Ne})}),$$

we deduce that

$$\int_\Omega |b(u_\epsilon)|dx \le (|\mu|(\Omega) + \|\tilde{d}\|_{L^1(\tilde{\Gamma}_{Ne})}) \tag{3.30}$$

and

$$\int_{\tilde{\Gamma}_{Ne}} |\tilde{\rho}(u_\epsilon)|d\sigma \le (|\mu|(\Omega) + \|\tilde{d}\|_{L^1(\tilde{\Gamma}_{Ne})}). \tag{3.31}$$

By Fatou's lemma, the continuity of b, $\tilde{\rho}$ and using Proposition 3.10, we have

$$\liminf_{\epsilon\to 0} \int_\Omega |b(u_\epsilon)|dx \ge \int_\Omega |b(u)|dx \tag{3.32}$$

and

$$\liminf_{\epsilon\to 0} \int_{\tilde{\Gamma}_{Ne}} |\tilde{\rho}(u_\epsilon)|d\sigma \ge \int_{\tilde{\Gamma}_{Ne}} |\tilde{\rho}(u)|d\sigma. \tag{3.33}$$

Using (3.30)-(3.33), we deduce that

$$\int_\Omega |b(u)|dx \le (|\mu|(\Omega) + \|\tilde{d}\|_{L^1(\tilde{\Gamma}_{Ne})})$$

and

$$\int_{\tilde{\Gamma}_{Ne}} |\tilde{\rho}(u)|d\sigma \le (|\mu|(\Omega) + \|\tilde{d}\|_{L^1(\tilde{\Gamma}_{Ne})}).$$

Therefore, $b(u) \in L^1(\Omega)$ and $\tilde{\rho}(u) \in L^1(\tilde{\Gamma}_{Ne})$. □

Lemma 3.12. *Assume* (1.4)-(1.8) *hold and* u_ϵ *be a weak solution of the problem* $P(\rho, \tilde{\mu}_\epsilon, \tilde{d}_\epsilon)$. *Then,*

(i) $\dfrac{\partial}{\partial x_i} u_\epsilon$ *converges in measure to* $\dfrac{\partial}{\partial x_i} u$.

(ii) $a_i\left(x, \dfrac{\partial T_k(u_\epsilon)}{\partial x_i}) \to a_i(x, \dfrac{\partial T_k(u)}{\partial x_i}\right)$ *strongly in* $L^1(\Omega)$ *and weakly in* $L^{p_i'(\cdot)}(\Omega)$, *for all i=1,...,N.*

In order to give the proof of Lemma 3.12, we need the following lemmas.

Lemma 3.13 (Cf [6])**.** *Let* $u \in \mathcal{T}^{1,\vec{p}(\cdot)}(\Omega)$. *Then, there exists a unique measurable function* $\nu_i : \Omega \to \mathbb{R}$ *such that*

$$\nu_i \chi_{\{|u|<k\}} = \frac{\partial}{\partial x_i} T_k(u) \text{ for a.e. } x \in \Omega,\ \forall k > 0 \text{ and } i = 1, ..., N;$$

where χ_A *denotes the characteristic function of a measurable set* A.
The functions ν_i *are denoted* $\dfrac{\partial}{\partial x_i} u$. *Moreover, if* u *belongs to* $W^{1,\vec{p}(\cdot)}(\Omega)$, *then* $\nu_i \in L^{p_i(\cdot)}(\Omega)$ *and coincides with the standard distributional gradient of* u *i.e.* $\nu_i = \dfrac{\partial}{\partial x_i} u$.

Lemma 3.14 (Cf [37], lemma 5.4)**.** *Let* $(v_n)_{n\in\mathbb{N}}$ *be a sequence of measurable functions. If* v_n *converges in measure to* v *and is uniformly bounded in* $L^{p(\cdot)}(\Omega)$ *for some* $1 << p(.) \in L^\infty(\Omega)$, *then* $v_n \to v$ *strongly in* $L^1(\Omega)$.

The third technical lemma is a standard fact in measure theory (Cf [16]).

Lemma 3.15. *Let* $(X, \mathcal{M}, \mu)$ *be a measurable space such that* $\mu(X) < \infty$.
Consider a measurable function $\gamma : X \to [0; \infty]$ *such that*

$$\mu(\{x \in X : \gamma(x) = 0\}) = 0.$$

Then, for every $\epsilon > 0$, *there exists* δ *such that*

$$\mu(A) < \epsilon, \text{ for all } A \in \mathcal{M} \text{ with } \int_A \gamma dx < \delta.$$

Proof of Lemma 3.12. (i) We claim that $\left(\dfrac{\partial}{\partial x_i} u_\epsilon\right)_{\epsilon\in\mathbb{N}}$ is Cauchy in measure. Indeed, let $s > 0$, consider
$A_{n,m} := \left\{\left|\dfrac{\partial}{\partial x_i} u_n\right| > h\right\} \cup \left\{\left|\dfrac{\partial}{\partial x_i} u_m\right| > h\right\}$, $B_{n,m} := \{|u_n - u_m| > k\}$ and
$C_{n,m} := \left\{\left|\dfrac{\partial}{\partial x_i} u_n\right| \le h, \left|\dfrac{\partial}{\partial x_i} u_m\right| \le h, |u_n - u_m| \le k, \left|\dfrac{\partial}{\partial x_i} u_n - \dfrac{\partial}{\partial x_i} u_m > s\right|\right\}$, where h and k will be chosen later. One has

$$\left\{\left|\frac{\partial}{\partial x_i} u_n - \frac{\partial}{\partial x_i} u_m\right| > s\right\} \subset A_{n,m} \cup B_{n,m} \cup C_{n,m}. \tag{3.34}$$

Let $\vartheta > 0$. By Lemma 3.9, we can choose $h = h(\vartheta)$ large enough such that $meas(A_{n,m}) \le \dfrac{\vartheta}{3}$ for all $n, m \ge 0$. On the other hand, by Proposition 3.10, we have that $meas(B_{n,m}) \le \dfrac{\vartheta}{3}$

for all $n, m \geq n_0(k, \vartheta)$. Moreover, by assumption (H_3), there exists a real valued function $\gamma : \Omega \to [0, \infty]$ such that $meas\{x \in \Omega : \gamma(x) = 0\} = 0$ and

$$(a_i(x,\xi) - a_i(x,\xi')).(\xi - \xi') \geq \gamma(x), \tag{3.35}$$

for all $i = 1, ..., N$, $|\xi|, |\xi'| \leq h$, $|\xi - \xi'| \geq s$, for a.e. $x \in \Omega$. Indeed, let's set $K = \{(\xi, \eta) \in \mathbb{R} \times \mathbb{R} : |\xi| \leq h, |\eta| \leq h, |\xi - \eta| \geq s\}$. We have $K \subset B(0,h) \times B(0,h)$ and so K is a compact set because it is closed in a compact set.
For all $x \in \Omega$ and for all $i = 1, ..., N$, let us define $\psi : K \to [0; \infty]$ such that

$$\psi(\xi,\eta) = (a_i(x,\xi) - a_i(x,\eta)).(\xi - \eta).$$

As for $a.e.$ $x \in \Omega$, $a_i(x, .)$ is continuous on $\mathbb{R}$, ψ is continuous on the compact K, by Weierstrass theorem, there exists $(\xi_0, \eta_0) \in K$ such that

$$\forall (\xi,\eta) \in K, \ \psi(\xi,\eta) \geq \psi(\xi_0,\eta_0).$$

Now let us define γ on Ω as follows.

$$\gamma(x) = \psi_i(\xi_0,\eta_0) = (a_i(x,\xi_0) - a_i(x,\eta)).(\xi - \eta_0).$$

Since $s > 0$, the function γ is such that $meas\left(\{x \in \Omega : \gamma(x) = 0\}\right) = 0$. Let $\delta = \delta(\epsilon)$ be given by Lemma 3.15, replacing ϵ and A by $\dfrac{\epsilon}{3}$ and $C_{n,m}$ respectively. Taking respectively $\tilde{\xi} = T_k(u_n - u_m)$ and $\tilde{\xi} = T_k(u_m - u_n)$ for the weak solutions u_n and u_m in (3.19) and after adding the two relations, we have

$$\begin{cases} \displaystyle\sum_{i=1}^{N} \int_{\{|u_n - u_m| < k\}} \left(a_i\left(x, \frac{\partial}{\partial x_i} u_n\right) - a_i\left(x, \frac{\partial}{\partial x_i} u_m\right)\right) \left(\frac{\partial}{\partial x_i}(u_n - u_m)\right) dx \\ \displaystyle + \int_Q \left(\left(\frac{1}{\epsilon^{p_i(x)}} \left|\frac{\partial u_n}{\partial x_i}\right|^{p_i(x)-2} \frac{\partial u_n}{\partial x_i} \right) - \left(\frac{1}{\epsilon^{p_i(x)}} \left|\frac{\partial u_m}{\partial x_i}\right|^{p_i(x)-2} \frac{\partial u_m}{\partial x_i} \right)\right) \left(\frac{\partial (u_n - u_m)}{\partial x_i}\right) dx \\ \displaystyle + \int_\Omega (|u_n|^{p_M(x)-2} u_n - u_m|^{p_M(x)-2} u_m)(T_k(u_n - u_m) dx + \int_{\tilde{\Gamma}_{Ne}} (\tilde{\rho}(u_n) - \tilde{\rho}(u_m)) T_k(u_n - u_m) d\sigma \\ \displaystyle = 2\left(\int_\Omega T_k(u_n - u_m) d\mu_\epsilon + \int_{\tilde{\Gamma}_{Ne}} \tilde{d}_\epsilon T_k(u_n - u_m) d\sigma \right), \end{cases}$$

where $Q = \{\tilde{\Omega} \setminus \Omega \cap \{|u_n - u_m| < k\}\}$. As the three last terms on the left hand side are non-negative and

$$\int_\Omega T_k(u_n - u_m) d\mu_\epsilon + \int_{\tilde{\Gamma}_{Ne}} \tilde{d}_\epsilon T_k(u_n - u_m) d\sigma \leq k(|\mu|(\Omega) + \|\tilde{d}\|_{L^1(\tilde{\Gamma}_{Ne})}),$$

we deduce that

$$\sum_{i=1}^{N} \int_{\{|u_n - u_m| < k\}} \left(a_i\left(x, \frac{\partial u_n}{\partial x_i}\right) - a_i\left(x, \frac{\partial u_m}{\partial x_i}\right)\right) \left(\frac{\partial (u_n - u_m)}{\partial x_i}\right) dx \leq 2k(|\mu|(\Omega) + \|\tilde{d}\|_{L^1(\tilde{\Gamma}_{Ne})}).$$

Therefore, using (H_3) we have

$$\begin{aligned}\int_{C_{n,m}} \gamma dx &\leq \int_{C_{n,m}} \left(a_i\left(x, \frac{\partial}{\partial x_i}u_n\right) - a_i\left(x, \frac{\partial}{\partial x_i}u_m\right)\right) \frac{\partial}{\partial x_i}\left(u_n - u_m\right) dx \\ &\leq \sum_{i=1}^{N} \int_{C_{n,m}} \left(a_i\left(x, \frac{\partial}{\partial x_i}u_n\right) - a_i\left(x, \frac{\partial}{\partial x_i}u_m\right)\right) \frac{\partial}{\partial x_i}(u_n - u_m) dx \\ &\leq 2k(\|\tilde{d}\|_{L^1(\tilde{\Gamma}_{Ne})} + |\mu|(\Omega)) < \delta,\end{aligned}$$

by choosing $k = \delta/4\left(\|\tilde{d}\|_{L^1(\tilde{\Gamma}_{Ne})} + |\mu|(\Omega)\right)$. From Lemma 3.15 again, it follows that $meas(C_{n,m}) < \dfrac{\vartheta}{3}$. Thus, using (3.35) and the estimates obtained for $A_{n,m}$, $B_{n,m}$ and $C_{n,m}$, it follows that

$$meas\left(\left\{\left|\frac{\partial}{\partial x_i}u_n - \frac{\partial}{\partial x_i}u_m\right| > s\right\}\right) \leq \vartheta, \tag{3.36}$$

for all $n, m \geq n_0(s, \vartheta)$, and then the claim is proved.

As consequence, $\left(\dfrac{\partial}{\partial x_i}u_\epsilon\right)_{\epsilon\in\mathbb{N}}$ converges in measure to some measurable function ν_i.

In order to end the proof of Lemma 3.12, we need the following lemma.

Lemma 3.16. *(a) For a.e.* $k \in \mathbb{R}$, $\dfrac{\partial}{\partial x_i}T_k(u_\epsilon)$ *converges in measure to* $\nu_i\chi_{\{|u|<k\}}$.

(b) For a.e. $k \in \mathbb{R}$, $\dfrac{\partial}{\partial x_i}T_k(u) = \nu_i\chi_{\{|u|<k\}}$.

(c) $\dfrac{\partial}{\partial x_i}T_k(u) = \nu_i\chi_{\{|u|<k\}}$ *holds for all* $k \in \mathbb{R}$.

Proof. (a) We know that $\dfrac{\partial}{\partial x_i}u_\epsilon \to \nu_i$ in measure. Thus $\dfrac{\partial}{\partial x_i}u_\epsilon\chi_{\{|u|<k\}} \to \nu_i\chi_{\{|u|<k\}}$ in measure.

Now, let us show that $\left(\chi_{\{|u_\epsilon|<k\}} - \chi_{\{|u|<k\}}\right)\dfrac{\partial}{\partial x_i}u_\epsilon \to 0$ in measure.

For that, it is sufficient to show that $\left(\chi_{\{|u_\epsilon|<k\}} - \chi_{\{|u|<k\}}\right) \to 0$ in measure. Now, for all $\delta > 0$, $\left\{|\chi_{\{|u_\epsilon|<k\}} - \chi_{\{|u|<k\}}|\left|\dfrac{\partial}{\partial x_i}u_\epsilon\right| > \delta\right\} \subset \left\{|\chi_{\{|u_\epsilon|<k\}} - \chi_{\{|u|<k\}}| \neq 0\right\} \subset \{|u| = k\} \cup \{u_\epsilon < k < u\} \cup \{u < k < u_\epsilon\} \cup \{u_\epsilon < -k < u\} \cup \{u < -k < u_\epsilon\}$. Thus,

$$\left\{\begin{aligned} & meas\left(\left\{|\chi_{\{|u_\epsilon|<k\}} - \chi_{\{|u|<k\}}|\left|\frac{\partial}{\partial x_i}u_\epsilon\right| > \delta\right\}\right) \\ \leq\; & meas\left(\{|u| = k\}\right) + meas\left(\{u_\epsilon < k < u\}\right) \\ & + meas\left(\{u < k < u_\epsilon\}\right) \\ & + meas\left(\{u_\epsilon < -k < u\}\right) \\ & + meas\left(\{u < -k < u_\epsilon\}\right). \end{aligned}\right. \tag{3.37}$$

Note that

$meas\left(\{|u| = k\}\right) \leq meas\left(\{k - h < u < k + h\}\right) + meas\left(\{-k - h < u < -k + h\}\right) \to 0$ as $h \to 0$ for a.e. $k > 0$, since u is fixed function.

Next, $meas\left(\{u_\epsilon < k < u\}\right) \leq meas\left(\{k < u < k + h\}\right) + meas\left(\{|u_\epsilon - u| > h\}\right)$, for all

$h > 0$.

Due to Proposition 3.10, we have for all fixed $h > 0$, $meas\left(\{|u_\epsilon - u| > h\}\right) \to 0$ as $\epsilon \to 0$. Since $meas\left(\{k < u < k+h\}\right) \to 0$ as $h \to 0$, for all $\vartheta > 0$, one can find N such that for all $n > N$, $meas\left(\{|u| = k\}\right) < \dfrac{\vartheta}{2} + \dfrac{\vartheta}{2} = \vartheta$ by choosing h and then N. Each of the other terms on the right-hand side of (3.37) can be treated in the same way as for $meas\left(\{u_\epsilon < k < u\}\right)$. Thus, $meas\left(\left\{|\chi_{\{|u_\epsilon|<k\}} - \chi_{\{|u|<k\}}|\left|\dfrac{\partial}{\partial x_i}u_\epsilon\right| > \delta\right\}\right) \to 0$ as $\epsilon \to 0$. Finally, since $\dfrac{\partial}{\partial x_i}T_k(u_\epsilon) = \dfrac{\partial}{\partial x_i}u_\epsilon\chi_{\{|u_\epsilon|<k\}}$, the claim (a) follows.

(b) Using the Proof of Proposition 3.10-(ii) we have $\dfrac{\partial}{\partial x_i}T_k(u_\epsilon) \rightharpoonup \dfrac{\partial}{\partial x_i}T_k(u)$ weakly in $L^{p_i^-}(\tilde{\Omega})$. The previous convergence also ensures that $\dfrac{\partial}{\partial x_i}T_k(u_\epsilon)$ converges to $\dfrac{\partial}{\partial x_i}T_k(u)$ weakly in $L^1(\Omega)$. On the other hand, by (a), $\dfrac{\partial}{\partial x_i}T_k(u_\epsilon)$ converges to $\nu_i\chi_{\{|u|<k\}}$ in measure. By Lemma 3.14, since $\dfrac{\partial}{\partial x_i}T_k(u_\epsilon)$ is uniformly bounded in $L^{p_i^-}(\tilde{\Omega})$ (see Lemma 3.8) hence in $L^{p_i^-}(\Omega)$, the convergence is actually strong in $L^1(\Omega)$; thus it is also weak in $L^1(\Omega)$. By the uniqueness of the weak L^1-limit, $\nu_i\chi_{\{|u|<k\}}$ coincides with $\dfrac{\partial}{\partial x_i}T_k(u)$.

(c) Let $0 < k < s$, and s be such that $\nu_i\chi_{\{|u|<s\}}$ coincides with $\dfrac{\partial}{\partial x_i}T_s(u)$. Then,

$$\begin{aligned}
\frac{\partial}{\partial x_i}T_k(u) &= \frac{\partial}{\partial x_i}T_k(T_s(u)) \\
&= \frac{\partial}{\partial x_i}T_s(u)\chi_{\{|T_s(u)|<k\}} \\
&= \nu_i\chi_{\{|u|<s\}}\chi_{\{|u|<k\}} \\
&= \nu_i\chi_{\{|u|<k\}}.
\end{aligned}$$

□

□

Now, we can end the proof of Lemma 3.12. Indeed, combining lemmas 3.16 (c) and 3.13; *(i)* follows.

Next, by lemmas 3.14 and 3.16, we have for all $k > 0$, $i = 1, ..., N$, $a_i\left(x, \dfrac{\partial}{\partial x_i}T_k(u_\epsilon)\right)$ converges to $a_i\left(x, \dfrac{\partial}{\partial x_i}T_k(u)\right)$ in $L^1(\Omega)$ strongly. Indeed, let $s, k > 0$, consider $E_4 = \left\{\left|\dfrac{\partial u_n}{\partial x_i} - \dfrac{\partial u_m}{\partial x_i}\right| > s, |u_n| \leq k, |u_m| \leq k\right\}$, $E_5 = \left\{\left|\dfrac{\partial u_m}{\partial x_i}\right| > s, |u_n| > k, |u_m| \leq k\right\}$, $E_6 = \left\{\left|\dfrac{\partial u_n}{\partial x_i}\right| > s, |u_n| \leq k, |u_m| > k\right\}$.

We have

$$\left\{\left|\frac{\partial T_k(u_n)}{\partial x_i} - \frac{\partial T_k(u_m)}{\partial x_i}\right| > s\right\} \subset E_4 \cup E_5 \cup E_6. \tag{3.38}$$

$\forall\vartheta > 0$, by Lemma 3.7, there exists $k(\vartheta)$ such that

$$meas(E_5) \leq \frac{\vartheta}{3} \text{ and } meas(E_6) \leq \frac{\vartheta}{3}. \tag{3.39}$$

Using (3.36)-(3.39), we get

$$meas\left(\left\{\left|\frac{\partial}{\partial x_i}T_k(u_n) - \frac{\partial}{\partial x_i}T_k(u_m)\right| > s\right\}\right) \leq \vartheta, \tag{3.40}$$

for all $n, m \geq n_1(s,\vartheta)$. Therefore, $\dfrac{\partial T_k(u_\epsilon)}{\partial x_i}$ converges in measure to $\dfrac{\partial T_k(u)}{\partial x_i}$. Using lemmas 3.8 and 3.14, we deduce that $\dfrac{\partial T_k(u_\epsilon)}{\partial x_i}$ converges to $\dfrac{\partial T_k(u)}{\partial x_i}$ in $L^1(\Omega)$. So, after passing to a suitable subsequence of $\left(\dfrac{\partial T_k(u_\epsilon)}{\partial x_i}\right)_{\epsilon>0}$, we can assume that $\dfrac{\partial T_k(u_\epsilon)}{\partial x_i}$ converges to $\dfrac{\partial T_k(u)}{\partial x_i}$ *a.e.* in Ω. By the continuity of $a_i(x,.)$, we deduce that $a_i\left(x, \dfrac{\partial T_k(u_\epsilon)}{\partial x_i}\right)$ converges to $a_i\left(x, \dfrac{\partial T_k(u)}{\partial x_i}\right)$ *a.e.* in Ω. As Ω is bounded, this convergence is in measure. Using lemmas 3.14 and 3.16, we deduce that for all $k > 0$, $i = 1, ..., N$, $a_i\left(x, \dfrac{\partial}{\partial x_i}T_k(u_\epsilon)\right)$ converges to $a_i\left(x, \dfrac{\partial}{\partial x_i}T_k(u)\right)$ in $L^1(\Omega)$ strongly and $a_i\left(x, \dfrac{\partial}{\partial x_i}T_k(u_\epsilon)\right)$ converges to $\chi_k \in L^{p_i'(.)}(\Omega)$ weakly in $L^{p_i'(.)}(\Omega)$. Since each of the convergences implies the weak L^1-convergence, χ_k can be identified with $a_i\left(x, \dfrac{\partial}{\partial x_i}T_k(u)\right)$; thus, $a_i\left(x, \dfrac{\partial}{\partial x_i}T_k(u)\right) \in L^{p_i'(.)}(\Omega)$

By using Lebesgue generalized convergence theorem and above results, we deduce the following result.

Proposition 3.17. *For any $k > 0$ and any $i = 1, ..., N$, as ϵ tends to 0, we have*

(i) $\dfrac{\partial T_k(u_\epsilon)}{\partial x_i} \to \dfrac{\partial T_k(u)}{\partial x_i}$ *a.e. in* Ω,

(ii) $a_i\left(x, \dfrac{\partial T_k(u_\epsilon)}{\partial x_i}\right)\dfrac{\partial T_k(u_\epsilon)}{\partial x_i} \to a_i\left(x, \dfrac{\partial T_k(u)}{\partial x_i}\right)\dfrac{\partial T_k(u)}{\partial x_i}$ *a.e. in* Ω *and strongly in* $L^1(\Omega)$,

(iii) $\dfrac{\partial T_k(u_\epsilon)}{\partial x_i} \to \dfrac{\partial T_k(u)}{\partial x_i}$ *strongly in* $L^{p_i(x)}(\Omega)$.

4 Existence and uniqueness of solution to $P(\rho, \mu, d)$

We are now able to prove Theorem 2.6.

Proof of Theorem 2.6

Thanks to the Proposition 3.10 and as $\forall k > 0$, $\forall i = 1, ..., N$, $\dfrac{\partial T_k(u)}{\partial x_i} = 0$ in $L^{p_i^-}(\tilde{\Omega} \setminus \Omega)$, then, $\forall k > 0$, $T_k(u) = constant$ *a.e.* on $\tilde{\Omega} \setminus \Omega$. Hence, we conclude that $u \in \mathcal{T}_{Ne}^{1,\vec{p}(.)}(\Omega)$.

We already state that $b(u) \in L^1(\Omega)$.

To show that u is an entropy solution of $P(\rho, \mu, d)$, we only have to prove the inequality in (2.9). Let $\varphi \in W_D^{1,\vec{p}(.)}(\Omega) \cap L^\infty(\Omega)$. We consider the function $\varphi_1 \in W_D^{1,\vec{p}(.)}(\tilde{\Omega}) \cap L^\infty(\Omega)$ such that

$$\varphi_1 = \varphi \chi_\Omega + \varphi_N \chi_{\tilde{\Omega}\setminus\Omega}.$$

We set $\tilde{\xi} = T_k(u_\epsilon - \varphi_1)$ in (3.19) to get

$$\begin{cases} \displaystyle\sum_{i=1}^N \int_\Omega \left(a_i\left(x, \frac{\partial}{\partial x_i} u_\epsilon\right) . \frac{\partial}{\partial x_i} T_k(u_\epsilon - \varphi) \right) dx \\ \displaystyle + \sum_{i=1}^N \int_{\tilde{\Omega}\setminus\Omega} \left(\frac{1}{\epsilon^{p_i(x)}} \left| \frac{\partial}{\partial x_i} u_\epsilon \right|^{p_i(x)-2} \frac{\partial}{\partial x_i} u_\epsilon . \frac{\partial}{\partial x_i} T_k(u_\epsilon - \varphi_N) \right) dx \\ \displaystyle \int_\Omega b(u_\epsilon) T_k(u_\epsilon - \varphi) dx = \int_\Omega T_k(u_\epsilon - \varphi) d\mu_\epsilon + \int_{\tilde{\Gamma}_{Ne}} (\tilde{d}_\epsilon - \tilde{\rho}(u_\epsilon)) T_k(u_\epsilon - \varphi_N) d\sigma. \end{cases} \tag{4.1}$$

The following convergence result hold true.

Lemma 4.1. *For any $k > 0$, for all $i = 1, ..., N$, as $\epsilon \to 0$,*

$$\frac{\partial}{\partial x_i} T_k(u_\epsilon - \varphi) \to \frac{\partial}{\partial x_i} T_k(u - \varphi) \textit{ strongly in } L^{p_i(\cdot)}(\Omega).$$

Proof. Let $k > 0$, $i = 1, ..., N$. We have

$$\begin{aligned} & \int_\Omega \left| \frac{\partial}{\partial x_i} T_k(u_\epsilon - \varphi) - \frac{\partial}{\partial x_i} T_k(u - \varphi) \right|^{p_i(x)} dx \\ = & \int_{\Omega \cap [|u_\epsilon - \varphi| \le k, |u - \varphi| \le k]} \left| \frac{\partial}{\partial x_i} u_\epsilon - \frac{\partial}{\partial x_i} u \right|^{p_i(x)} dx \\ \le & \int_{\Omega \cap [|u_\epsilon| \le l, |u| \le l]} \left| \frac{\partial u_\epsilon}{\partial x_i} - \frac{\partial u}{\partial x_i} \right|^{p_i(x)} dx, \text{ with } l = k + \|\varphi\|_\infty \\ = & \int_\Omega \left| \frac{\partial}{\partial x_i} T_l(u_\epsilon) - \frac{\partial}{\partial x_i} T_l(u) \right|^{p_i(x)} dx \\ \to & \ 0 \text{ as } \epsilon \to 0 \text{ by Proposition 3.17} - (iii). \end{aligned}$$

□

We need to pass to the limit in (4.1) as $\epsilon \to 0$. We have

$$\sum_{i=1}^N \int_\Omega \left(a_i\left(x, \frac{\partial}{\partial x_i} u_\epsilon\right) \frac{\partial}{\partial x_i} T_k(u_\epsilon - \varphi) \right) dx = \sum_{i=1}^N \int_\Omega \left(a_i\left(x, \frac{\partial T_l(u_\epsilon)}{\partial x_i}\right) \frac{\partial}{\partial x_i} T_k(u_\epsilon - \varphi) \right) dx,$$

with $l = k + \|\varphi\|_\infty$, then, by Lemma 3.12- (ii) and Lemma 4.1, we have

$$\lim_{\epsilon \to 0} \sum_{i=1}^N \int_\Omega \left(a_i\left(x, \frac{\partial T_l(u_\epsilon)}{\partial x_i}\right) \frac{\partial}{\partial x_i} T_k(u_\epsilon - \varphi) \right) dx = \sum_{i=1}^N \int_\Omega \left(a_i\left(x, \frac{\partial T_l(u)}{\partial x_i}\right) \frac{\partial}{\partial x_i} T_k(u - \varphi) \right) dx;$$

that is

$$\lim_{\epsilon \to 0} \sum_{i=1}^N \int_\Omega \left(a_i\left(x, \frac{\partial}{\partial x_i} u_\epsilon\right) \frac{\partial}{\partial x_i} T_k(u_\epsilon - \varphi) \right) dx = \sum_{i=1}^N \int_\Omega \left(a_i\left(x, \frac{\partial T_l(u)}{\partial x_i}\right) \frac{\partial}{\partial x_i} T_k(u - \varphi) \right) dx. \tag{4.2}$$

For the second term in the left hand side of (4.1), we have

$$\limsup_{\epsilon\to 0}\sum_{i=1}^{N}\int_{\tilde\Omega\setminus\Omega}\left(\frac{1}{\epsilon^{p_i(x)}}|\frac{\partial}{\partial x_i}u_\epsilon|^{p_i(x)-2}\frac{\partial}{\partial x_i}u_\epsilon\frac{\partial}{\partial x_i}T_k(u_\epsilon-\varphi_N)\right)dx\geq 0. \tag{4.3}$$

Indeed

$$\begin{cases} & \displaystyle\sum_{i=1}^{N}\int_{\tilde\Omega\setminus\Omega}\left(\frac{1}{\epsilon^{p_i(x)}}\left|\frac{\partial}{\partial x_i}u_\epsilon\right|^{p_i(x)-2}\frac{\partial}{\partial x_i}u_\epsilon\frac{\partial}{\partial x_i}T_k(u_\epsilon-\varphi_N)\right)dx\\ = & \displaystyle\sum_{i=1}^{N}\int_{\tilde\Omega\setminus\Omega\cap[|u_\epsilon-\varphi|\leq k]}\left(\frac{1}{\epsilon^{p_i(x)}}|\frac{\partial}{\partial x_i}u_\epsilon|^{p_i(x)-2}\frac{\partial}{\partial x_i}u_\epsilon\frac{\partial}{\partial x_i}(u_\epsilon-\varphi_N)\right)dx\\ = & \displaystyle\sum_{i=1}^{N}\int_{\tilde\Omega\setminus\Omega\cap[|u_\epsilon-\varphi|\leq k]}\left(\frac{1}{\epsilon^{p_i(x)}}|\frac{\partial}{\partial x_i}u_\epsilon|^{p_i(x)}\right)dx\geq 0.\end{cases}$$

Hence, we get (4.3).

Let us examine the last term in the left hand side of (4.1).

we have

$$\int_\Omega b(u_\epsilon)T_k(u_\epsilon-\varphi)dx=\int_\Omega(b(u_\epsilon)-b(\varphi))T_k(u_\epsilon-\varphi)dx+\int_\Omega b(\varphi)T_k(u_\epsilon-\varphi)dx.$$

As b is non-decreasing,

$$(b(u_\epsilon)-b(\varphi))T_k(u_\epsilon-\varphi)\geq 0 \text{ a.e. in } \Omega$$

and we get by Fatou's lemma that

$$\liminf_{\epsilon\to 0}\int_\Omega(b(u_\epsilon)-b(\varphi))T_k(u_\epsilon-\varphi)dx\geq\int_\Omega(b(u)-b(\varphi))T_k(u-\varphi)dx.$$

As $\varphi\in L^\infty(\Omega)$, we obtain $b(\varphi)\in L^\infty(\Omega)$ and so $b(\varphi)\in L^1(\Omega)$ (as Ω is bounded) and by Lebesgue dominated convergence theorem, we deduce that

$$\lim_{\epsilon\to 0}\int_\Omega b(\varphi)T_k(u_\epsilon-\varphi)dx=\int_\Omega b(\varphi)T_k(u-\varphi)dx.$$

Consequently,

$$\limsup_{\epsilon\to 0}\int_\Omega b(u_\epsilon)T_k(u_\epsilon-\varphi)dx\geq\int_\Omega b(u)T_k(u-\varphi)dx. \tag{4.4}$$

As $f_\epsilon\to f$ strongly in $L^1(\Omega)$ and $T_k(u_\epsilon-v)\rightharpoonup^* T_k(u-v)$ in $L^\infty(\Omega)$, using the Lebesgue generalized convergence theorem we have

$$\begin{cases}\displaystyle\lim_{\epsilon\to 0}\int_\Omega f_\epsilon T_k(u_\epsilon-\varphi)dx=\int_\Omega T_k(u-\varphi)dx,\\ \displaystyle\lim_{\epsilon\to 0}\int_{\tilde\Gamma_{Ne}}\tilde d_\epsilon T_k(u_\epsilon-\varphi_N)d\sigma=\int_\Omega\tilde d T_k(u-\varphi_N)d\sigma.\end{cases} \tag{4.5}$$

Since $\nabla T_k(u_\epsilon-\varphi)\rightharpoonup\nabla T_k(u-\varphi)$ in $(L^{p_m(\cdot)}(\Omega))^N$ and $F\in(L^{p'_m(\cdot)}(\Omega))^N$,

$$\lim_{\epsilon\to 0}\int_\Omega F.\nabla T_k(u_\epsilon-\varphi)dx=\int_\Omega F.\nabla T_k(u-\varphi)dx. \tag{4.6}$$

We know that $\forall k>0$, $T_k(u)=constant$ on $\tilde\Omega\setminus\Omega$, then, it yields that $u=constant$ on $\tilde\Omega\setminus\Omega$. So, one has

$$\lim_{\epsilon\to 0}\int_{\tilde\Gamma_{Ne}}\tilde d_\epsilon T_k(u_\epsilon-\varphi)dx=dT_k(u_N-\varphi_N). \tag{4.7}$$

At last, we have

$$\int_{\tilde{\Gamma}_{Ne}} \tilde{\rho}(u_\epsilon)T_k(u_\epsilon - \varphi_N)d\sigma = \int_{\tilde{\Gamma}_{Ne}} (\tilde{\rho}(u_\epsilon) - \tilde{\rho}(\varphi_N))T_k(u_\epsilon - \varphi_N)d\sigma + \int_{\tilde{\Gamma}_{Ne}} \tilde{\rho}(\varphi_N)T_k(u_\epsilon - \varphi_N)d\sigma.$$

As $\tilde{\rho}$ is non-decreasing,

$$(\tilde{\rho}(u_\epsilon) - \tilde{\rho}(\varphi_N))T_k(u_\epsilon - \varphi_N) \geq 0 \text{ a.e. in } \tilde{\Gamma}_{Ne}$$

and we get by Fatou's lemma that

$$\begin{aligned} \liminf_{\epsilon\to 0} \int_{\tilde{\Gamma}_{Ne}} (\tilde{\rho}(u_\epsilon) - \tilde{\rho}(\varphi_N))T_k(u_\epsilon - \varphi_N)d\sigma &\geq \int_{\tilde{\Gamma}_{Ne}} (\tilde{\rho}(u_N) - \tilde{\rho}(\varphi_N))T_k(u_N - \varphi_N)d\sigma \\ &= (\rho(u_N) - \rho(\varphi_N))T_k(u_N - \varphi_N). \end{aligned}$$

As $\varphi_N \in L^\infty(\tilde{\Gamma}_{Ne})$, we obtain $\tilde{\rho}(\varphi_N) \in L^\infty(\tilde{\Gamma}_{Ne})$ and so $\tilde{\rho}(\varphi_N) \in L^1(\tilde{\Gamma}_{Ne})$ (as $\tilde{\Gamma}_{Ne}$ is bounded) and by the Lebesgue dominated convergence theorem, we deduce that

$$\lim_{\epsilon\to 0} \int_{\tilde{\Gamma}_{Ne}} \tilde{\rho}(\varphi_N)T_k(u_\epsilon - \varphi_N)d\sigma = \int_{\tilde{\Gamma}_{Ne}} \tilde{\rho}(\varphi_N)T_k(u_N - \varphi_N)d\sigma = \rho(\varphi_N)T_k(u_N - \varphi_N).$$

Hence,

$$\limsup_{\epsilon\to 0} \int_{\tilde{\Gamma}_{Ne}} \tilde{\rho}(u_\epsilon)T_k(u_\epsilon - \varphi_N)d\sigma \geq \rho(\varphi_N)T_k(u_N - \varphi_N). \tag{4.8}$$

Passing to the limit as $\epsilon \to 0$ in (4.1) and using (4.2)-(4.8), we see that u is an entropy solution of $P(\rho, \mu, d)$.

We now prove the uniqueness part of Theorem 2.6.
Let u and v be two entropy solutions of $P(\rho, \mu, d)$.
Let $h > 0$. For u, we take $\xi = T_h(v)$ as test function and for v, we take $\xi = T_h(u)$ as test function in (2.9), to get for any $k > 0$ with $k < h$,

$$\begin{cases} \int_\Omega \left(\sum_{i=1}^N a_i\left(x, \frac{\partial}{\partial x_i}u\right)\frac{\partial}{\partial x_i}T_k(u - T_h(v))\right)dx + \int_\Omega b(u)T_k(u - T_h(v))dx \leq \\ \int_\Omega fT_k(u - T_h(v))dx + \int_\Omega F.\nabla T_k(u - T_h(v))dx + (d - \rho(u_{Ne}))T_k(u_{Ne} - T_h(v)) \end{cases} \tag{4.9}$$

and

$$\begin{cases} \int_\Omega \left(\sum_{i=1}^N a_i\left(x, \frac{\partial}{\partial x_i}v\right)\frac{\partial}{\partial x_i}T_k(v - T_h(u))\right)dx + \int_\Omega b(v)T_k(v - T_h(u))dx \leq \\ \int_\Omega fT_k(v - T_h(u))dx + \int_\Omega F.\nabla T_k(v - T_h(u))dx + (d - \rho(v_{Ne}))T_k(v_{Ne} - T_h(u)). \end{cases} \tag{4.10}$$

By adding (4.9) and (4.10), we obtain

$$\begin{cases} \int_\Omega \left(\sum_{i=1}^N a_i \left(x, \frac{\partial}{\partial x_i} u \right) \frac{\partial}{\partial x_i} T_k(u - T_h(v)) \right) dx & \\ + \int_\Omega \left(\sum_{i=1}^N a_i \left(x, \frac{\partial}{\partial x_i} v \right) \frac{\partial}{\partial x_i} T_k(v - T_h(u)) \right) dx & := A(h,k) \\ + \int_\Omega b(u) T_k(u - T_h(v)) dx + \int_\Omega b(v) T_k(v - T_h(u)) dx & := B(h,k) \\ + \rho(u_{Ne}) T_k(u_{Ne} - T_h(v)) + \rho(v_{Ne}) T_k(v_{Ne} - T_h(u)) & := C(h,k) \\ \leq \int_\Omega f T_k(u - T_h(v)) dx + \int_\Omega f T_k(v - T_h(u)) dx & := D(h,k) \\ + \int_\Omega F.\nabla T_k(u - T_h(v)) dx + \int_\Omega F.\nabla T_k(v - T_h(u)) dx & := T(h,k) \\ + d T_k(u_{Ne} - T_h(v)) + d T_k(v_{Ne} - T_h(u)) & := E(h,k). \end{cases} \tag{4.11}$$

Let us introduce the following subsets of Ω.

$$\begin{aligned} A_0 &:= [|u-v| < k, |u| < h, |v| < h] \\ A_1 &:= [|u - T_h(v)| < k, |v| \geq h] \\ A_1' &:= [|v - T_h(u)| < k, |u| \geq h] \\ A_2 &:= [|u - T_h(v)| < k, |u| \geq h, |v| < h] \\ A_2' &:= [|v - T_h(u)| < k, |v| \geq h, |u| < h]. \end{aligned}$$

We have the following assertion (see [22] for the proof).

Assertion 4.2. *If u is an entropy solution of $P(\rho, \mu, d)$, then $A_2 \subset F_{h,k}$ and $A_1 \subset F_{h-k,2k}$, where*

$$F_{h,k} = \{h \leq |u| < h + k, h > 0, k > 0\}.$$

Assertion 4.3. *Let u be an entropy solution of $P(\rho, \mu, d)$. On A_2 (and on A_1) we have according to Hölder inequality.*

(1)

$$\int_{A_2} F.\nabla u dx \leq \left(\int_{A_2} |F|^{(p_m')^-} dx \right)^{\frac{1}{(p_m')^-}} \left(\int_{A_2} |\nabla u|^{p_m^-} \right)^{\frac{1}{p_m^-}} dx, \tag{4.12}$$

with $\lim_{h\to\infty} \left(\int_{A_2} |F|^{(p_m')^-} dx \right)^{\frac{1}{(p_m')^-}} \left(\int_{A_2} |\nabla u|^{p_m^-} dx \right)^{\frac{1}{p_m^-}} = 0.$

(2)

$$\int_{A_1} F.\nabla u dx \leq \left(\int_{A_1} |F|^{(p_m')^-} dx \right)^{\frac{1}{(p_m')^-}} \left(\int_{A_1} |\nabla u|^{p_m^-} dx \right)^{\frac{1}{p_m^-}}, \tag{4.13}$$

with $\lim_{h\to\infty} \left(\int_{A_1} |F|^{(p_m')^-} dx \right)^{\frac{1}{(p_m')^-}} \left(\int_{A_1} |\nabla u|^{p_m^-} dx \right)^{\frac{1}{p_m^-}} = 0.$

Proof. (1) $\lim_{h\to\infty}\left(\int_{A_2}|F|^{(p'_m)^-}dx\right)^{\frac{1}{(p'_m)^-}} = 0$ (see [22]).

Now, it remains to prove that $\left(\int_{A_2}|\nabla u|^{p_m^-}dx\right)^{\frac{1}{p_m^-}}$ is bounded with respect to h.

We make the following notations:

$\mathcal{I}=\left\{i\in\{1,...,N\}:\left\{\left|\frac{\partial}{\partial x_i}u\right|\right\}\leq 1\right\}$ and $\mathcal{J}=\left\{i\in\{1,...,N\}:\left\{\left|\frac{\partial}{\partial x_i}u\right|\right\}>1\right\}$.

We have

$$\begin{aligned}
\sum_{i=1}^N\int_{F_{h,k}}\left|\frac{\partial}{\partial x_i}u\right|^{p_i(x)}dx &= \sum_{i\in\mathcal{I}}\left(\int_{F_{h,k}}\left|\frac{\partial}{\partial x_i}u\right|^{p_i(x)}dx\right)+\sum_{i\in\mathcal{J}}\left(\int_{F_{h,k}}\left|\frac{\partial}{\partial x_i}u\right|^{p_i(x)}dx\right)\\
&\geq \sum_{i\in\mathcal{J}}\left(\int_{F_{h,k}}\left|\frac{\partial}{\partial x_i}u\right|^{p_i(x)}dx\right)\\
&\geq \sum_{i\in\mathcal{J}}\left(\int_{F_{h,k}}\left|\frac{\partial}{\partial x_i}u\right|^{p_m^-}dx\right)\\
&\geq \sum_{i=1}^N\left(\int_{F_{h,k}}\left|\frac{\partial}{\partial x_i}u\right|^{p_m^-}dx\right)-\sum_{i\in\mathcal{I}}\left(\int_{F_{h,k}}\left|\frac{\partial}{\partial x_i}u\right|^{p_m^-}dx\right)\\
&\geq \sum_{i=1}^N\left(\int_{F_{h,k}}\left|\frac{\partial}{\partial x_i}u\right|^{p_m^-}\right)-Nmeas(\Omega)\\
&\geq \sum_{i=1}^N\left\|\frac{\partial}{\partial x_i}u\right\|^{p_m^-}_{(L^{p_m^-}(F_{h,k}))^N}-Nmeas(\Omega)\\
&\geq C\|\nabla u\|^{p_m^-}_{(L^{p_m^-}(F_{h,k}))^N}-Nmeas(\Omega).
\end{aligned}$$

We deduce that

$$\sum_{i=1}^N\int_{F_{h,k}}\left|\frac{\partial}{\partial x_i}u\right|^{p_i(x)}dx\geq C\int_{F_{h,k}}|\nabla u|^{p_m^-}dx-Nmeas(\Omega). \tag{4.14}$$

Choosing $T_h(u)$ as test function in (2.9), we get

$$\begin{cases}\int_\Omega\left(\sum_{i=1}^N a_i\left(x,\frac{\partial}{\partial x_i}u\right))\frac{\partial}{\partial x_i}T_k(u-T_h(u))\right)dx+\int_\Omega|u|^{p_M(x)-2}uT_k(u-T_h(u))dx\leq\\ \int_\Omega fT_k(u-T_h(u))dx+\int_\Omega F.\nabla T_k(u-T_h(u))dx+(d-\rho(u_{Ne}))T_k(u_{Ne}-T_h(u_{Ne})).\end{cases} \tag{4.15}$$

According to the fact that $\nabla T_k(u-T_h(u))=\nabla u$ on $\{h\leq|u|<h+k\}$ and zero elsewhere, $\int_\Omega|u|^{p_M(x)-2}uT_k(u-T_h(u))dx\geq 0$ and $\rho(u_{Ne})T_k(u_{Ne}-T_h(u_{Ne}))\geq 0$, we deduce from (4.15) that

$$\begin{cases}\int_{F_{h,k}}\left(\sum_{i=1}^N a_i\left(x,\frac{\partial}{\partial x_i}u\right)\frac{\partial}{\partial x_i}T_k(u-T_h(u))\right)dx\leq\\ k\int_{|u|\geq h}|f|dx+\int_{F_{h,k}}\left|\left(\frac{2}{Cp_m^-}\right)^{\frac{1}{p_m^-}}F\right|\left|\left(\frac{Cp_m^-}{2}\right)^{\frac{1}{p_m^-}}\nabla u\right|dx+k|d|.\end{cases} \tag{4.16}$$

Using (1.7) (in the left hand side of (4.16)), Young inequality (in the right hand side of(4.16)) and setting

$$c = \left(\frac{2}{Cp_m^-}\right)^{\frac{(p'_m)^-}{p_m^-}} \frac{p_m^- - 1}{p_m^-},$$

we obtain

$$\begin{cases} \displaystyle\sum_{i=1}^{N} \int_{F_{h,k}} \left|\frac{\partial}{\partial x_i} u\right|^{p_i(x)} dx \leq \\ \displaystyle k\int_{|u|\geq h} |f|dx + c\int_{F_{h,k}} |F|^{(p')^-_m} dx + \frac{C}{2}\int_{F_{h,k}} |\nabla u|^{p_m^-} dx + k|d|. \end{cases} \tag{4.17}$$

From (4.14) and (4.17), we deduce

$$\begin{cases} \displaystyle C\int_{F_{h,k}} |\nabla u|^{p_m^-} dx \leq \\ \displaystyle k\int_{|u|\geq h} |f|dx + c\int_{F_{h,k}} |F|^{(p')^-_m} dx + \frac{C}{2}\int_{F_{h,k}} |\nabla u|^{p_m^-} dx + k|d| + N meas(\Omega). \end{cases}$$

Therefore,

$$\begin{cases} \displaystyle \frac{C}{2}\int_{F_{h,k}} |\nabla u|^{p_m^-} dx \leq \\ \displaystyle k\int_{\{|u|\geq h\}} |f|dx + c\int_{F_{h,k}} |F|^{(p')^-_m} dx + k|d| + N meas(\Omega). \end{cases} \tag{4.18}$$

Since $A_2 \subset F_{h,k}$, we deduce from (4.18) that $\displaystyle\int_{A_2} |\nabla u|^{p_m^-} dx$ is bounded.

(2) $\displaystyle\lim_{h\to\infty} \left(\int_{A_1} |F|^{(p'_m)^-} dx\right)^{\frac{1}{(p'_m)^-}} = 0$ (see [22]).

Now, it remains to prove that $\displaystyle\left(\int_{A_1} |\nabla u|^{p_m^-} dx\right)^{\frac{1}{p_m^-}}$ is bounded with respect to h.

Since $A_1 \subset F_{h-k,2k}$, we deduce from (4.18) that $\displaystyle\int_{A_2} |\nabla u|^{p_m^-} dx$ is bounded.

□

Remark 4.4. *Similarly, we prove that if v is an entropy solution of $P(\rho, f, d)$, then*

$$\lim_{h\to\infty} \int_{A'_2} F.\nabla v dx \leq 0$$

and

$$\lim_{h\to\infty} \int_{A'_1} F.\nabla v dx \leq 0.$$

Now, we have

$$\begin{cases} \displaystyle A(h,k) = \int_{A_0} \left(\sum_{i=1}^{N} \left(a_i\left(x, \frac{\partial}{\partial x_i} u\right) - a_i\left(x, \frac{\partial}{\partial x_i} v\right)\right) \frac{\partial}{\partial x_i}(u-v)\right) dx & := I_0(h,k) \\ \displaystyle + \int_{A_1} \left(\sum_{i=1}^{N} a_i\left(x, \frac{\partial}{\partial x_i} u\right) \frac{\partial}{\partial x_i} u\right) dx + \int_{A'_1} \left(\sum_{i=1}^{N} a_i\left(x, \frac{\partial}{\partial x_i} v\right) \frac{\partial}{\partial x_i} v\right) dx & := I_1(h,k) \\ \displaystyle + \int_{A_2} \left(\sum_{i=1}^{N} a_i\left(x, \frac{\partial}{\partial x_i} u\right) \frac{\partial}{\partial x_i}(u-v)\right) dx + \int_{A'_2} \left(\sum_{i=1}^{N} a_i\left(x, \frac{\partial}{\partial x_i} v\right) \frac{\partial}{\partial x_i}(v-u)\right) dx & := I_2(h,k). \end{cases}$$

The term $I_1(h,k)$ is non-negative since each term in $I_1(h,k)$ is non-negative.
For the term $I_2(h,k)$, as

$$I_2(h,k)+\int_{A_2}\left(\sum_{i=1}^{N}a_i\left(x,\frac{\partial}{\partial x_i}u\right)\frac{\partial}{\partial x_i}v\right)dx+\int_{A_2'}\left(\sum_{i=1}^{N}a_i\left(x,\frac{\partial}{\partial x_i}v\right)\frac{\partial}{\partial x_i}u\right)dx=I_1(h,k),$$

so,

$$I_2(h,k)\geq -\left(\int_{A_2}\left(\sum_{i=1}^{N}a_i\left(x,\frac{\partial}{\partial x_i}u\right)\frac{\partial}{\partial x_i}v\right)dx+\int_{A_2'}\left(\sum_{i=1}^{N}a_i\left(x,\frac{\partial}{\partial x_i}v\right)\frac{\partial}{\partial x_i}u\right)dx\right).$$

Let us show that $-\left(\int_{A_2}\left(\sum_{i=1}^{N}a_i\left(x,\frac{\partial}{\partial x_i}u\right)\frac{\partial}{\partial x_i}v\right)dx\right)$ goes to 0 as $h\to\infty$.
We have

$$\begin{cases}\left|\int_{A_2}\left(\sum_{i=1}^{N}a_i\left(x,\frac{\partial}{\partial x_i}u\right)\frac{\partial}{\partial x_i}(v)\right)dx\right|\leq\\ C\sum_{i=1}^{N}\left(|j_i|_{p_i'(.)}+\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)-1}_{L^{p_i(.)}(\{h<|u|\leq h+k\})}\right)\left|\frac{\partial v}{\partial x_i}\right|_{L^{p_i(.)}(\{h-k<|v|\leq h\})}\end{cases}.$$

For all $i=1,...N$, the quantity $\left(|j_i|_{p_i'(.)}+\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)-1}_{L^{p_i(.)}(\{h<|u|\leq h+k\})}\right)$ is finite since $u=T_{h+k}(u)\in\mathcal{T}^{1,\vec{p}(.)}_{Ne}(\Omega)$ and $j_i\in L^{p_i'(.)}(\Omega)$; then by Lemma 3.8, the last expression converges to zero as h tends to infinity.

Similarly we can show that $-\left(\int_{A_2}\left(\sum_{i=1}^{N}a_i\left(x,\frac{\partial}{\partial x_i}v\right)\frac{\partial}{\partial x_i}(u)\right)dx\right)$ goes to 0 as $h\to\infty$, hence, we obtain

$$\limsup_{h\to\infty}A(h,k)\geq\int_{[|u-v|<k]}\left[\sum_{i=1}^{N}\left(a_i\left(x,\frac{\partial}{\partial x_i}u\right)-a_i\left(x,\frac{\partial}{\partial x_i}v\right)\right)\frac{\partial}{\partial x_i}(u-v)\right]dx. \tag{4.19}$$

By using the Lebesgue dominated convergence theorem, it yields that

$$\lim_{h\to\infty}B(h,k)=\int_{\Omega}(b(u)-b(v))\,T_k(u-v)dx \text{ and } \lim_{h\to\infty}D(h,k)=0. \tag{4.20}$$

For h large enough, we get

$$\lim_{h\to\infty}C(h,k)=(\rho(u_N)-\rho(v_N))\,T_k(u_N-v_N) \text{ and } \lim_{h\to\infty}E(h,k)=0. \tag{4.21}$$

$$\begin{cases}T(h,k)=\int_{A_1}F.\nabla u dx+\int_{A_1'}F.\nabla v dx\\ +\int_{A_2}F.\nabla(u-v)dx+\int_{A_2'}F.\nabla(v-u)dx.\end{cases}$$

$$\begin{cases}T(h,k)=\int_{A_1}F.\nabla u dx+\int_{A_1'}F.\nabla v dx\\ +\int_{A_2}F.\nabla u dx-\int_{A_2}F.\nabla v dx+\int_{A_2'}F.\nabla v dx-\int_{A_2'}F.\nabla u dx.\end{cases}$$

Using Assertion 4.3 and Remark 4.4, it is easy to see that $\lim_{h\to\infty} |T(h,k)| = 0$.
Letting h go to ∞ in (4.11) and combining (4.20)-(4.21), we obtain

$$\begin{cases} \displaystyle\int_{[|u-v|<k]} \left[\sum_{i=1}^{N} \left(a_i\left(x, \frac{\partial}{\partial x_i} u\right) - a_i\left(x, \frac{\partial}{\partial x_i} v\right)\right) \frac{\partial}{\partial x_i}(u-v) \right] dx \\ \displaystyle + \int_\Omega (b(u)-b(v))\, T_k(u-v)dx + (\rho(u_N)-\rho(v_N))\, T_k(u_N - v_N) \leq 0. \end{cases} \tag{4.22}$$

All the terms in the left hand side of (4.22) are non-negative so that we get $\forall k > 0$,

$$\int_{[|u-v|<k]} \left[\sum_{i=1}^{N} \left(a_i\left(x, \frac{\partial}{\partial x_i} u\right) - a_i\left(x, \frac{\partial}{\partial x_i} v\right)\right) \frac{\partial}{\partial x_i}(u-v) \right] dx = 0 \tag{4.23}$$

and

$$\begin{cases} \displaystyle\int_\Omega (b(u)-b(v))\, T_k(u-v)dx = 0 \\ (\rho(u_N)-\rho(v_N))\, T_k(u_N - v_N) = 0. \end{cases} \tag{4.24}$$

Relation (4.23) gives $\frac{\partial}{\partial x_i}(u-v) = 0$ a.e. in Ω; we deduce that there exists a constant c such that $u - v = c$ a.e. in Ω. According to (4.24), $b(u) = b(v)$. Since b is invertible, we deduce that $u = v$ in Ω and so

$$\begin{cases} u = v \text{ a.e. in } \Omega \\ \rho(u_N) = \rho(v_N); \end{cases}$$

which prove the uniqueness part.

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Cubo
A Mathematical Journal

CUBO, A Mathematical Journal
Vol. 23, no. 01, pp. 63–85, April 2021
DOI: 10.4067/S0719-06462021000100063

Convolutions in (μ,ν)-pseudo-almost periodic and (μ,ν)-pseudo-almost automorphic function spaces and applications to solve integral equations

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ABSTRACT

In this paper we give sufficient conditions on $k \in L^1(\mathbb{R})$ and the positive measures μ, ν such that the doubly-measure pseudo-almost periodic (respectively, doubly-measure pseudo-almost automorphic) function spaces are invariant by the convolution product $\zeta f = k * f$. We provide an appropriate example to illustrate our convolution results. As a consequence, we study under Acquistapace-Terreni conditions and exponential dichotomy, the existence and uniqueness of (μ,ν)-pseudo-almost periodic (respectively, (μ,ν)- pseudo-almost automorphic) solutions to some nonautonomous partial evolution equations in Banach spaces like neutral systems.

RESUMEN

En este artículo damos condiciones suficientes sobre $k \in L^1(\mathbb{R})$ y las medidas positivas μ, ν tales que los espacios de funciones pseudo-casi periódicas que duplican la medida (respectivamente, pseudo-casi automorfas que duplican la medida) son invariantes por el producto de convolución $\zeta f = k * f$. Entregamos un ejemplo apropiado para ilustrar nuestros resultados de convolución. Como consecuencia, estudiamos bajo condiciones de Acquistapace-Terreni y dicotomía exponencial, la existencia y unicidad de soluciones (μ,ν)- pseudo-casi periódicas (respectivamente, (μ,ν)- pseudo-casi automorfas) de algunas ecuaciones de evolución parciales no autónomas en espacios de Banach como sistemas neutrales.

Keywords and Phrases: Measure theory, (μ,ν)-ergodic, (μ,ν)-pseudo almost periodic and automorphic functions, evolution families, nonautonomous equations, neutral systems.

2020 AMS Mathematics Subject Classification: 34C27, 34K14, 35B15, 35K57, 37A30, 43A60.

Accepted: 09 January, 2021
Received: 22 August, 2020

1 Introduction

The existence and uniqueness of pseudo almost periodic and pseudo almost automorphic solutions is one of the most powerful tools in the qualitative theory of differential equations due to applications in mathematical biology, control theory and physical sciences. Recently, Diagana, Ezzinbi and Miraoui [11] applied the abstract measure theory to define the notion of double-weight pseudo almost periodicity (respectively double-weight pseudo almost automorphy) functions, and thus the classical theory of μ-pseudo almost periodic (respectively μ-pseudo almost automorphic) introduced by [4, 5], and double-weight pseudo almost periodicity [8] become particular cases of this approach. See the section 2.1 for technical details about this concept of double-weight pseudo almost periodicity (respectively double-weight pseudo almost automorphy) functions. We note that for $f \in PAP(\mathbb{R} \times X, X, \mu, \nu)$ or $f \in PAA(\mathbb{R} \times X, X, \mu, \nu)$, $k \in L^1(\mathbb{R})$, $k * f = k * g + k * \phi$. We have that $k * g$ is almost periodic or almost automorphic function, but $k * \phi$ is not necessarily in $\mathcal{E}(\mathbb{R}, X, \mu, \nu)$. Then, the convolution invariance of the spaces $PAP(\mathbb{R} \times X, X, \mu, \nu)$ (resp. $PAA(\mathbb{R} \times X, X, \mu, \nu)$) is equivalent to the convolution invariance of $\mathcal{E}(\mathbb{R}, X, \mu, \nu)$.

During the last decade, many research results about pseudo almost periodic and pseudo almost atomorphic was produce see [4, 5, 7, 9, 10]. Inspired by the work of Ezzinbi *et al.* [11] who studied the translation invariance of $PAA(\mathbb{R} \times X, X, \mu, \nu)$ (resp. $PAP(\mathbb{R} \times X, X, \mu, \nu)$) functions and the recent work of Mbounja *et al.* [15] who gave some several hypotheses for convolution invariance of $PAP(\mathbb{R} \times X, X, \mu)$ and $PAA(\mathbb{R} \times X, X, \mu)$, in this work we established new sufficient conditions on $\mu, \nu \in \mathcal{M}$ and $k \in L^1(\mathbb{R})$ ensuring that, the space $PAP(\mathbb{R}, X, \mu, \nu)$ of (μ, ν)-pseudo almost periodic functions and the space $PAA(\mathbb{R}, X, \mu, \nu)$ of (μ, ν)-pseudo almost automorphic functions are invariant by the convolution product $\zeta f = k * f$. Our obtained conditions are more general than [15] and helped to show that the integral solution of some differential equations is a (μ, ν)-pseudo almost periodic (respectively (μ, ν)-pseudo almost automorphic) solutions. To illustrate our investigation, we show the existence and uniqueness of (μ, ν)-pseudo almost periodic (respectively (μ, ν)-pseudo almost automorphic) solutions of the following nonautomous differential equations,

$$\frac{d}{dt}u(t) = A(t)u(t) + F(t, u(t)), \quad t \in \mathbb{R}, \tag{1.1}$$

and

$$\frac{d}{dt}\left(u(t) - G(t, u(t)\right) = A(t)\left(u(t) - G(t, u(t)\right) + F(t, u(t)), \quad t \in \mathbb{R}, \tag{1.2}$$

where $A(t) : D(A(t)) \subset X \longmapsto X$ for $t \in \mathbb{R}$ is a family of closed linear operators on a Banach space X, satisfying the well-known Acquitaspace-Terreni conditions developed in [1, 2], and $F, G : \mathbb{R} \times X \longmapsto X$ are jointly continuous functions satisfying some additional conditions. The study of equation (1.1) in an non-autonomous case is new even in the case of one measure, $\mu = \nu$. Also, equation (1.2) is treated here.

The rest of this work is organized as follows. In section 2, we recall some basic results which will be used throughout this work. In section 3, we state and prove main results about the convolution invariance. In section 4 we study the existence and uniqueness of (μ,ν)-pseudo almost periodic (respectively (μ,ν)-pseudo almost automorphic) solutions to both equation (1.1) and equation (1.2) which illustrate our new results.

2 Preliminaries

2.1 Notation and terminology

Let $(X, \|\cdot\|)$ a Banach space and let $BC(\mathbb{R}, X)$ be the space of bounded continuous functions $f : \mathbb{R} \longrightarrow X$. The space $BC(\mathbb{R}, X)$, equipped with the supremum norm $\|f\|_\infty = \sup\limits_{t\in\mathbb{R}}\|f(t)\|$, is a Banach space.

We denote by $\mathcal{B}$ the Lebesgue σ-field of $\mathbb{R}$ and by $\mathcal{M}$ the space of all positive measures ϑ on $\mathcal{B}$ satisfying $\vartheta(\mathbb{R}) = +\infty$ and $\vartheta([a, b]) < \infty$, for all $a, b \in \mathbb{R}$ $(a \leq b)$.

Definition 2.1 ([6]). *A continuous function $f : \mathbb{R} \to X$ is said to be almost periodic if for every $\varepsilon > 0$ there exists a positive number l_ε such that every interval of length l_ε contains a number τ such that:*

$$\|f(t + \tau) - f(t)\| < \varepsilon, \quad \forall t \in \mathbb{R}.$$

Let $AP(\mathbb{R}, X)$ denote the collection of almost periodic functions from $\mathbb{R}$ to X. We recall that $(AP(\mathbb{R}, X), \|\cdot\|_\infty)$ is a Banach space.

Definition 2.2 ([11]). *Let $\mu, \nu \in \mathcal{M}$. A bounded continuous function $f : \mathbb{R} \to X$ is said to be (μ,ν)-ergodic if*

$$\lim_{r\to+\infty} \frac{1}{\nu([-r, r])} \int_{-r}^{r} \|f(t)\| d\mu(t) = 0.$$

We denote the space of all such functions by $\mathcal{E}(\mathbb{R}, X, \mu, \nu)$.

The space $(\mathcal{E}(\mathbb{R}, X, \mu, \nu), \|.\|_\infty)$ is a Banach space for the supremum norm.

Definition 2.3 ([11]). *Let $\mu, \nu \in \mathcal{M}$. A continuous function $f : \mathbb{R} \to X$ is said to be (μ,ν)-pseudo almost periodic if f admits the following decomposition:*

$$f = g + \phi,$$

where $g \in AP(\mathbb{R}, X)$ and $\phi \in \mathcal{E}(\mathbb{R}, X, \mu, \nu)$.

We denote the space of all such functions by $PAP(\mathbb{R}, X, \mu, \nu)$.

We have $AP(\mathbb{R},X) \subset PAP(\mathbb{R},X,\mu,\nu) \subset BC(\mathbb{R},X)$.

Let $(Y,\|\cdot\|)$ a Banach space and let $BC(\mathbb{R}\times Y,X)$ be the space of jointly bounded continuous functions $f:\mathbb{R}\times Y \longrightarrow X$. The space $BC(\mathbb{R}\times Y,X)$ equipped with the supremum norm $\|f\|_\infty = \sup_{t\in\mathbb{R},x\in Y}\|f(t,x)\|$ is a Banach space.

Definition 2.4 ([12]). *A jointly continuous function $f:\mathbb{R}\times Y\to X$ is said to be almost periodic in t uniformly with respect to $x\in Y$, if for every $\varepsilon>0$, and any compact subset K of Y, there exists a positive number $l_K(\varepsilon)$ such that every interval of length $l_K(\varepsilon)$ contains a number τ such that:*

$$\|f(t+\tau,x)-f(t,x)\|<\varepsilon, \quad \forall (t,x)\in\mathbb{R}\times K.$$

We denote the space of such functions by $APU(\mathbb{R}\times Y,X)$.

Definition 2.5 ([11]). *Let $\mu,\nu\in\mathcal{M}$. A continuous function $f:\mathbb{R}\times Y\to X$ is said to be (μ,ν)-ergodic in t uniformly with respect to $x\in Y$, if the following two conditions are true:*

(i) *f is uniformly continuous on each compact set K in Y with respect to the second variable x.*

(ii) *$\forall x\in Y,\ f(.,x)\in\mathcal{E}(\mathbb{R},X,\mu,\nu)$.*

The space of such functions is denoted by $\mathcal{E}U(\mathbb{R}\times Y,X,\mu,\nu)$.

Definition 2.6 ([11]). *Let $\mu,\nu\in\mathcal{M}$. A continuous function $f:\mathbb{R}\times Y\to X$ is said to be (μ,ν)-pseudo almost periodic in t uniformly for $x\in Y$, if f admits the following decomposition:*

$$f=g+\phi, \tag{2.1}$$

where $g\in APU(\mathbb{R}\times Y,X)$ and $\phi\in\mathcal{E}U(\mathbb{R}\times Y,X,\mu,\nu)$.
The collection of such functions is denoted by $PAPU(\mathbb{R}\times Y,X,\mu,\nu)$.

We have $APU(\mathbb{R}\times Y,X)\subset PAPU(\mathbb{R}\times Y,X,\mu,\nu)\subset BC(\mathbb{R}\times Y,X,\mu,\nu)$.

Definition 2.7 ([16]). *A continuous function $f:\mathbb{R}\to X$ is said to be almost automorphic if for every sequence of real numbers $(s'_n)_{n\in\mathbb{N}}$, there exists a subsequence $(s_n)_{n\in\mathbb{N}}\subset(s'_n)_{n\in\mathbb{N}}$ such that:*

$$\lim_{n,m\to\infty} f(t+s_n-s_m)=f(t), \quad for \quad each \quad t\in\mathbb{R}.$$

Equivalently,

$$g(t)=\lim_{n\to\infty}f(t+s_n) \quad exists \quad \forall t\in\mathbb{R} \quad and \quad f(t)=\lim_{n\to\infty}g(t-s_n) \quad \forall t\in\mathbb{R}.$$

We denote the space of such functions by $AA(\mathbb{R},X)$.

We recall that $(AA(\mathbb{R},X), \|\cdot\|_\infty)$ is a Banach space.

Definition 2.8 ([11]). *Let $\mu,\nu \in \mathcal{M}$. A continuous function $f : \mathbb{R} \to X$ is said to be (μ,ν)-pseudo almost automorphic if f admits the following decomposition:*

$$f = g + \phi,$$

where $g \in AA(\mathbb{R}, X)$ and $\phi \in \mathcal{E}(\mathbb{R}, X, \mu, \nu)$.
We denote the space of all such functions by $PAA(\mathbb{R}, X, \mu, \nu)$.

We have $AA(\mathbb{R}, X) \subset PAA(\mathbb{R}, X, \mu, \nu) \subset BC(\mathbb{R}, X)$.

Definition 2.9 ([16]). *A continuous function $f : \mathbb{R} \times Y \to X$ is said to be almost automorphic in t uniformly for $x \in Y$, if the following conditions hold:*

(i) *f is uniformly continuous on each compact set K in Y with respect to the second variable x, namely, for each compact set K in Y, for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x_1, x_2 \in K$, one has:*

$$\|x_1 - x_2\| \leq \delta \Rightarrow \sup_{t\in\mathbb{R}} \|f(t, x_1) - f(t, x_2)\| \leq \varepsilon.$$

(ii) *for all $x \in Y$, $f(., x) \in AA(\mathbb{R}, X)$.*

Denote by $AAU(\mathbb{R} \times Y, X)$ the set of all such functions.

Definition 2.10 ([11]). *Let $\mu,\nu \in \mathcal{M}$. A continuous function $f : \mathbb{R} \times Y \to X$ is said to be (μ,ν)-pseudo almost periodic in t uniformly for $x \in Y$, if f admits the following decomposition:*

$$f = g + \phi, \tag{2.2}$$

where $g \in AAU(\mathbb{R} \times Y, X)$ and $\phi \in \mathcal{E}U(\mathbb{R} \times Y, X, \mu, \nu)$.
The collection of such functions is denoted by $PAAU(\mathbb{R} \times Y, X, \mu, \nu)$

We have $AAU(\mathbb{R} \times Y, X) \subset PAAU(\mathbb{R} \times Y, X, \mu, \nu) \subset BC(\mathbb{R} \times Y, X, \mu, \nu)$

2.2 Some useful results on the space functions

For $\mu \in \mathcal{M}$ and $\tau \in \mathbb{R}$, we denote by μ_τ the positive measure on $(\mathbb{R}, \mathcal{B})$ defined by:

$$\mu_\tau(A) = \mu(\{a + \tau : a \in A\}), \quad \forall A \in \mathcal{B}.$$

Now we introduce the following hypotheses on $\mu, \nu \in \mathcal{M}$.

($\mathbf{H}_0$): For all $\tau \in \mathbb{R}$, there exists $\delta > 0$ and a bounded interval I such that

$$\mu_\tau(A) \leq \delta\mu(A), \quad \nu_\tau(A) \leq \delta\nu(A), \quad \forall A \in \mathcal{B} \quad \text{satisfied} \quad A \cap I = \emptyset.$$

($\mathbf{H}_1$):

$$\limsup_{r\to\infty} \frac{\mu([-r,r])}{\nu([-r,r])} < \infty.$$

Remark 2.11.

i) *Without assumptions on μ and ν, like ($\boldsymbol{H}_0$), the decomposition* (2.1) *(resp.* (2.2)*) of the* (μ,ν)*-pseudo almost periodic and automorphic functions is not unique, (see [11]).*

ii) *The spaces $\mathcal{E}(\mathbb{R}, X, \mu, \nu)$, $\mathcal{E}(\mathbb{R}\times Y, X, \mu, \nu)$, $PAP(\mathbb{R}, X, \mu, \nu)$, $PAP(\mathbb{R}\times Y, X, \mu, \nu)$, $PAA(\mathbb{R}, X, \mu, \nu)$, and $PAA(\mathbb{R} \times Y, X, \mu, \nu)$ coincides when $\mu = \nu$, with the spaces $\mathcal{E}(\mathbb{R}, X, \mu)$, $\mathcal{E}(\mathbb{R} \times Y, X, \mu)$, $PAP(\mathbb{R}, X, \mu)$, $PAP(\mathbb{R} \times Y, X, \mu)$, $PAA(\mathbb{R}, X, \mu)$, and $PAA(\mathbb{R} \times Y, X, \mu)$.*

We recall the following six theorems proved in [11].

Theorem 2.12 ([11])**.** *Consider that $\mu, \nu \in \mathcal{M}$ and $k \in L^1(\mathbb{R})$ and $f \in PAP(\mathbb{R}, X, \mu, \nu)$ (respectively $f \in PAA(\mathbb{R}, X, \mu, \nu)$. If ($\boldsymbol{H}_0$) is valid then $PAP(\mathbb{R}, X, \mu, \nu)$ (respectively $PAA(\mathbb{R}, X, \mu, \nu)$) is translation invariant. Moreover,*

$$\{g(t) : t \in \mathbb{R}\} \subset \overline{\{f(t) : t \in \mathbb{R}\}}, \textit{ (the closure of the range of } f\textit{)}.$$

Theorem 2.13 ([11])**.** *If ($\boldsymbol{H}_0$) is valid, then the decomposition* (2.1)*(resp.* (2.2)*) of $PAP(\mathbb{R}, X, \mu, \nu)$ and $PAA(\mathbb{R}, X, \mu)$ is unique.*

Theorem 2.14 ([11])**.** *If ($\boldsymbol{H}_1$) holds, then $(\mathcal{E}(\mathbb{R}, X, \mu, \nu), \|\cdot\|_\infty)$ is a Banach space with respect to the sup norm.*

Theorem 2.15 ([11])**.** *Let $\mu, \nu \in \mathcal{M}$ satisfy ($\boldsymbol{H}_1$). If ($\boldsymbol{H}_0$) holds, then $PAP(\mathbb{R}, X, \mu, \nu)$ and $PAA(\mathbb{R}, X, \mu, \nu)$ are Banach spaces with respect to the sup norm.*

Theorem 2.16 ([11])**.** *Let $\mu, \nu \in \mathcal{M}$, $F \in PAPU(\mathbb{R} \times Y, X, \mu, \nu)$ and $h \in PAP(\mathbb{R}, X, \mu, \nu)$. Assume that ($\boldsymbol{H}_1$) and the following hypothesis holds:*
For all bounded subsets B of X, F is bounded on $\mathbb{R} \times B$.
Then $t \longmapsto F(t, h(t)) \in PAP(\mathbb{R}, X, \mu, \nu)$.

Theorem 2.17 ([11])**.** *Let $\mu, \nu \in \mathcal{M}$, $F \in PAAU(\mathbb{R} \times Y, X, \mu, \nu)$ and $h \in PAA(\mathbb{R}, X, \mu, \nu)$. Assume that for all bounded subsets B of X, F is bounded on $\mathbb{R} \times B$. Then $t \longmapsto F(t, h(t)) \in PAA(\mathbb{R}, X, \mu, \nu)$.*

2.3 Measure theory results

Let $\mu, \nu \in \mathcal{M}$; if $f : \mathbb{R} \longrightarrow X$ is a bounded continuous function, we define the following doubly-weight mean, if the limit exists, by:

$$\mathbf{M}(f, \mu, \nu) := \lim_{r \to +\infty} \frac{1}{\nu([-r, r])} \int_{-r}^{r} \|f(t)\|_X d\mu(t).$$

Definition 2.18 ([17]). *Let $(E, \mathcal{B})$ be a Borel space. If μ and ν are two measures defined on $(E, \mathcal{B})$, we say that:*

(i) *μ and ν are mutually singular, if there are disjoint sets A and B in $\mathcal{B}$ such that $E = A \cup B$ and*

$$\nu(A) = \mu(B) = 0.$$

(ii) *ν is absolutely continuous with respect to μ, if for each $A \in \mathcal{B}$,*

$$(\mu(A) = 0) \Longrightarrow (\nu(A) = 0).$$

We recall the following theorems of measure theory.

Theorem 2.19 (Radon-Nikodym [17]). *Let $(E, \mathcal{B}, \mu)$ be a σ-finite measure space, and let ν be a measure defined on $\mathcal{B}$ which is absolutely continuous with respect to μ. Then there is a unique nonnegative measurable function f such that for each set B in $\mathcal{B}$ we have:*

$$\nu(B) = \int_B f d\mu.$$

The function f is called the Radon-Nikodym derivative of ν with respect of μ.

Example 2.20.

Let ρ be a nonnegative $\mathcal{B}$-measurable function. Denote by μ the positive measure defined by:

$$\mu(A) = \int_A \rho(t)dt, \quad \text{for} \quad A \in \mathcal{B}$$

where dt is the Lebesgue measure on $\mathbb{R}$. The function ρ is the Radon-Nikodym derivative of μ with respect to the Lebesgue measure dt on $\mathbb{R}$, i.e. $d\mu(t) = \rho(t)dt$. In this case, $\mu \in \mathcal{M}$ if and only if its Radon-Nikodym derivative ρ is locally Lebesgue integrable on $\mathbb{R}$ and it satisfies

$$\int_{-\infty}^{+\infty} \rho(t)dt = +\infty.$$

Theorem 2.21 (Lebesgue-Radon-Nikodym [17]). *Let $(X, \mathcal{B}, \vartheta)$ be a σ-finite measure space, and μ a σ-finite measure defined on $\mathcal{B}$. Then, we can find a measure μ_0, singular with respect to ϑ, and a measure μ_1, absolutely continuous with respect to ϑ, such that $\mu = \mu_0 + \mu_1$. The measures μ_0 and μ_1 are unique.*

In this section, by using the previous theorem, we consider that for a given $\mu \in \mathcal{M}$, $\mu = \mu_0 + \mu_1$ where μ_0 is the μ-measure component which is absolutely continuous with respect to the Lebesgue measure and its Radon-Nikodym derivative is ρ, that is $d\mu_0(t) = \rho(t)dt$ and μ_1 is the μ-measure component such that μ_1 is singular to Lebesgue measure.
We give new general hypotheses on $\mu, \nu \in \mathcal{M}$ and $k \in L^1(\mathbb{R})$ such that:

$$(\zeta f)(t) = \int_{-\infty}^{+\infty} k(t-s)f(s)ds, \quad \forall k \in L^1(\mathbb{R}) \tag{2.3}$$

maps $\mathcal{E}(\mathbb{R}, X, \mu, \nu)$ into itself.
In particular, our hypotheses on $\mu, \nu \in \mathcal{M}$ and $k \in L^1(\mathbb{R})$ will imply that for every $f \in \mathcal{E}(\mathbb{R}, X, \mu, \nu)$, the (μ, ν)-mean,

$$\mathbf{M}(\zeta f, \mu, \nu) := \lim_{r \to +\infty} \frac{1}{\nu([-r,r])} \int_{-r}^{r} \left\| \int_{-\infty}^{+\infty} k(t-s)f(s)ds \right\|_X d\mu(t)$$

exists.

3 Main results of convolution and translation invariance

3.1 Convolution invariance on $\mathcal{E}(\mathbb{R}, X, \mu, \nu)$

Theorem 3.1. *Let $k \in L^1(\mathbb{R})$ and $\nu \in \mathcal{M}$. Consider that $\mu \in \mathcal{M}$, with Radon-Nikodym derivative ρ with respect to dt and ζ is defined in* (2.3). *Assume that ρ, μ, ν and k satisfy the following requirements:*

$$\begin{cases} \sup\limits_{|s| \le r, r \in \mathbb{R}_+} \dfrac{1}{\rho(s)} \int_s^r |k(t-s)| d\mu(t) < \infty, & (3.1.1) \\ \sup\limits_{|s| \le r, r \in \mathbb{R}_+} \dfrac{1}{\rho(s)} \int_{-r}^s |k(t-s)| d\mu(t) < \infty, & (3.1.2) \end{cases} \tag{3.1}$$

$$\begin{cases} \lim\limits_{r \to +\infty} \dfrac{1}{\nu([-r,r])} \int_{-\infty}^{-r} \left(\int_{-r}^{r} |k(t-s)| d\mu(t) \right) ds = 0, & (3.2.1) \\ \lim\limits_{r \to +\infty} \dfrac{1}{\nu([-r,r])} \int_{r}^{+\infty} \left(\int_{-r}^{r} |k(t-s)| d\mu(t) \right) ds = 0. & (3.2.2) \end{cases} \tag{3.2}$$

If $f \in \mathcal{E}(\mathbb{R}, X, \mu, \nu)$, then $\zeta f \in \mathcal{E}(\mathbb{R}, X, \mu, \nu)$.

Proof. We adapt the proof in [15], Theorem 3.5. By the properties of convolution we have that $f \in BC(\mathbb{R}, X)$ implies that $k * f \in BC(\mathbb{R}, X)$, $\forall k \in L^1(\mathbb{R})$. Then, in order to get that $k * f \in \mathcal{E}(\mathbb{R}, X, \mu, \nu)$ we must prove that $\mathbf{M}(\zeta f, \mu, \nu) = 0$.
We consider $\mu \in \mathcal{M}$ and ρ its Radon-Nikodym derivative, $\nu \in \mathcal{M}$. In the first stage, we assume that $k(t) = 0$ on $\mathbb{R}_-^*$. From $\nu(\mathbb{R}) = +\infty$, we deduce the existence of $r_0 \ge 0$ such that $\nu([-r,r]) > 0, \quad \forall r \ge r_0$. Then by applying the Fubini's Theorem, we deduce that for $f \in BC(\mathbb{R}, X)$, $\forall r \ge r_0$.

We notice that

$$
\begin{aligned}
\mathbf{M}(\zeta f,\mu,\nu) &= \lim_{r\to+\infty}\frac{1}{\nu([-r,r])}\int_{-r}^{r}\|k*f\|_X d\mu(t)\\
&\le \lim_{r\to+\infty}\frac{1}{\nu([-r,r])}\int_{-r}^{r}\int_{-\infty}^{t}\| f(s)\|_X|\,k(t-s)\,|\,dsd\mu(t)\\
&= \lim_{r\to+\infty}\frac{1}{\nu([-r,r])}\int_{-r}^{r}\left(\int_{-\infty}^{-r}\| f(s)\|_X|\,k(t-s)\,|\,ds\right)d\mu(t)\\
&+ \lim_{r\to+\infty}\frac{1}{\nu([-r,r])}\int_{-r}^{r}\left(\int_{-r}^{t}\| f(s)\|_X|\,k(t-s)\,|\,ds\right)d\mu(t)\\
&\le \| f\|_\infty \lim_{r\to+\infty}\frac{1}{\nu([-r,r])}\int_{-r}^{r}\left(\int_{-\infty}^{-r}|\,k(t-s)\,|\,ds\right)d\mu(t)\\
&+ \lim_{r\to+\infty}\frac{1}{\nu([-r,r])}\int_{-r}^{r}\| f(s)\|_X\left(\int_{s}^{r}|\,k(t-s)\,|\,d\mu(t)\right)ds\\
&\le \| f\|_\infty \lim_{r\to+\infty}\frac{1}{\nu([-r,r])}\int_{-\infty}^{-r}\left(\int_{-r}^{r}|\,k(t-s)\,|\,d\mu(t)\right)ds\\
&+ \lim_{r\to+\infty}\frac{1}{\nu([-r,r])}\int_{-r}^{r}\| f(s)\|_X\left[\frac{1}{\rho(s)}\int_{s}^{r}|\,k(t-s)\,|\,d\mu(t)\right]\rho(s)ds\\
&\le \sup_{|s|\le r, r\in\mathbb{R}_+}\frac{1}{\rho(s)}\int_{s}^{r}|k(t-s)|d\mu(t)\lim_{r\to+\infty}\frac{1}{\nu([-r,r])}\int_{-r}^{r}\|f(s)\|_X\rho(s)ds\\
&+ \| f\|_\infty \lim_{r\to+\infty}\frac{1}{\nu([-r,r])}\int_{-\infty}^{-r}\left(\int_{-r}^{r}|\,k(t-s)\,|\,d\mu(t)\right)ds\\
&\le \sup_{|s|\le r, r\in\mathbb{R}_+}\frac{1}{\rho(s)}\int_{s}^{r}|k(t-s)|d\mu(t)\lim_{r\to+\infty}\frac{1}{\nu([-r,r])}\int_{-r}^{r}\|f(s)\|_X d\mu(s)\\
&+ \| f\|_\infty \lim_{r\to+\infty}\frac{1}{\nu([-r,r])}\int_{-\infty}^{-r}\left(\int_{-r}^{r}|\,k(t-s)\,|\,d\mu(t)\right)ds.
\end{aligned}
$$

Using assumptions (3.1.1), (3.2.1) and the fact that $f\in\mathcal{E}(\mathbb{R},X,\mu,\nu)$, we have proved that $\mathbf{M}(\zeta f,\mu,\nu)=0+0=0$. This settles the first stage for every $k\in L^1(\mathbb{R})$ such that $k(t)=0$ on $\mathbb{R}_-^*$. Now, in the **second stage**, proceeding similarly like in the first stage, we assume that $k(t)=0$ on $\mathbb{R}_+^*$ we obtain:

$$
\begin{aligned}
\mathbf{M}(\zeta f,\mu,\nu) &= \lim_{r\to+\infty}\frac{1}{\nu([-r,r])}\int_{-r}^{r}\|k*f\|_X d\mu(t)\\
&\le \| f\|_\infty \lim_{r\to+\infty}\frac{1}{\nu([-r,r])}\int_{r}^{+\infty}\left(\int_{-r}^{r}|\,k(t-s)\,|\,d\mu(t)\right)ds\\
&+ \sup_{|s|\le r, r\in\mathbb{R}_+}\frac{1}{\rho(s)}\int_{-r}^{s}|k(t-s)|d\mu(t)\lim_{r\to+\infty}\frac{1}{\nu([-r,r])}\int_{-r}^{r}\|f(s)\|_X d\mu(s).
\end{aligned}
$$

Then, using the fact that $f\in\mathcal{E}(\mathbb{R},X,\mu,\nu)$ and hypotheses (3.1.2), (3.2.2), we have that $\mathbf{M}(\zeta f,\mu,\nu)=0$.

In the general case of k, we deduce the result using the fact that $k(t)=k\chi_{t\ge0}(t)+k\chi_{t<0}(t)$.

Theorem 3.2. *Assume that $\mu,\nu\in\mathcal{M}$ and ($\boldsymbol{H}_1$) holds. Then the condition* (3.2.1) *(resp.* (3.2.2)*)* *is valid for every $k \in L^1(\mathbb{R})$ if and only if the following condition (3.3.1) (resp. (3.3.2)) is true:*

$$\begin{cases} \lim\limits_{r\to+\infty} \dfrac{\mu([-r,\sigma-r])}{\nu([-r,r])} = 0, \quad \forall\sigma>0 \quad (3.3.1) \\ \lim\limits_{r\to+\infty} \dfrac{\mu([\sigma+r,r])}{\nu([-r,r])} = 0, \quad \forall\sigma<0 \quad (3.3.2). \end{cases} \tag{3.3}$$

Proof. **We first prove that** (3.3.1) $\Longrightarrow$ (3.2.1), for every $k \in L^1(\mathbb{R})$.
In the first stage, we assume that $k(t) = 0$ on $\mathbb{R}_-^*$. Let $\sigma = t - s > 0$ fixed. From $\nu(\mathbb{R}) = +\infty$, we deduce the existence of $r_0 \geq 0$ such that $\nu([-r,r]) > 0, \quad \forall r \geq r_0$. In the sequel, for all $r \geq r_0$, we shall assume that

$$B := \frac{1}{\nu([-r,r])}\int_{-r}^{r}\left(\int_{-\infty}^{-r} \mid k(t-s) \mid ds\right) d\mu(t).$$

Then, by applying the Fubini's Theorem we deduce that:

$$\begin{aligned} B &= \frac{1}{\nu([-r,r])}\int_{-r}^{r}\left(\int_{t+r}^{+\infty} \mid k(\sigma) \mid d\sigma\right) d\mu(t) \\ &= \int_{0}^{+\infty}\left(\frac{\int_{-r}^{\min(\sigma-r,r)} d\mu(t)}{\nu([-r,r])}\right) \mid k(\sigma) \mid d\sigma \\ &= \int_{0}^{+\infty}\left(\frac{\mu([-r,\min(\sigma-r,r)])}{\nu([-r,r])}\right) \mid k(\sigma) \mid d\sigma. \end{aligned}$$

By using assumption (3.3.1), we have:

$$\lim_{r\to+\infty} \frac{\mu([-r,\min(\sigma-r,r)])}{\nu([-r,r])} = 0, \quad \forall\sigma>0.$$

Since $\mu([-r,\min(\sigma-r,r)]) \leq \mu([-r,r])$ and the fact that $(\mathbf{H}_1)$ holds, there exists $\beta > 0$ such that:

$$0 \leq \left(\frac{\mu([-r,\min(\sigma-r,r)])}{\nu([-r,r])}\right) \mid k(\sigma) \mid \leq \beta \mid k(\sigma) \mid,$$

where $k \in L^1(\mathbb{R})$, $\forall\sigma > 0$. Then, by the Lebesgue dominated convergence Theorem, we obtain:

$$\lim_{r\to+\infty}\int_{0}^{+\infty}\left(\frac{\mu([-r,\min(\sigma-r,r)])}{\nu([-r,r])}\right) \mid k(\sigma) \mid d\sigma = 0$$

This concludes this stage of (3.2.1).

Now, in the **second stage**, proceeding similarly like the first stage, we assume that $k(t) = 0$ on $\mathbb{R}_+^*$. Let $\sigma = t - s < 0$ fixed. From $\nu(\mathbb{R}) = +\infty$, we deduce the existence of $r_0 \geq 0$ such that $\nu([-r,r]) > 0$, $\nu([-r,r]) > 0, \quad \forall r \geq r_0$. We set:

$$A := \frac{1}{\nu([-r,r])}\int_{-r}^{r}\left(\int_{r}^{+\infty} \mid k(t-s) \mid ds\right) d\mu(t)$$

Then, by applying the Fubini's Theorem, we deduce that:

$$\begin{aligned}
A &= \frac{1}{\nu([-r,r])}\int_{-r}^{r}\left(\int_{-\infty}^{t-r} \mid k(\sigma)\mid d\sigma\right)d\mu(t)\\
&= \int_{-\infty}^{0}\left(\frac{\int_{\max(\sigma+r,r)}^{r} d\mu(t)}{\nu([-r,r])}\right)\mid k(\sigma)\mid d\sigma\\
&= \int_{-\infty}^{0}\left(\frac{\mu([\max(\sigma+r,r),r])}{\nu([-r,r])}\right)\mid k(\sigma)\mid d\sigma.
\end{aligned}$$

Like in the first part, we use assumptions (3.3.2), and the Lebesgue dominated convergence Theorem. This concludes this second stage of (3.2.2).

Let us prove (3.2.1) $\Longrightarrow$ (3.3.1)**.**

Let $\sigma = t-s$, by (3.2.1) and Fubini's Theorem we have that:

$$\begin{aligned}
0 &= \lim_{r\to+\infty}\frac{1}{\nu([-r,r])}\int_{-r}^{r}\left(\int_{-\infty}^{-r}\mid k(t-s)\mid ds\right)d\mu(t)\\
&= \lim_{r\to+\infty}\int_{0}^{+\infty}\left(\frac{\mu([-r,\min(\sigma-r,r)])}{\nu([-r,r])}\right)\mid k(\sigma)\mid d\sigma.
\end{aligned}$$

Let $\tau > 0$ such that $\sigma\in[\tau,\tau+1]$ and $r > \dfrac{\tau}{2}$. We have also $[-r,\tau-r]\subseteq[-r,\sigma-r]$ and $[-r,\tau-r]\subseteq[-r,r]$, that implies $\mu([-r,\tau-r])\le\min\{\mu([-r,\sigma-r]),\mu([-r,r])\}$, i.e.

$$\frac{\mu([-r,\tau-r])}{\nu([-r,r])}\le\frac{\mu([-r,\min(\sigma-r,r)])}{\nu([-r,r])}.$$

Let $k(\sigma)=\chi_{[\tau,\tau+1]}(\sigma)$. We have that:

$$\begin{aligned}
0 &\le \frac{\mu([-r,\tau-r])}{\nu([-r,r])}\int_{\tau}^{\tau+1}d\sigma\\
&\le \int_{\tau}^{\tau+1}\frac{\mu([-r,\min(\sigma-r,r)])}{\nu([-r,r])}d\sigma\\
&= \lim_{r\to+\infty}\frac{1}{\nu([-r,r])}\int_{-r}^{r}\left(\int_{-\infty}^{-r}\mid k(t-s)\mid ds\right)d\mu(t).
\end{aligned}$$

Then by (3.2.1):

$$\limsup_{r\to+\infty}\frac{\mu([-r,\tau-r])}{\nu([-r,r])}\int_{\tau}^{\tau+1}d\sigma\le\lim_{r\to+\infty}\int_{\tau}^{\tau+1}\frac{\mu([-r,\min(\sigma-r,r)])}{\nu([-r,r])}d\sigma=0$$

then:

$$\lim_{r\to+\infty}\frac{\mu([-r,\tau-r])}{\nu([-r,r])}=0.$$

So (3.3.1) is verified. In the second stage (3.3.2), we do the same proof as above.

Remark 3.3. *Hypothesis ($\boldsymbol{H}_1$) was used only in the proof of the implication* (3.3.1) $\Longrightarrow$ (3.2.1).

Corollary 3.4. *Let $\mu, \nu \in \mathcal{M}$ be such that the nonnegative $\mathcal{B}$-measurable function ρ be the Radon-Nikodym derivative of μ. Assume that for all $k \in L^1(\mathbb{R})$ the requirements* (3.1) *and* (3.2) *are satisfied. Then $\mathcal{E}(\mathbb{R}, X, \mu, \nu)$ is convolution invariant.*

Corollary 3.5. *Consider that $\mu, \nu \in \mathcal{M}$, such that the nonnegative $\mathcal{B}$-measurable function ρ be the Radon-Nikodym derivative of μ. Assume that ($\boldsymbol{H}_1$) holds and the requirements* (3.1) *and* (3.3) *are satisfied. Then $\mathcal{E}(\mathbb{R}, X, \mu, \nu)$ is convolution invariant.*

Example 3.6.

We check that Theorem 3.1 and Corollary 3.5 hold.

Let

$$k(t) = \begin{cases} \dfrac{1}{10} e^{-2t}, \text{ for } t \in [0, +\infty[\\ 0, \text{ for } t \in]-\infty, 0[. \end{cases}$$

We take $d\mu_{k,\eta}(t) = e^{\sigma t} dt + \eta \sum_{n=-\infty}^{\infty} e^{\sigma n} \delta_n$, where $0 \le \sigma < 2$, $\eta > 0$ and δ_n denotes the Dirac measure at the integer n ($\sum_{n=-\infty}^{\infty} e^{\sigma n} \delta_n$ is a 'generalized Dirac comb', it is called a Dirac comb when $\sigma = 0$). Then $\mu_{\sigma,\eta} \in \mathcal{M}$ and its Radon-Nikodym derivative is $\rho_{\sigma,\eta}(t) = e^{\sigma t}$. Let $\nu_{\sigma,\eta} = \gamma \mu_{\sigma,\eta}$, where $\gamma > 0$. Then $\nu_{\sigma,\eta} \in \mathcal{M}$.

First, if for $|s| \le r,\ r > 0$, we write:

$$J_{\sigma,\eta}(r,s) := \frac{1}{\rho_{\sigma,\eta}(s)} \int_s^r |k(t-s)|\, d\mu_\sigma(t),$$

we must prove that:

$$\sup_{|s| \le r,\ r>0} J_{\sigma,\eta}(r,s) < \infty.$$

In fact,

$$\begin{aligned} J_{\sigma,\eta}(r,s) &= \frac{1}{10 e^{\sigma s}} \left(\int_s^r e^{-2(t-s)} e^{\sigma t} dt + \eta \sum_{s \le n \le r} e^{-2(n-s)} e^{\sigma n} \right) \\ &= \frac{1}{10} \left(\int_s^r e^{-(2-\sigma)(t-s)} dt + \eta \sum_{s \le n \le r} e^{-(2-\sigma)(n-s)} \right) \\ &\le \frac{1}{10} \left(\int_0^{r-s} e^{-(2-\sigma)u} du + \eta \sum_{[s] \le n \le [r]} e^{-(2-\sigma)(n-[s]-1)} \right), \end{aligned}$$

where we applied the change of integral $u = t - s$ in the integral and we denoted $[x]$ the integral part of the real number x. We next apply the change of index $m = n - [s]$ in the latter sum; this implies:

$$J_{\sigma,\eta}(r,s) \leq \frac{1}{10}\left(\int_0^{r-s} e^{-(2-\sigma)(u)}du + \eta e^{2-\sigma}\sum_{m=0}^{[r]-[s]} e^{-(2-\sigma)m}\right).$$

Then:

$$\begin{aligned}\sup_{|s|\leq r,\ r>0} J_{\sigma,\eta}(r,s) &\leq \frac{1}{10}\left(\int_0^{\infty} e^{-(2-\sigma)(u)}du + \eta e^{2-\sigma}\sum_{m=0}^{\infty} e^{-(2-\sigma)m}\right)\\ &= \frac{1}{10(2-\sigma)} + \eta e^{2-\sigma}\frac{1}{10(1-e^{-(2-\sigma)})} < \infty.\end{aligned}$$

This proves the estimate (3.1.1).

Secondly, we shall show that for all $\alpha > 0$, we have:

$$\lim_{r\to\infty}\frac{\mu_{\sigma,\eta}([-r,\alpha-r])}{\nu_{\sigma,\eta}([-r,r])} = 0.$$

It actually suffices to prove this estimate when α is a positive integer. In fact,

$$\begin{aligned}\frac{\mu_{\sigma,\eta}([-r,\alpha-r])}{\nu_{\sigma,\eta}([-r,r])} &= \frac{\displaystyle\int_{-r}^{\alpha-r} e^{\sigma t}dt + \eta \sum_{-r\leq n\leq \alpha-r} e^{\sigma n}}{\gamma\left(\displaystyle\int_{-r}^{r} e^{\sigma t}dt + \eta\sum_{-r\leq n\leq r} e^{\sigma n}\right)}\\ &\leq \frac{\dfrac{1}{\sigma}\left(e^{\sigma(\alpha-r)} - e^{-\sigma r}\right) + \eta \displaystyle\sum_{-[r]-1\leq n\leq \alpha-[r]} e^{\sigma n}}{\gamma\left(\dfrac{1}{\sigma}\left(e^{\sigma r} - e^{-\sigma r}\right) + \eta\displaystyle\sum_{-[r]\leq n\leq [r]} e^{\sigma n}\right)}\\ &= \frac{\dfrac{1}{\sigma}e^{-\sigma r}\left(e^{\sigma\alpha}-1\right) + \eta e^{-\sigma([r]+1)}\displaystyle\sum_{m=0}^{\alpha+1} e^{\sigma m}}{\gamma\left(\dfrac{1}{\sigma}\left(e^{\sigma r} - e^{-\sigma r}\right) + \eta e^{-\sigma[r]}\displaystyle\sum_{m=0}^{2[r]} e^{\sigma m}\right)},\end{aligned}$$

where we applied the change of index $m = n + [r] + 1$ on the numerator and the change of index $m = n + [r]$ on the denominator. So

$$\frac{\mu_{\sigma,\eta}([-r,\alpha-r])}{\mu_{\sigma,\eta}([-r,r])} \leq \frac{\dfrac{1}{\sigma}e^{-\sigma r}\left(e^{\sigma\alpha}-1\right) + \eta\dfrac{e^{\sigma(\alpha+2)}-1}{e^{\sigma}-1}e^{-\sigma([r]+1)}}{\dfrac{\gamma}{\sigma}e^{\sigma r}\left(1-e^{-2\sigma r}\right) + \gamma\eta e^{-\sigma[r]}\dfrac{e^{\sigma(2[r]+1)}-1}{e^{\sigma}-1}}.$$

The estimation (3.3.1) easily follows.

Thirdly, we show that $(\mathbf{H}_1)$ holds.

$$\limsup_{r\to\infty}\frac{\mu_{\sigma,\eta}([-r,r])}{\nu_{\sigma,\eta}([-r,r])} = \frac{1}{\gamma} < \infty.$$

Then, Theorem 3.1 and Corollary 3.5 hold.

3.2 Translation invariance and convolution invariance of $PAP(\mathbb{R}, X, \mu, \nu)$ and $PAA(\mathbb{R}, X, \mu, \nu)$

Theorem 3.7. *Assume that $\mu, \nu \in \mathcal{M}$ and $(\boldsymbol{H}_1)$ holds. If the space $\mathcal{E}(\mathbb{R}, X, \mu, \nu)$ is translation invariant, then $\mathcal{E}(\mathbb{R}, X, \mu, \nu)$ is convolution invariant.*

Proof. Let $f \in \mathcal{E}(\mathbb{R}, X, \mu, \nu)$. Let us prove that if $f(t-\tau) \in \mathcal{E}(\mathbb{R}, X, \mu, \nu)$, for $\tau \in \mathbb{R}$, then $\zeta f \in \mathcal{E}(\mathbb{R}, X, \mu, \nu)$, i.e. $\mathbf{M}(\zeta f, \mu, \nu) = 0$. By the properties of convolution we have that $f \in BC(\mathbb{R}, X)$ implies that $k * f \in BC(\mathbb{R}, X)$, $\forall k \in L^1(\mathbb{R})$. By the Fubini's Theorem we have,

$$\begin{aligned} \mathbf{M}(\zeta f, \mu, \nu) &= \lim_{r\to+\infty} \frac{1}{\nu([-r,r])} \int_{-r}^{r} \|k * f\|_X d\mu(t) \\ &\leq \lim_{r\to+\infty} \int_{-\infty}^{\infty} \frac{|k(s)|}{\nu([-r,r])} \left(\int_{-r}^{r} \|f(t-s)\|_X d\mu(t) \right) ds \end{aligned}$$

Since f is invariant by translation we have for all $s \in \mathbb{R}$:

$$\lim_{r\to+\infty} \frac{1}{\nu([-r,r])} \int_{-r}^{r} \|f(t-s)\|_X d\mu(t) = 0.$$

Since $(\mathbf{H}_1)$ holds for all $s \in \mathbb{R}$, we have that:

$$0 \leq \frac{|k(s)|}{\nu([-r,r])} \int_{-r}^{r} \|f(t-s)\|_X d\mu(t) ds \leq \beta |k(s)| \|f\|_\infty,$$

where $k \in L^1(\mathbb{R})$. Then by the Lebesgue dominated convergence Theorem, we obtain that $\mathbf{M}(\zeta f, \mu, \nu) = 0$.

Theorem 3.8. *Let $(\boldsymbol{H}_1)$ holds. If the space $PAP(\mathbb{R}, X, \mu, \nu)$ (resp. $PAA(\mathbb{R}, X, \mu, \nu)$) is translation invariant, then $\mathcal{E}(\mathbb{R}, X, \mu, \nu)$ is convolution invariant.*

Proof. For $f \in AP(\mathbb{R}, X)$ or $f \in AA(\mathbb{R}, X)$, then f is invariant by ζ i.e. $\zeta f \in AP(\mathbb{R}, X)$ or $\zeta f \in AA(\mathbb{R}, X)$. We use the previous theorem to conclude.

Corollary 3.9. *Let $(\boldsymbol{H}_0)$ and $(\boldsymbol{H}_1)$ hold. Then $\mathcal{E}(\mathbb{R}, X, \mu, \nu)$, $PAP(\mathbb{R}, X, \mu, \nu)$ and $PAA(\mathbb{R}, X, \mu, \nu)$ are convolution invariant.*

Proof. Combine Theorem 2.12 and Theorem 3.8.

4 Existence, Uniqueness results and Applications

This section is similar to section 3 in [11], but here we applied our new results obtained in the above section.

4.1 Evolution Families and Exponential Dichotomy

($\mathbf{H}_2$): A family of closed linear operators $A(t)$ for $t \in \mathbb{R}$ on X with domain $D(A(t))$ (possibly not densely defined), is said to satisfy the so-called Acquistapace-Terreni conditions, if there exist constants $\omega \in \mathbb{R}$, $\theta \in (\frac{\pi}{2}, \pi)$, $K, L \geq 0$ and $\mu_0, \nu_0 \in (0, 1]$, with $1 < \mu_0 + \nu_0$ such that

$$\Sigma_\theta \cup \{0\} \subset \rho(A(t) - \omega) \ni \lambda, \|R(\lambda, A(t) - \omega)\| \leq \frac{K}{1 + |\lambda|}, \tag{4.1}$$

and

$$\|(A(t) - \omega)R(\lambda, A(t) - \omega)[R(\omega, A(t)) - R(\omega, A(s))]\| \leq L\frac{|t - s|^{\mu_0}}{|\lambda|^{\nu_0}}, \tag{4.2}$$

for $t, s \in \mathbb{R}, \lambda \in \Sigma_\theta := \{\lambda \in \mathbb{C}/\{0\} : |arg\lambda| \leq \theta\}$.

For a given family of linear operators $A(t)$, the existence of an evolution family associated with it is not always guaranteed. However, if $A(t)$ satisfied Acquistapace-Terreni conditions, then there exists a unique evolution family

$$\mathcal{U} = \{\mathcal{U}(t, s) : t, s \in \mathbb{R}, t \geq s\}$$

on X associated with $A(t)$ such that $\mathcal{U}(t, s)X \subseteq D(A(t))$ for all $t, s \in \mathbb{R}$ with $t \geq s$, and,

i) $\mathcal{U}(t, r)\mathcal{U}(r, s) = \mathcal{U}(t, s)$ and $\mathcal{U}(s, s) = I$ $\forall t \geq r \geq s$ and $t, r, s \in \mathbb{R}$;

ii) the map $(t, s) \longrightarrow \mathcal{U}(t, s)x$ is continuous for all $x \in X$, $t \geq s$ and $t, s \in \mathbb{R}$;

iii) $\mathcal{U}(., s) \in C^1((s, \infty), B(X))$, $\dfrac{\partial \mathcal{U}}{\partial t}(t, s) = A(t)\mathcal{U}(t, s)$ and

$$\|A(t)^k\mathcal{U}(t, s)\| \leq K(t - s)^{-k}$$

for $0 < t - s \leq 1, k = 0, 1$.

Definition 4.1 ([3]). *An evolution family $(\mathcal{U}(t, s))_{t\geq s}$ on a Banach space X is called hyperbolic (or has exponential dichotomy) if there exist projections $P(t)$, $t \in \mathbb{R}$, uniformly bounded and strongly continuous in t, and constants $N \geq 1$, $\delta > 0$ such that*

i) *$\mathcal{U}(t, s)P(s) = P(t)\mathcal{U}(t, s)$ for $t \geq s$;*

ii) *the restriction $\mathcal{U}_Q(t, s) : Q(s)X \longrightarrow Q(t)X$ for $\mathcal{U}(t, s)$ is inversible for $t, s \in \mathbb{R}$ and we set $\mathcal{U}_Q(t, s) = \mathcal{U}(s, t)^{-1}$;*

iii)
$$\|\mathcal{U}(t,s)P(s)\| \leq Ne^{-\delta(t-s)} \tag{4.3}$$

and
$$\|\mathcal{U}_Q(s,t)Q(t)\| \leq Ne^{-\delta(t-s)} \tag{4.4}$$

for $t \geq s$ and $t, s \in \mathbb{R}$, where $Q(t) := I - P(t)$

4.2 Existence Results

To study the existence and uniqueness of (μ,ν)- pseudo-almost periodic (respectively, (μ,ν)-pseudo-almost automorphic) solutions to equation (1.1), we also assume that the next hypothesis holds:

$(\mathbf{H}_3)$ The evolution family $\mathbb{U}$ generated by $A(.)$ has an exponential dichotomy with constants $N \geq 1$, $\delta > 0$ and dichotomy projections $P(t)$.

We recall the following sufficient conditions to fulfill the assumption $(\mathbf{H}_3)$.

$(\mathbf{H}_{3.1})$ Let $(A(t), D(A(t)))_{t\in\mathbb{R}}$ be generators of analytic semigroups on X of the same type. Suppose that $D(A(t)) = D(A(0))$, $A(t)$ is inversible, $\sup_{t,s\in\mathbb{R}} \|A(t)A(s)^{-1}\|$ is finite, and
$$\|A(t)A(s)^{-1} - I\| \leq L_0|t-s|^{\mu_1}$$
for $t, s \in \mathbb{R}$ and constants $L_0 \geq 0$ and $0 \leq \mu_1 \leq 1$.

$(\mathbf{H}_{3.2})$ The semigroup $(e^{\tau A(t)})_{\tau\geq 0}$, $t \in \mathbb{R}$, are hyperbolic with projection P_t and constants $N, \delta > 0$. Moreover, let
$$\|A(t)(e^{\tau A(t)}P_t)\| \leq \Psi(\tau), \quad \|A(t)(e^{\tau A(t)}Q_t)\| \leq \Psi(-\tau)$$
for $\tau > 0$ and a function Ψ such that $\mathbb{R} \ni s \longrightarrow \varphi(s) := |s|^\mu \Psi(s)$ is integrable with $L_0\|\varphi\|_{L^1(\mathbb{R})} < 1$.

We introduce here the defnition of the mild solution of equation (1.1).

Definition 4.2 ([3]). *A continuous function $u : \mathbb{R} \longmapsto X$ is called a bounded mild solution of equation* (1.1) *if:*
$$u(t) = \mathcal{U}(t,s)u(s) + \int_s^t \mathcal{U}(t,\tau)F(\tau, u(\tau))d\tau, \ \forall t, s \in \mathbb{R}, with \quad t \geq s. \tag{4.5}$$

Theorem 4.3 ([11]). *Assume that ($\boldsymbol{H}_2$) and ($\boldsymbol{H}_3$) hold. If there exists $0 < K_F < \frac{\delta}{2N}$ such that*
$$\|F(t,u) - F(t,v)\| \leq K_F\|u - v\|,$$
for all $u, v \in X$ and $t \in \mathbb{R}$, then the equation (1.1) *has a unique bounded mild solution $u : \mathbb{R} \longmapsto X$ given by*
$$u(t) = \int_{\mathbb{R}} \Gamma(t,s)F(s,u(s))ds, \quad t \in \mathbb{R},$$

where the operator family $\Gamma(t,s)$*, called Green's function corresponding to* $\mathcal{U}$ *and* $P(\cdot)$*, is given by*

$$\Gamma(t,s) = \mathcal{U}(t,s)P(s), \quad \forall t,s \in \mathbb{R}, with \quad t \geq s,$$
$$\Gamma(t,s) = -\mathcal{U}_Q(t,s)Q(s), \quad \forall t,s \in \mathbb{R}, with \quad t < s.$$

Denote by Γ_1 and Γ_2 the nonlinear integral operators defined by,

$$(\Gamma_1 u)(t) := \int_{-\infty}^{t} \mathcal{U}(t,s)P(s)F(s,u(s))ds,$$

and

$$(\Gamma_2 u)(t) := \int_{t}^{+\infty} \mathcal{U}_Q(t,s)Q(s)F(s,u(s))ds.$$

In the rest of this work, we fix $\mu,\nu \in \mathcal{M}$ to satisfy $(\mathbf{H}_1)$.

4.3 Existence of (μ,ν)-pseudo-almost periodic solutions

In addition to the previous assumptions, we require the following additional ones:
$(\mathbf{H}_4)$: $R(\omega, A(.)) \in AP(\mathbb{R}, \mathcal{L}(X))$.
$(\mathbf{H}_5)$: We propose $F : \mathbb{R}\times X \longmapsto X$ belongs to $PAP(\mathbb{R}\times X, X, \mu, \nu)$ and there exists $K_F > 0$ such that

$$\|F(t,u) - F(t,v)\| \leq K_F\|u - v\|,$$

for all $u, v \in X$ and $t \in \mathbb{R}$.
The following Lemma plays an important role to prove the main results of this study.

Lemma 4.4 ([13])**.** *Assume that* $(\boldsymbol{H}_2)$*-*$(\boldsymbol{H}_4)$ *hold. Then* $r \longrightarrow \Gamma(t+r, s+r)$ *belongs to* $AP(\mathbb{R}, \mathcal{L}(X))$ *for all* $t, s \in \mathbb{R}$*, where we may take the same pseudo periods for* t, s *with* $|t - s| \geq h > 0$*. If* $f \in AP(\mathbb{R}, \mathcal{L}(X))$*, then the unique bounded mild solution* $u(t) = \int_{\mathbb{R}} \Gamma(t,s)f(s)ds$ *of the following equation*

$$u'(t) = A(t)u(t) + f(t), \quad t \in \mathbb{R},$$

is almost periodic.

Lemma 4.5. *Assume that* $(\boldsymbol{H}_2)$*-*$(\boldsymbol{H}_5)$ *hold. If* (3.1) *and* (3.2)*, or* (3.1) *and* (3.3) *hold, then the integral operators* Γ_1 *and* Γ_2 *defined above map* $PAP(\mathbb{R}, X, \mu, \nu)$ *into itself.*

Proof. Let $u \in PAP(\mathbb{R}, X, \mu, \nu)$. setting $h(t) = F(t, u(t))$, using the assumption $(\mathbf{H}_5)$ and Theorem 2.16 it follows that $h \in PAP(\mathbb{R}, X, \mu, \nu)$. Now write $h = \Psi_1 + \Psi_2$ where $\Psi_1 \in AP(\mathbb{R}, X)$ and $\Psi_2 \in \mathcal{E}(\mathbb{R}, X, \mu, \nu)$. That is, $\Gamma_1 h = \Xi(\Psi_1) + \Xi(\Psi_2)$ where

$$\Xi\Psi_i(t) := \int_{-\infty}^{t} U(t,s)P(s)\Psi_i(s)ds, for \quad i \in \{1,2\}.$$

From Lemma 4.4, we have $\Xi(\Psi_1) \in AP(\mathbb{R}, X)$. To complete the proof, we will prove that $\Xi(\Psi_2) \in \mathcal{E}(\mathbb{R}, X, \mu, \nu)$. Now, let $r > 0$. From equation (4.3), we have:

$$\begin{aligned}
\frac{1}{\nu([r,-r])}\int_{-r}^{r}\|(\Xi(\Psi_2)(t)\|d\mu(t) &\leq \frac{1}{\nu([r,-r])}\int_{-r}^{r}\int_{-\infty}^{t}U(t,s)P(s)\Psi_2(s)dsd\mu(t)\\
&\leq \frac{N}{\nu([r,-r])}\int_{-r}^{r}\int_{-\infty}^{t}e^{-\delta(t-s)}\|\Psi_2(s)\|dsd\mu(t)
\end{aligned}$$

Since μ and ν satisfy (3.1.1) and (3.2.1), (3.1.1) and (3.3.1), with $k(t) = e^{-\delta t}$, then by Theorem 3.1 or Corollary 3.5, we conclude that:

$$\lim_{r\to+\infty}\frac{1}{\nu([r,-r])}\int_{-r}^{r}\|(\Xi(\Psi_2)(t)\|d\mu(t) = 0.$$

The proof for $\Gamma_2 u(.)$ is similar to that of $\Gamma_1 u(.)$ except that one makes use of equation (4.4) instead of (4.3), (3.1.2) and (3.2.2), or (3.1.2) and (3.3.2).

Theorem 4.6. *Assume that ($\boldsymbol{H}_2$)-($\boldsymbol{H}_5$) hold. If* (3.1) *and* (3.2)*, or* (3.1) *and* (3.3) *hold, then equation* (1.1) *has a unique* (μ,ν)*-pseudo almost periodic mild solution whenever* K_F *is small enough.*

Proof. Consider the nonlinear operator $\mathbb{K}$ defined on $PAP(\mathbb{R}, X, \mu, \nu)$ by

$$\mathbb{K}u(t) = \int_{-\infty}^{t}U(t,s)P(s)F(s,u(s))ds - \int_{t}^{+\infty}U_Q(t,s)Q(s)F(s,u(s))ds, \quad \forall t \in \mathbb{R}.$$

By Lemma 4.5, it follows that $\mathbb{K}$ maps $PAP(\mathbb{R}, X, \mu, \nu)$ into itself. To complete the proof one has to show that $\mathbb{K}$ is a contraction map on $PAP(\mathbb{R}, X, \mu, \nu)$.
Let $u, v \in PAP(\mathbb{R}, X, \mu, \nu)$. Firstly, we have that:

$$\begin{aligned}
\|\Gamma_1(v)(t) - \Gamma_1(u)(t)\| &\leq \int_{-\infty}^{t}\|U(t,s)P(s)[F(s,v(s)) - F(s,u(s))]\|ds\\
&\leq NK_F\int_{-\infty}^{t}e^{-\delta(t-s)}\|v(s) - u(s)\|ds\\
&\leq NK_F\delta^{-1}\|v-u\|_\infty.
\end{aligned}$$

Next, we have that:

$$\begin{aligned}
\|\Gamma_2(v)(t) - \Gamma_2(u)(t)\| &\leq \int_{t}^{+\infty}\|U_Q(t,s)Q(s)[F(s,v(s)) - F(s,u(s))]\|ds\\
&\leq NK_F\int_{t}^{+\infty}e^{-\delta(t-s)}\|v(s) - u(s)\|ds\\
&\leq NK_F\delta^{-1}\|v-u\|_\infty\int_{t}^{+\infty}e^{-\delta(t-s)}ds\\
&= NK_F\delta^{-1}\|v-u\|_\infty.
\end{aligned}$$

Finally, combining previous approximations it follows that:

$$\|\mathbb{K}v - \mathbb{K}u\|_\infty < 2NK_F\delta^{-1}\|v-u\|_\infty.$$

Thus if K_F is small enough, that is, $K_F < \delta(2N)^{-1}$, then $\mathbb{K}$ is a contraction map on $PAP(\mathbb{R}, X, \mu, \nu)$. Therefore, $\mathbb{K}$ has a unique fixed point in $PAP(\mathbb{R}, X, \mu, \nu)$, that is, there exists a unique function u satisfying $\mathbb{K}u = u$, which is the unique (μ,ν)-pseudo almost periodic mild solution to equation (1.1).

Theorem 4.7 ([11])**.** *Assume that ($\boldsymbol{H}_2$)-($\boldsymbol{H}_5$) hold. If ($\boldsymbol{H}_0$) holds, then equation* (1.1) *has a unique (μ,ν)-pseudo almost periodic mild solution whenever K_F is small enough.*

4.4 Existence of (μ,ν)-pseudo-almost automorphic solutions

In this section we consider the following assumptions:

($\mathbf{H}_6$): $R(\omega, A(.)) \in AA(\mathbb{R}, \mathcal{L}(X))$.

($\mathbf{H}_7$): We propose $F : \mathbb{R} \times X \longmapsto X$ belongs to $PAA(\mathbb{R} \times X, X, \mu, \nu)$ and there exists $K_F > 0$ such that

$$\|F(t,u) - F(t,v)\| \leq K_F \|u - v\|_\infty,$$

for all $u, v \in X$ and $t \in \mathbb{R}$.

Lemma 4.8 ([14])**.** *Assume that ($\boldsymbol{H}_2$), ($\boldsymbol{H}_3$) and ($\boldsymbol{H}_6$) hold. Let a sequence $(s'_l)_{l\in\mathbb{N}} \subset \mathbb{R}$ there is a sub-sequence $(s_l)_{l\in\mathbb{N}}$ such that for every $h > 0$*

$$\|\Gamma(t + s_l - s_k, s + s_l - s_k) - \Gamma(t,s)\| \longrightarrow 0, \quad k, l \longrightarrow \infty.$$

Lemma 4.9. *Assume that ($\boldsymbol{H}_2$), ($\boldsymbol{H}_3$), ($\boldsymbol{H}_6$) and ($\boldsymbol{H}_7$) hold. If* (3.1) *and* (3.2) *or* (3.1) *and* (3.3) *or ($\boldsymbol{H}_0$) hold, then the integral operators Γ_1 and Γ_2 defined above map $PAA(\mathbb{R} \times X, X, \mu, \nu)$ into itself.*

Proof. Let $u \in PAA(\mathbb{R}, X, \mu, \nu)$. Setting $g(t) = F(t, u(t))$, by assumption ($\mathbf{H}_7$) and Theorem 2.17 we obtain that $g \in PAA(\mathbb{R}, X, \mu, \nu)$. Now write $g = u_1 + u_2$ where $u_1 \in AA(\mathbb{R}, X)$ and $u_2 \in \mathcal{E}(\mathbb{R}, X, \mu, \nu)$. That is, $\Gamma_1 g = \mathcal{S}u_1 + \mathcal{S}u_2$, where

$$\mathcal{S}u_1(t) := \int_{-\infty}^{t} U(t,s)P(s)u_1(s)ds, \quad \mathcal{S}u_2(t) := \int_{-\infty}^{t} U(t,s)P(s)u_2(s)ds.$$

From equation (4.3), we obtain:

$$\|\mathcal{S}u_1(t)\| \leq N\delta^{-1}\|u_1\|_\infty, \quad \|\mathcal{S}u_2(t)\| \leq N\delta^{-1}\|u_2\|_\infty, \quad \forall t \in \mathbb{R}.$$

Then $\mathcal{S}u_1(t), \mathcal{S}u_2(t) \in BC(\mathbb{R}, X)$. Now, we prove that $\mathcal{S}u_1(t) \in AA(\mathbb{R}, X)$. Since $u_1 \in AA(\mathbb{R}, X)$, then for every sequence $(\tau'_n)_{n\in\mathbb{N}} \in \mathbb{R}$ there exists a subsequence $(\tau_n)_{n\in\mathbb{N}}$ such that:

$$v_1(t) := \lim_{n\to\infty} u_1(t + \tau_n), \tag{4.6}$$

is well defined for each $t \in \mathbb{R}$, and

$$\lim_{n\to\infty} v_1(t - \tau_n) = u_1(t), \ \forall t \in \mathbb{R}. \tag{4.7}$$

Set for $t \in \mathbb{R}$,

$$M(t) := \int_{-\infty}^{t} U(t,s)P(s)u_1(s)ds, \text{ and } N(t) := \int_{-\infty}^{t} U(t,s)P(s)v_1(s)ds.$$

Now, we have

$$\begin{aligned}
M(t+\tau_n) - N(t) &= \int_{-\infty}^{t+\tau_n} U(t+\tau_n,s)P(s)u_1(s)ds - \int_{-\infty}^{t} U(t,s)P(s)v_1(s)ds \\
&= \int_{-\infty}^{t} U(t+\tau_n,s+\tau_n)P(s+\tau_n)u_1(s+\tau_n)ds \\
&- \int_{-\infty}^{t} U(t,s)P(s)v_1(s)ds \\
&= \int_{-\infty}^{t} U(t+\tau_n,s+\tau_n)P(s+\tau_n)[u_1(s+\tau_n) - v_1(s)]ds \\
&+ \int_{-\infty}^{t} [U(t+\tau_n,s+\tau_n)P(s+\tau_n) - U(t,s)P(s)]v_1(s)ds.
\end{aligned}$$

Using equation (4.3), equation (4.6) and the Lebesgue's Dominated Convergence Theorem, it follows that:

$$\lim_{n\to+\infty} \left|\left| \int_{-\infty}^{t} U(t+\tau_n,s+\tau_n)P(s+\tau_n)[u_1(s+\tau_n) - v_1(s)]ds \right|\right| = 0, \text{ for } t \in \mathbb{R}. \tag{4.8}$$

Similary, using Lemma 4.8 it follows that:

$$\lim_{n\to+\infty} \left|\left| \int_{-\infty}^{t} [U(t+\tau_n,s+\tau_n)P(s+\tau_n) - U(t,s)P(s)]v_1(s)ds \right|\right| = 0, \text{ for } \quad t \in \mathbb{R}. \tag{4.9}$$

Therefore, we have that:

$$N(t) := \lim_{n\to\infty} M(t+\tau_n), \forall t \in \mathbb{R}. \tag{4.10}$$

Using similar ideas as the previous ones, then:

$$M(t) := \lim_{n\to\infty} N(t-\tau_n), \forall t \in \mathbb{R}. \tag{4.11}$$

Therefore, $\mathcal{S}u_1(t) \in AA(\mathbb{R}, X)$. Arguing as in Lemma 4.5, we get that $\mathcal{S}u_2(t) \in \mathcal{E}(\mathbb{R}, X, \mu, \nu)$. The proof for $\Gamma_2 u(.)$ is similar to that of $\Gamma_1 u(.)$ except that one makes use of equation (4.4) instead of equation (4.3) and, (3.1.2) and (3.2.2), (3.1.2) and (3.3.2).

Theorem 4.10. *Under assumptions ($\boldsymbol{H}_2$), ($\boldsymbol{H}_3$), ($\boldsymbol{H}_6$) and ($\boldsymbol{H}_7$), if* (3.1) *and* (3.2) *or* (3.1) *and* (3.3) *or ($\boldsymbol{H}_0$) then equation* (1.1) *has a unique* (μ,ν)*-pseudo almost automorphic mild solution whenever* K_F *is small enough.*

Proof The proof of Theorem 4.10 is similar to that Theorem 4.6 except that one makes use of Lemma 4.9 instead of Lemma 4.5.

4.5 Neutral Systems

In this subsection, we establish the existence and uniqueness of (μ,ν)-pseudo almost periodic (respectively (μ,ν)-pseudo almost automorphic) solutions for the nonautonomous neutral partial evolution equation (1.2). For that, we need the following assumptions:
$(\mathbf{H}_8)$: We suppose $G:\mathbb{R}\times X\longrightarrow X$ belongs to $PAP(\mathbb{R}\times X, X,\mu,\nu)$ and there exists $K_G>0$ such that:

$$\|G(t,u)-G(t,v)\|\leq K_G\|u-v\|,$$

for all $u,v\in X$ and $t\in\mathbb{R}$.
$(\mathbf{H}_9)$ We suppose $G:\mathbb{R}\times X\longrightarrow X$ belongs to $PAA(\mathbb{R}\times X, X,\mu,\nu)$ and there exists $K_G>0$ such that:

$$\|G(t,u)-G(t,v)\|\leq K_G\|u-v\|,$$

for all $u,v\in X$ and $t\in\mathbb{R}$.

Definition 4.11. *A function $v:\mathbb{R}\longmapsto X$ is said a mild solution of* (1.2) *on $\mathbb{R}$ if :*

$$v(t)=G(t,v(t))+\int_{-\infty}^{t}U(t,s)P(s)F(s,v(s))ds-\int_{t}^{+\infty}U_Q(t,s)Q(s)F(s,v(s))ds,$$

for all $t\in\mathbb{R}$.

Theorem 4.12. *Assume that assumptions $(\boldsymbol{H}_2)$-$(\boldsymbol{H}_5)$ and $(\boldsymbol{H}_8)$ hold. If* (3.1) *and* (3.2) *or* (3.1) *and* (3.3) *or $(\boldsymbol{H}_0)$ hold, and $(K_G+2NK_F\delta^{-1})<1$, then equation* (1.2) *has a unique (μ,ν)-pseudo almost periodic mild solution.*

Proof. We consider the nonlinear operator $\mathbb{W}$ defined on $PAP(\mathbb{R},X,\mu,\nu)$ by:

$$\mathbb{W}v(t)=G(t,v(t))+\int_{-\infty}^{t}U(t,s)P(s)F(s,v(s))ds-\int_{t}^{+\infty}U_Q(t,s)Q(s)F(s,v(s))ds$$

for all $t\in\mathbb{R}$. From $(\mathbf{H}_9)$, Theorem 2.16, and Lemma 4.5 it follows that $\mathbb{W}$ maps $PAP(\mathbb{R},X,\mu,\nu)$ into itself. To complete the proof we need to show that $\mathbb{W}$ is a contraction map on $PAP(\mathbb{R},X,\mu,\nu)$. For that, letting $u,v\in PAP(\mathbb{R},X,\mu,\nu)$, we obtain:

$$\|\mathbb{W}v-\mathbb{W}u\|_\infty\leq(K_G+2NK_F\delta^{-1})\|v-u\|_\infty,$$

which yields $\mathbb{W}$ is a contraction map on $PAP(\mathbb{R},X,\mu,\nu)$. Therefore, $\mathbb{W}$ has unique fixed point in $PAP(\mathbb{R},X,\mu,\nu)$. Therefore, equation (1.2) has unique (μ,ν)-pseudo almost periodic mild solution.

Theorem 4.13. *Assume that $(\boldsymbol{H}_2)$, $(\boldsymbol{H}_3)$,$(\boldsymbol{H}_6)$, $(\boldsymbol{H}_7)$ and $(\boldsymbol{H}_9)$ hold and $(K_G+2NK_F\delta^{-1})<1$. If* (3.1) *and* (3.2)*or* (3.1) *and* (3.3) *or $(\boldsymbol{H}_0)$ hold, then equation* (1.2) *has a unique (μ,ν)-pseudo almost automorphic mild solution.*

Proof. Similarly, we can show, by using the assumption $(\mathbf{H}_9)$, Theorem 2.17 and Lemma 4.9, that the equation (1.2) has a unique (μ,ν)-pseudo almost automorphic mild solution.

Acknowledgments

The authors are grateful to the anonymous referee for the careful reading of this paper, for very helpful suggestions and comments which improved of quality of this article.

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Cubo
A Mathematical Journal

CUBO, A Mathematical Journal
Vol. 23, no. 01, pp. 87–96, April 2021
DOI: 10.4067/S0719-06462021000100087

Hyper generalized pseudo Q-symmetric semi-Riemannian manifolds

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ABSTRACT

The object of the present paper is to study the properties of a hyper generalized pseudo Q-symmetric semi-Riemannian manifold, proving that under certain assumptions, it is a perfect fluid spacetime.

RESUMEN

El objetivo del presente artículo es estudiar las propiedades de una variedad semi-Riemanniana hiper generalizada pseudo Q-simétrica, probando que bajo ciertas condiciones, es un espacio-tiempo fluido perfecto.

Keywords and Phrases: Q-curvature tensor, perfect fluid spacetime.

2020 AMS Mathematics Subject Classification: 53C15, 53C25.

Accepted: 21 January, 2021
Received: 04 July, 2020

1 Introduction

Let R, S, L and r denote the curvature tensor, Ricci tensor, Ricci operator and the scalar curvature of a (semi)-Riemannian manifold, respectively. It is Mantica and Suh [5] who have introduced the notion of *Q-curvature tensor*. In an n-dimensional Riemannian or semi-Riemannian manifold (M^n, g) $(n > 2)$, the Q-curvature tensor is defined as

$$R(Y,U,V,W) = Q(Y,U,V,W) + \frac{\psi}{n-1}[g(Y,W)g(U,V) - g(Y,V)g(U,W)], \tag{1.1}$$

where Y, U, V, W are arbitrary vector fields on M^n and ψ is a scalar function. Semi-Riemannian manifolds with Ricci tensor S of the form

$$S(Y,V) = ag(Y,V) + bT(Y)T(V),$$

for any vector fields Y, V, are often termed as perfect fluid spacetimes, where a and b are scalars and the vector field ϱ, metrically equivalent to the 1-form T (that is, $g(Y,\varrho) = T(Y)$), is a unit time like vector field (that is, $g(\varrho,\varrho) = -1$).

An n-dimensional semi-Riemannian manifold is said to be *hyper generalized pseudo Q-symmetric* (which will be abbreviated hereafter as $(HGPQS)_n$) if it satisfies the equation

$$\begin{aligned} & (\nabla_X Q)(Y,U,V,W) \\ = \; & 2A_1(X)Q(Y,U,V,W) + A_1(Y)Q(X,U,V,W) \\ & + A_1(U)Q(Y,X,V,W) + A_1(V)Q(Y,U,X,W) \\ & + A_1(W)Q(Y,U,V,X) + 2A_2(X)(g \wedge S)(Y,U,V,W) \\ & + A_2(Y)(g \wedge S)(X,U,V,W) + A_2(U)(g \wedge S)(Y,X,V,W) \\ & + A_2(V)(g \wedge S)(Y,U,X,W) + A_2(W)(g \wedge S)(Y,U,V,X), \end{aligned} \tag{1.2}$$

where

$$\begin{aligned} (g \wedge S)(Y,U,V,W) = \; & g(Y,W)S(U,V) + g(U,V)S(Y,W) \\ & - g(Y,V)S(U,W) - g(U,W)S(Y,V) \end{aligned} \tag{1.3}$$

and A_1, A_2 are non-zero 1-forms whose g-dual vector fields will be denoted by θ_1 and θ_2, i.e. $A_1(X) = g(X,\theta_1)$ and $A_2(X) = g(X,\theta_2)$.

We organized our paper as follows: section 2 is concerned with preliminaries. After preliminaries, some curvature properties of $(HGPQS)_n$ manifolds are studied in section 3. It is obtained that the Q-curvature tensor in a $(HGPQS)_n$ manifold satisfies 2nd Bianchi's identity. It is further obtained that the scalar function ψ is always constant. In section 4 we investigate properties of divergence-free $(HGPQS)_n$ manifolds and we prove that a divergence-free $(HGPQS)_n$ manifold $(n > 2)$ under the assumption $A_1(Q(Y,U)V) = 0$ is a perfect fluid spacetime as well as the integral curves of the vector field ϱ are geodesics and the vector field ϱ is irrotational, if the associated vector fields ϱ and σ corresponding to the 1-forms T_1 and T_2 are related by $(r-1)\varrho + n\sigma = 0$.

2 Preliminaries

In this section, some relations useful to the study of $(HGPQS)_n$ manifolds are obtained. Let $\{e_i\}$ be an orthonormal basis of the tangent space at each point of the manifold, where $1 \leq i \leq n$.

From (1.1) we can easily verify that the tensor Q satisfies the following properties:

$$\begin{aligned} &(i) \quad Q(Y,U)V + Q(U,Y)V = 0, \\ &(ii) \quad Q(Y,U)V + Q(U,V)Y + Q(V,Y)U = 0, \end{aligned} \tag{2.1}$$

where $g(Q(X,Y)U,V) = Q(X,Y,U,V)$.

Also from (1.1) we have

$$\sum_{i=1}^{n} \epsilon_i Q(X,Y,e_i,e_i) = 0 = \sum_{i=1}^{n} \epsilon_i Q(e_i,e_i,W,U) \tag{2.2}$$

and

$$\begin{aligned} \sum_{i=1}^{n} \epsilon_i Q(e_i,Y,V,e_i) &= \sum_{i=1}^{n} \epsilon_i Q(Y,e_i,e_i,V) = S(Y,V) - \psi g(Y,V) \\ &=: Z(Y,V), \end{aligned} \tag{2.3}$$

where

$$\epsilon_i = g(e_i,e_i) = \pm 1, \ S(X,Y) = \sum_{i=1}^{n} \epsilon_i g(R(X,e_i)e_i,Y), \ r = \sum_{i=1}^{n} \epsilon_i S(e_i,e_i).$$

From (1.1) and (2.1) it follows that

$$\begin{aligned} &(i) \quad Q(X,Y,U,V) + Q(X,Y,V,U) = 0, \\ &(ii) \quad Q(X,Y,U,V) - Q(U,V,X,Y) = 0. \end{aligned} \tag{2.4}$$

3 Some curvature properties of $(HGPQS)_n$ manifolds

In this section we prove that in a $(HGPQS)_n$ manifold, the Q-curvature tensor satisfies 2nd Bianchi's identity, that is,

$$(\nabla_X Q)(Y,U,V,W) + (\nabla_Y Q)(U,X,V,W) + (\nabla_U Q)(X,Y,V,W) = 0. \tag{3.1}$$

In view of (1.1), (1.2) and (3.1) we get

$$\begin{aligned}
&(\nabla_X Q)(Y,U,V,W)+(\nabla_Y Q)(U,X,V,W)+(\nabla_U Q)(X,Y,V,W) \\
= \; & A_1(V)[Q(Y,U,X,W)+Q(U,X,Y,W)+Q(X,Y,U,W)] \\
&+A_1(W)[Q(Y,U,V,X)+Q(U,X,V,Y)+Q(X,Y,V,U)] \\
&+A_2(V)[(g\wedge S)(Y,U,X,W)+(g\wedge S)(U,X,Y,W) \\
&+(g\wedge S)(X,Y,U,W)]+A_2(W)[(g\wedge S)(Y,U,V,X) \\
&+(g\wedge S)(U,X,V,Y)+(g\wedge S)(X,Y,V,U)].
\end{aligned} \tag{3.2}$$

Using (1.3) and 1st Bianchi's identity for the Q-curvature tensor in (3.2) and then simplifying, we obtain (3.1).

Thus we can state the following:

Theorem 3.1. *The Q-curvature tensor in a $(HGPQS)_n$ manifold satisfies 2nd Bianchi's identity.*

Using (1.1) in (3.1), we have

$$\begin{aligned}
(\nabla_X R)(Y,U,V,W) \; + \; & (\nabla_Y R)(U,X,V,W)+(\nabla_U R)(X,Y,V,W) \\
- \; & \frac{d\psi(X)}{(n-1)}[g(Y,W)g(U,V)-g(Y,V)g(U,W)] \\
- \; & \frac{d\psi(Y)}{(n-1)}[g(U,W)g(X,V)-g(U,V)g(X,W)] \\
- \; & \frac{d\psi(U)}{(n-1)}[g(X,W)g(Y,V)-g(X,V)g(Y,W)]=0.
\end{aligned} \tag{3.3}$$

By virtue of 2nd Bianchi's identity for the Riemannian curvature tensor, (3.3) yields

$$\begin{aligned}
&\frac{d\psi(X)}{(n-1)}[g(Y,W)g(U,V)-g(Y,V)g(U,W)] \\
+&\frac{d\psi(Y)}{(n-1)}[g(U,W)g(X,V)-g(U,V)g(X,W)] \\
+&\frac{d\psi(U)}{(n-1)}[g(X,W)g(Y,V)-g(X,V)g(Y,W)]=0.
\end{aligned} \tag{3.4}$$

Contracting U and V in (3.4), we have

$$(n-2)[d\psi(X)g(Y,W)-d\psi(Y)g(X,W)]=0 \tag{3.5}$$

which yields after further contraction

$$(n-1)(n-2)d\psi(X)=0.$$

This implies that $d\psi(X)=0$, that is, ψ is constant since $n>2$ and leads to the following:

Theorem 3.2. *In a $(HGPQS)_n$ manifold, the scalar function ψ is always constant.*

Consequently, one can easily bring out the following:

Theorem 3.3. *In a $(HGPQS)_n$ manifold, $(divQ)(X,Y)Z$ and $(divR)(X,Y)Z$ are equivalent.*

In view of (1.1), (1.2) and Theorem 3.2 we have

$$\begin{aligned} & (\nabla_X R)(Y,U,V,W) \\ = \; & 2A_1(X)Q(Y,U,V,W) + A_1(Y)Q(X,U,V,W) \\ & +A_1(U)Q(Y,X,V,W) + A_1(V)Q(Y,U,X,W) \\ & +A_1(W)Q(Y,U,V,X) + 2A_2(X)(g \wedge S)(Y,U,V,W) \\ & +A_2(Y)(g \wedge S)(X,U,V,W) + A_2(U)(g \wedge S)(Y,X,V,W) \\ & +A_2(V)(g \wedge S)(Y,U,X,W) + A_2(W)(g \wedge S)(Y,U,V,X) \end{aligned} \tag{3.6}$$

which yields

$$\begin{aligned} & (\nabla_X S)(U,V) \\ = \; & [F_1(X) + F_2(X)]S(U,V) + F_2(U)S(X,V) + F_2(V)S(U,X) \\ & +[F_3(X) + F_4(X)]g(U,V) + F_4(U)g(X,V) + F_4(V)g(U,X) \\ & +A_1(Q(X,U)V) - A_1(Q(V,X)U) \end{aligned} \tag{3.7}$$

after contraction over Y and W, where

$$\begin{aligned} F_1(X) &= A_1(X) + (n+1)A_2(X), \\ F_2(X) &= A_1(X) + (n-3)A_2(X), \\ F_3(X) &= rA_2(X) - \psi A_1(X) + 3A_2(LX), \\ F_4(X) &= rA_2(X) - \psi A_1(X) - A_2(LX), \end{aligned}$$

where L is the Ricci operator defined by $g(LX,Y) = S(X,Y)$.

Definition 3.4. *An n-dimensional semi-Riemannian manifold is called almost generalized pseudo Ricci symmetric if the non-flat Ricci curvature tensor satisfies the equation*

$$\begin{aligned} & (\nabla_X S)(U,V) \\ = \; & [A(X) + B(X)]S(U,V) + A(U)S(X,V) + A(V)S(U,X) \\ & +[C(X) + D(X)]g(U,V) + C(U)g(X,V) + C(V)g(U,X), \end{aligned}$$

where A, B, C and D are non-zero 1-forms whose g-dual vector fields will be denoted by γ_1, γ_2, δ_1 and δ_2, i.e. $A(X) = g(X,\gamma_1)$, $B(X) = g(X,\gamma_2)$, $C(X) = g(X,\delta_1)$ and $D(X) = g(X,\delta_2)$.

Thus we can state the following:

Theorem 3.5. *A* $(HGPQS)_n$ *manifold* $(n > 2)$ *under the assumption* $A_1(Q(X, U)V) = A_1(Q(V, X)U)$ *is necessarily almost generalized pseudo Ricci symmetric.*

Making use of (2.3) in (3.7), we get

$$\begin{aligned} & (\nabla_X Z)(U, V) \\ = \; & [F_1(X) + F_2(X)]Z(U, V) + F_2(U)Z(X, V) + F_2(V)Z(U, X) \\ & +[F_3(X) + \psi F_1(X) + F_4(X) + \psi F_2(X)]g(U, V) \\ & +[F_4(U) + \psi F_2(U)]g(X, V) + [F_4(V) + \psi F_2(V)]g(U, X), \end{aligned} \tag{3.8}$$

where $Z = S - \psi g$ is the tensor considered in ([4], [6], [7]). This leads to the following:

Theorem 3.6. *A* $(HGPQS)_n$ *manifold* $(n > 2)$ *under the assumption* $A_1(Q(X, U)V) = A_1(Q(V, X)U)$ *is necessarily almost generalized pseudo* Z*-symmetric.*

4 $(\mathbf{HGPQS})_n$ manifolds $(n > 2)$ with $div\mathbf{Q} = 0$

Let (M^n, g) be a semi-Riemannian manifold of dimension n and let $\{e_i\}$ be an orthonormal basis of the tangent space T_pM at any point $p \in M$ and $\epsilon_i = \pm 1$. Then the divergence of a vector field U is defined as

$$divU = \sum_{i=1}^{n} \epsilon_i g(\nabla_{e_i} U, e_i),$$

and the divergence of a tensor field of type $(1, 3)$, which is a tensor field of type $(0, 3)$, is defined as

$$(divK)(X, Y)Z = \sum_{i=1}^{n} \epsilon_i g((\nabla_{e_i} K)(X, Y)Z, e_i).$$

Now

$$\begin{aligned} (divQ)(Y, U)V &= \sum_{i=1}^{n} \epsilon_i g((\nabla_{e_i} Q)(Y, U)V, e_i) \\ &= \sum_{i=1}^{n} \epsilon_i [2A_1(e_i)Q(Y, U, V, e_i) + A_1(Y)Q(e_i, U, V, e_i) \\ &\quad +A_1(U)Q(Y, e_i, V, e_i) + A_1(V)Q(Y, U, e_i, e_i) \\ &\quad +A_1(e_i)Q(Y, U, V, e_i) + 2A_2(e_i)(g \wedge S)(Y, U, V, e_i) \\ &\quad +A_2(Y)(g \wedge S)(e_i, U, V, e_i) + A_2(U)(g \wedge S)(Y, e_i, V, e_i) \\ &\quad +A_2(V)(g \wedge S)(Y, U, e_i, e_i) + A_2(e_i)(g \wedge S)(Y, U, V, e_i)] \end{aligned}$$

$$\begin{aligned}
&= 3A_1(Q(Y,U)V) + A_1(Y)[S(U,V) - \psi g(U,V)] \\
&\quad -A_1(U)[S(Y,V) - \psi g(Y,V)] + 3A_2(Y)S(U,V) \\
&\quad +3A_2(LY)g(U,V) - 3A_2(LU)g(Y,V) - 3A_2(U)S(Y,V) \\
&\quad +A_2(Y)[(n-2)S(U,V) + rg(U,V)] \\
&\quad -A_2(U)[(n-2)S(Y,V) + rg(Y,V)] \\
&= 3A_1(Q(Y,U)V) + S(U,V)[A_1(Y) + (n+1)A_2(Y)] \\
&\quad -S(Y,V)[A_1(U) + (n+1)A_2(U)] \\
&\quad +g(U,V)[3A_2(LY) + rA_2(Y) - \psi A_1(Y)] \\
&\quad -g(Y,V)[3A_2(LU) + rA_2(U) - \psi A_1(U)] \\
&= 3A_1(Q(Y,U)V) + T_1(Y)S(U,V) - T_1(U)S(Y,V) \\
&\quad +T_2(Y)g(U,V) - T_2(U)g(Y,V),
\end{aligned}$$

hence

$$\begin{aligned}
(divQ)(Y,U)V &= 3A_1(Q(Y,U)V) + T_1(Y)S(U,V) - T_1(U)S(Y,V) \\
&\quad +T_2(Y)g(U,V) - T_2(U)g(Y,V),
\end{aligned} \tag{4.1}$$

where

$$\begin{aligned}
T_1(Y) &= A_1(Y) + (n+1)A_2(Y) =: g(Y,\varrho), \text{ for } \varrho = \theta_1 + (n+1)\theta_2, \\
T_2(Y) &= 3A_2(LY) + rA_2(Y) - \psi A_1(Y) =: g(Y,\sigma), \text{ for } \sigma = 3L\theta_2 + r\theta_2 - \psi\theta_1.
\end{aligned}$$

Assuming $(divQ)(Y,U)V = 0$ and $A_1(Q(Y,U)V) = 0$, we get from the above equation

$$T_1(Y)S(U,V) + T_2(Y)g(U,V) = T_1(U)S(Y,V) + T_2(U)g(Y,V). \tag{4.2}$$

Now contracting (4.2) over U and V we get

$$S(Y,\varrho) = rT_1(Y) + (n-1)T_2(Y). \tag{4.3}$$

Again putting $V = \varrho$ in (4.2) we get

$$(n-2)[T_1(Y)T_2(U) - T_1(U)T_2(Y)] = 0, \tag{4.4}$$

which under the assumption $n > 2$ implies $T_1(Y)T_2(U) = T_1(U)T_2(Y)$.

Now putting $U = \varrho$ in (4.2) and using (4.3) and (4.4) we get

$$T_1(\varrho)S(Y,V) + T_2(\varrho)g(Y,V) = T_1(Y)[rT_1(V) + nT_2(V)] \tag{4.5}$$

and we can state:

Theorem 4.1. *A divergence-free* $(HGPQS)_n$ *manifold* $(n > 2)$ *under the assumption* $A_1(Q(Y,U)V) = 0$ *is a perfect fluid spacetime with unit timelike vector field* ϱ*, provided the associated vector fields* ϱ *and* σ *corresponding to the 1-forms* T_1 *and* T_2 *are related by* $(r-1)\varrho + n\sigma = 0$.

In this case, (4.5) becomes

$$S(Y,V) = ag(Y,V) - T_1(Y)T_1(V), \tag{4.6}$$

where $a =: T_2(\varrho)$.

Again, $(divQ)(Y,U)V = 0$ gives

$$(\nabla_Y S)(U,V) - (\nabla_U S)(Y,V) = 0. \tag{4.7}$$

Now using (4.6) in (4.7) we find

$$\begin{aligned} & da(Y)g(U,V) - da(U)g(Y,V) \\ & -[T_1(V)(\nabla_Y T_1)(U) + T_1(U)(\nabla_Y T_1)(V)] \\ & +[T_1(V)(\nabla_U T_1)(Y) + T_1(Y)(\nabla_U T_1)(V)] = 0. \end{aligned} \tag{4.8}$$

Taking a frame field and contracting Y and V we get

$$(n-1)da(U) + [T_1(U)(\delta T_1) + (\nabla_\varrho T_1)(U)] = 0, \tag{4.9}$$

where

$$\delta T_1 = \sum_{i=1}^{n} \epsilon_i (\nabla_{e_i} T_1)(e_i).$$

Setting $V = Y = \varrho$ in (4.8) we find

$$(\nabla_\varrho T_1)(U) = -da(U) - da(\varrho)T_1(U). \tag{4.10}$$

Substituting (4.10) in (4.9) we get

$$(n-2)da(U) + T_1(U)(\delta T_1) - da(\varrho)T_1(U) = 0 \tag{4.11}$$

which yields

$$\delta T_1 = (n-1)da(\varrho) \tag{4.12}$$

for $U = \varrho$.

Using (4.12) in (4.11) we obtain

$$da(U) = -T_1(U)da(\varrho), \tag{4.13}$$

provided $n > 2$.

Putting $V = \varrho$ in (4.8) and using (4.13) we get

$$(\nabla_Y T_1)(U) - (\nabla_U T_1)(Y) = 0.$$

This means that the 1-form T_1 is closed, that is,

$$dT_1(Y, U) = 0.$$

Hence

$$g(\nabla_U \varrho, Y) = g(\nabla_Y \varrho, U) \text{ for all } U, Y, \tag{4.14}$$

which yields

$$g(\nabla_\varrho \varrho, Y) = g(\nabla_Y \varrho, \varrho), \tag{4.15}$$

for $U = \varrho$. Since $g(\nabla_Y \varrho, \varrho) = 0$, from (4.15) it follows that $g(\nabla_\varrho \varrho, Y) = 0$ for all Y. Hence $\nabla_\varrho \varrho = 0$. This implies that the integral curves of the vector field ϱ are geodesics. Therefore we can state the following:

Theorem 4.2. *In a divergence-free $(HGPQS)_n$ manifold $(n > 2)$ under the assumption $A_1(Q(Y, U)V) = 0$, the integral curves of the unit timelike vector field ϱ are geodesics, provided the associated vector fields ϱ and σ corresponding to the 1-forms T_1 and T_2 are related by $(r - 1)\varrho + n\sigma = 0$.*

Taking into account that the divergence of the conformal curvature tensor of a Riemannian manifold (M^n, g) is ([3], [6]):

$$\begin{aligned}(divC)(X, Y)Z &= \frac{n-3}{n-2}[(\nabla_X S)(Y, Z) - (\nabla_Y S)(X, Z)] \\ &= \frac{n-3}{n-2}(divQ)(X, Y)Z,\end{aligned} \tag{4.16}$$

for any vector fields X, Y, Z on M^n, from the Lemma 2.1 of [2] we infer

Theorem 4.3. *Let (M, g) be a $(HGPQS)_n$ perfect fluid spacetime $(n > 2)$. If $(divQ)(X, Y)Z = 0$, for any vector fields X, Y, Z on M, then the unit timelike vector field ϱ is irrotational.*

Also, in [2] was proved the following result:

Theorem 4.4. [2] *Let (M, g) be a $(HGPQS)_n$ perfect fluid spacetime $(n > 2)$. If $(divQ)(X, Y)Z = 0$, for any vector fields X, Y, Z on M, then (M, g) is a GRW spacetime whose fiber is Einstein.*

Acknowledgements. The authors are grateful to the referees for the valuable suggestions and remarks that definitely improved the paper.

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Cubo
A Mathematical Journal

CUBO, A Mathematical Journal
Vol. 23, no 01, pp. 97–108, April 2021
DOI: 10.4067/S0719-06462021000100097

Extended domain for fifth convergence order schemes

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ABSTRACT

We provide a local as well as a semi-local analysis of a fifth convergence order scheme involving operators valued on Banach space for solving nonlinear equations. The convergence domain is extended resulting a finer convergence analysis for both types. This is achieved by locating a smaller domain included in the older domain leading this way to tighter Lipschitz type functions. These extensions are obtained without additional hypotheses. Numerical examples are used to test the convergence criteria and also to show the superiority for our results over earlier ones. Our idea can be utilized to extend other schemes using inverses in a similar way.

RESUMEN

Entregamos un análisis local y uno semi-local de un esquema de quinto orden de convergencia que involucra operadores con valores en un espacio de Banach para resolver ecuaciones no-lineales. El dominio de convergencia es extendido resultando en un análisis de convergencia más fino para ambos tipos. Esto se logra ubicando un dominio más pequeño incluido en el dominio antiguo, entregando funciones de tipo Lipschitz más ajustadas. Estas extensiones se obtienen sin hipótesis adicionales. Se usan ejemplos numéricos para verificar los criterios de convergencia y también para mostrar que nuestros resultados son superiores a otros anteriores. Nuestra idea se puede utilizar para extender otros esquemas usando inversos de manera similar.

Keywords and Phrases: Fifth order convergence scheme, w-continuity, convergence analysis, Fréchet derivative, Banach space.

2020 AMS Mathematics Subject Classification: 65H10, 47H17, 49M15, 65D10, 65G99.

Accepted: 25 January, 2021
Received: 02 June, 2020

1 Introduction

In this article, B_1, B_2 are standing for Banach spaces, $D \subset B_1$ is denoting a convex and open set, and $F : D \longrightarrow B_2$ is considered differentiable according to the Fréchet notion. One of the most important tasks is the location of a solution x_* of nonlinear equation

$$F(x) = 0. \tag{1.1}$$

Solving equation $F(x) = 0$ is useful because using modeling (Mathematical) problems from many areas can be formulated as (1.1). The explosion of technology requires the development of higher convergence schemes. Starting from the quadratically convergent Newton's method higher order schemes develop all the time [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19].

Recently, Singh et al. [13] provided a semi-local convergence for efficient fifth order scheme under Lipschitz continuity on F'' defined as follows

$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1}F(x_n) \\ z_n &= y_n - F'(x_n)^{-1}F(y_n) \\ x_{n+1} &= z_n - F'(y_n)^{-1}F(z_n). \end{aligned} \tag{1.2}$$

Later in [14] the applicability of scheme (1.2) was extended using w- continuity conditions. In general, the convergence domain is small. That is why we develop a technique where a tighter domain than before is obtained containing the iterates. This way the new w-functions are tighter leading to a finer semi-local convergence analysis. It is worth noticing that these extensions do not involve new hypotheses because the new w-functions are specializations of the old one. Hence, we extend the applicability of the method. It turns out that the local convergence analysis can be extended too.

For example: Let $B_1 = B_2 = \mathbb{R}$, $\Omega = [-\frac{1}{2}, \frac{3}{2}]$. Define G on Ω by

$$G(x) = \begin{cases} x^3 \log x^2 + x^5 - x^4, & x \neq 0 \\ 0, & x = 0. \end{cases}$$

Then, we get $x_* = 1$, and

$$G'(x) = 3x^2 \log x^2 + 5x^4 - 4x^3 + 2x^2,$$

$$G''(x) = 6x \log x^2 + 20x^3 - 12x^2 + 10x,$$

$$G'''(x) = 6 \log x^2 + 60x^2 - 24x + 22.$$

Obviously $G'''(x)$ is not bounded on Ω. So, the convergence of scheme (1.2) is not guaranteed by the analysis in [13, 14]. In this study we use only assumptions on the first derivative to prove our results. Relevant studies can be found in [6, 19].

The structure of the rest of the article involves local and semi-local convergence analysis in Section 2 and Section 3, respectively. The numerical experiments appear in Section 4.

2 Local convergence

It is easier for the local convergence of method (1.2), if we develop some real functions. We start with a function ω_0 defined on the interval $I = [0, \infty)$ with values in I satisfying $\omega_0(0) = 0$. Assume equation

$$\omega_0(t) = 1 \tag{2.1}$$

has a least positive solution called ρ_0. Assume the existence of function ω, continuous increasing defined on $I_0 = [0, \rho_0)$ with values in I satisfying $\omega(0) = 0$. Define functions λ_1 and μ_1 on I_0 as follows

$$\lambda_1(t) = \frac{\int_0^1 \omega((1-\theta)t)d\theta}{1 - \omega_0(t)}$$

and

$$\mu_1(t) = \lambda_1(t) - 1.$$

These definitions lead to $\mu_1(0) = -1$ and $\mu_1(t) \longrightarrow \infty$ as $t \longrightarrow \rho_0^-$. Then, the theorem on intermediate value assure the existence of solutions for the equation $\mu_1(t) = 0$ in $(0, \rho_0)$. Set R_1 to be the least such solution. Assume equation

$$\omega_0(\lambda_1(t)t) = 1 \tag{2.2}$$

has a least positive solution called ρ_1. Set $I_1 = [0, \rho_2),\ \rho_2 = \min\{\rho_0, \rho_1\}$. Define functions λ_2 and μ_2 on I_1 as follows

$$\lambda_2(t) = \frac{\int_0^1 \omega((1-\theta)\lambda_1(t)t)d\theta\lambda_1(t)}{1 - \omega_0(\lambda_1(t)t)}$$

and

$$\mu_2(t) = \lambda_2(t) - 1.$$

This time we also have $\lambda_2(0) = -1$ and $\lambda_2(t) \longrightarrow \infty$ as $t \longrightarrow \rho_2^-$. Call R_2 the smallest solution of equation $\lambda_2(t) = 0$ in $(0, \rho_2)$. Assume equation

$$\omega_0(\lambda_2(t)t) = 1 \tag{2.3}$$

has a least positive solution called ρ_3. Set $I_2 = [0, \rho_4)$, $\rho_4 = \min\{\rho_2, \rho_3\}$. Consider functions λ_3 and μ_3 on I_2 as follows

$$\lambda_3(t) = \left[\frac{\int_0^1 \omega((1-\theta)\lambda_2(t)t)d\theta}{1 - \omega_0(\lambda_2(t)t)} + \frac{(\omega_0(\lambda_2(t)t) + \omega_0(\lambda_1(t)t) \int_0^1 v(\theta\lambda_2(t)t)d\theta}{(1 - \omega_0(\lambda_2(t)t))(1 - \omega_0(\lambda_1(t)t))}\right]\lambda_2(t)$$

and

$$\mu_3(t) = \lambda_3(t) - 1,$$

where $v : I_2 \longrightarrow I$ is an increasing and continuous function. By these functions, we obtain $\mu_3(0) = -1$ and $\mu_3(t) \longrightarrow \infty$ as $t \longrightarrow \rho_4^-$. Let R_3 stand for the smallest solution of equation $\mu_3(t) = 0$ in $(0, \rho_4)$. A radius of convergence can be given as follows

$$R = \min\{R_i\}, \ i = 1, 2, 3. \tag{2.4}$$

Then, for all $t \in [0, R)$.

$$0 \leq \omega_0(t) < 1 \tag{2.5}$$
$$0 \leq \omega_0(\lambda_1(t)t) < 1 \tag{2.6}$$
$$0 \leq \omega_0(\lambda_1(t)t) < 1 \tag{2.7}$$
$$0 \leq \omega_0(\lambda_2(t)t) < 1 \tag{2.8}$$

and

$$0 \leq \lambda_i(t) < 1. \tag{2.9}$$

Denote by $U(x_*, \gamma)$ a ball of center x_* and with a radius $\gamma > 0$. Then, $\bar{U}(x_*, \gamma)$ stands for the closure of $U(x_*, \gamma)$.

We base the local convergence on this notation and the conditions (C).

(c1) $F : D \longrightarrow B_2$ is differentiable according to Fréchet, and $x_* \in D$ with $F(x_*) = 0$ is a simple solution.

(c2) There exists an increasing and continuous real function ω_0 on I satisfying $\omega_0(0) = 0$ and such that for all $x \in D$

$$\|F'(x_*)^{-1}(F'(x) - F'(x_*))\| \leq \omega_0(\|x - x_*\|).$$

Set $U_0 = D \cap U(x_*, \rho_0)$.

(c3) There exists a function ω on I_0 continuous and increasing satisfying $\omega(0) = 0$ such that for all $x, y \in U_0$

$$\|F'(x_*)^{-1}(F'(y) - F'(x))\| \leq \omega(\|y - x\|).$$

Set $U_1 = D \cap U(x_*, \rho_4)$.

(c4) There exists a function v on I_2 continuous and increasing, such that for all $x \in U_1$

$$\|F'(x_*)^{-1}F'(x)\| \leq v(\|x - x_*\|).$$

(c5) $\bar{U}(x_*, R) \subseteq D$.

(c6) There exists $R_1 \geq R$ such that

$$\int_0^1 \omega_0(\theta R_1) d\theta < 1.$$

Set $U_2 = D \cap \bar{U}(x_*, R_1)$.

Theorem 2.1. *Assume hypotheses (C) hold and starting point $x_0 \in U(x^*, R) - \{x^*\}$. Then the following assertions are valid, sequence $\{x_n\}$ belongs in $U(x_*, R) - \{x_*\}$ and converges to $x_* \in U(x_*, R)$ so that this limit point uniquely solves equation $F(x) = 0$ in the set U_2.*

Proof. Let $z \in U(x^*, R) - \{x_*\}$ and utilize (c2), (2.4) and (2.5) to obtain

$$\|F'(x_*)^{-1}(F'(z) - F'(x_*))\| \leq \omega_0(\|z - x_*\|) \leq \omega_0(R) < 1,$$

which together with a result by Banach [12] for linear operators whose inverse exists imply

$$\|F'(z)^{-1}F'(x_*)\| \leq \frac{1}{1 - \omega_0(\|z - x^*\|)}. \tag{2.10}$$

In particular, by scheme (1.2) y_0, z_0 are well defined since if we set $z = x_0 \in U(x_*, R) - \{x_*\}$, and $F'(x_0)$ is invertible. Then, by (2.4), (2.8) (for $k = 1$), (c1), (c3) and (2.10) (for $z = x_0$), we have

$$\begin{aligned}
\|y_0 - x_*\| &= \|x_0 - x_* - F'(x_0)^{-1}F(x_0)\| \\
&\leq \|F'(x_0)^{-1}F'(x_*)\| \left[\int_0^1 \|F'(x_*)^{-1}[F'(x_0 + \theta(x_0 - x_*)) - F'(x_0)](x_0 - x_*)d\theta\|\right] \\
&\leq \frac{\int_0^1 \omega((1-\theta)\|x_0 - x_*\|)d\theta}{1 - \omega_0(\|x_0 - x_*\|)}\|x_0 - x_*\| \\
&\leq \lambda_1(\|x_0 - x_*\|)\|x_0 - x_*\| \leq \|x_0 - x_*\| < R. \qquad (2.11)
\end{aligned}$$

Hence, $y_0 \in U(x_*, R)$. Using the second substep of method (1.2) and replacing x_0, y_0, by y_0, z_0, respectively as in (2.10) and (2.11), we get

$$\begin{aligned}
\|z_0 - x_*\| &\leq \frac{\int_0^1 \omega((1-\theta)\|y_0 - x_*\|)d\theta}{1 - \omega_0(\|y_0 - x_*\|)}\|y_0 - x_*\| \\
&\leq \frac{\int_0^1 \omega((1-\theta)\lambda_1(\|x_0 - x_*\|)\|x_0 - x_*\|)d\theta\lambda_1(\|x_0 - x_*\|)\|x_0 - x_*\|}{1 - \omega_0(\lambda_1(\|x_0 - x_*\|)\|x_0 - x_*\|} \\
&\leq \lambda_2(\|x_0 - x_*\|)\|x_0 - x_*\| \leq \|x_0 - x_*\|. \qquad (2.12)
\end{aligned}$$

That is $z_0 \in U(x_*, R)$ and also x_1 exists (for $y_0 = z$, in (2.10)). Notice that (c1), (c4), (2.12) and

$$F(z_0) = F(z_0) - F(x_*) = \int_0^1 F'(x_* + \theta(z_0 - x_*))d\theta(z_0 - x_*),$$

we obtain that

$$\begin{aligned}
&\|F'(x_*)^{-1}F'(z_0)\| \\
\leq\ & \int_0^1 v(\theta\|z_0 - x_*\|)d\theta\|z_0 - x_*\| \\
\leq\ & \int_0^1 v(\theta\lambda_2(\|x_0 - x_*\|)\|x_0 - x_*\|d\theta\lambda_2(\|x_0 - x_*\|)\|x_0 - x_*\|. \qquad (2.13)
\end{aligned}$$

Moreover, by the last substep of method (1.2), (2.4), (2.5), (2.8) (for $k = 3$), (2.10), (2.13) (for

$z = x_0, y_0$), (2.11) and (2.12), we have in turn that

$$
\begin{aligned}
\|x_1 - x_*\| &\leq \|z_0 - x_* - F'(z_0)^{-1}F(z_0)\| &(2.14)\\
&\quad +\|F'(z_0)^{-1}[(F'(y_0) - F'(x_*)) + (F'(x_*) - F'(z_0))]F'(y_0)^{-1}F(z_0)\|\\
&\leq \left[\frac{\int_0^1 \omega((1-\theta)\|z_0 - x_*\|)d\theta}{1 - \omega_0(\|z_0 - x_*\|)}\right.\\
&\quad \left. + \frac{(\omega_0(\|z_0 - x_*\|) + \omega_0(\|y_0 - x_*\|)) \int_0^1 v(\theta\|z_0 - x_*\|)d\theta}{(1 - \omega_0(\|z_0 - x_*\|))(1 - \omega_0(\|y_0 - x_*\|))}\right] \|z_0 - x_*\|\\
&\leq \lambda_3(\|x_0 - x_*\|)\|x_0 - x_*\| \leq \|x_0 - x_*\|, &(2.15)
\end{aligned}
$$

so $x_1 \in U(x_*, R)$. Replacing x_0, y_0, z_0, x_1 by x_k, y_k, z_k, x_{k+1}, in the previous computations we obtain

$$\|y_k - x_*\| \leq \lambda_1(\|x_k - x_*\|)\|x_k - x_*\| \leq \|x_k - x_*\| < R, \tag{2.16}$$

$$\|z_k - x_*\| \leq \lambda_2(\|x_k - x_*\|)\|x_k - x_*\| \leq \|x_k - x_*\| \tag{2.17}$$

and

$$\|x_{k+1} - x^*\| \leq \lambda_3(\|x_k - x^*\|)\|x_k - x_*\| \leq \|x_k - x_*\|, \tag{2.18}$$

so y_k, z_k, x_{k+1} stay in $U(x_*, R)$ and $\lim_{k\longrightarrow\infty} x_k = x_*$. Furthermore, let $x_*^1 \in U_2$ with $F(x_*^1) = 0$. In view of (c2) and (c6) we obtain

$$
\begin{aligned}
\left\|F'(x_*)^{-1}\left(\int_0^1 F'(x_* + \theta(x_*^1 - x_*))d\theta - F'(x_*)\right)\right\| &\leq \int_0^1 \omega_0(\theta\|x_*^1 - x_*\|)d\theta\\
&\leq \int_0^1 \omega_0(\theta R_1)d\theta < 1,
\end{aligned}
$$

so $x_*^1 = x_*$, since $T = \int_0^1 F'(x_* + \theta(x_*^1 - x_*))d\theta$ is invertible and

$$0 = F(x_*^1) - F(x_*) = T(x_*^1 - x_*).$$

□

Remark 2.2. *1. In view of (2.10) and the estimate*

$$
\begin{aligned}
\|F'(x^*)^{-1}F'(x)\| &= \|F'(x^*)^{-1}(F'(x) - F'(x^*)) + I\|\\
&\leq 1 + \|F'(x^*)^{-1}(F'(x) - F'(x^*))\| \leq 1 + L_0\|x - x^*\|
\end{aligned}
$$

condition (2.13) can be dropped and M can be replaced by

$$M(t) = 1 + L_0 t$$

or

$$M(t) = M = 2,$$

since $t \in [0, \frac{1}{L_0})$.

2. *The results obtained here can be used for operators F satisfying autonomous differential equations [2] of the form*

$$F'(x) = P(F(x))$$

where P is a continuous operator. Then, since $F'(x^*) = P(F(x^*)) = P(0)$, we can apply the results without actually knowing x^*. For example, let $F(x) = e^x - 1$. Then, we can choose: $P(x) = x + 1$.

3. *Let $\omega_0(t) = L_0 t$, and $\omega(t) = Lt$. In [2, 3] we showed that $r_A = \frac{2}{2L_0+L}$ is the convergence radius of Newton's method:*

$$x_{n+1} = x_n - F'(x_n)^{-1}F(x_n) \text{ for each } n = 0, 1, 2, \cdots \tag{2.19}$$

under the conditions (2.11) and (2.12). It follows from the definition of R in (2.4) that the convergence radius R of the method (1.2) cannot be larger than the convergence radius r_A of the second order Newton's method (2.19). As already noted in [2, 3] r_A is at least as large as the convergence radius given by Rheinboldt [12]

$$r_R = \frac{2}{3L}, \tag{2.20}$$

where L_1 is the Lipschitz constant on D. The same value for r_R was given by Traub [15]. In particular, for $L_0 < L_1$ we have that

$$r_R < r_A$$

and

$$\frac{r_R}{r_A} \to \frac{1}{3} \text{ as } \frac{L_0}{L_1} \to 0.$$

That is the radius of convergence r_A is at most three times larger than Rheinboldt's.

4. *It is worth noticing that method (1.2) is not changing when we use the conditions of Theorem 2.1 instead of the stronger conditions used in [13, 14]. Moreover, we can compute the computational order of convergence (COC) defined by*

$$\xi = \ln\left(\frac{\|x_{n+1} - x^*\|}{\|x_n - x^*\|}\right) / \ln\left(\frac{\|x_n - x^*\|}{\|x_{n-1} - x^*\|}\right)$$

or the approximate computational order of convergence

$$\xi_1 = \ln\left(\frac{\|x_{n+1} - x_n\|}{\|x_n - x_{n-1}\|}\right) / \ln\left(\frac{\|x_n - x_{n-1}\|}{\|x_{n-1} - x_{n-2}\|}\right).$$

This way we obtain in practice the order of convergence in a way that avoids the bounds involving estimates using estimates higher than the first Fréchet derivative of operator F. Note also that the computation of ξ_1 does not require the usage of the solution x^*.

3 Semi-local convergence analysis

Let $\Gamma_0 = F'(x_0)^{-1} \in \mathcal{L}(B_2, B_1)$ exists at $x_0 \in D$, where $\mathcal{L}(B_2, B_1)$ denotes the set of bounded linear operators from B_2, B_1 and the following conditions hold.

(1) $\|\Gamma_0\| \le \beta_0$.

(2) $\|\Gamma_0 F(x_0)\| \le \eta_0$.

(3)' $\|F'(x) - F'(x_0)\| \le M_0\|x - x_0\|$ for all $x \in D$. Set $D_0 = D \cap U\left(x_0, \frac{1}{\beta_0 M_0}\right)$.

(3) $\|F''(x)\| \le M$ for all $x \in D_0$.

(4) $\|F''(x) - F''(y)\| \le \omega(\|x - y\|)$ for all $x, y \in D_0$ for a continuous nondecreasing function ω, $\omega(0) \ge 0$ such that $\omega(tx) \le t^p\omega(x)$ for $t \in [0,1], x \in (0,\infty)$ and $p \in [0,1]$.

Then, as in [13, 14], let $r_0 = M\beta_0\eta_0$, $s_0 = \beta_0\eta_0\omega(\eta_0)$ and define sequences $\{r_k\}, \{s_k\}$ and $\{\eta_k\}$ for $k = 0, 1, 2, \ldots$, by

$$r_{k+1} = r_k\varphi(r_k)^2\psi(r_k, s_k), \tag{3.1}$$

$$s_{k+1} = s_k\varphi(r_k)^{2+p}\psi(r_k, s_k)^{1+p}, \tag{3.2}$$

$$\eta_{k+1} = \eta_k\varphi(r_k)\psi(r_k, s_k), \tag{3.3}$$

where

$$\varphi(t) = \frac{1}{1 - tg(t)} \tag{3.4}$$

$$g(t) = \left(1 + \frac{t}{2} + \frac{t^2}{2(1-t)}\left(1 + \frac{t}{4}\right)\right) \tag{3.5}$$

and

$$\begin{aligned}\psi(t,s) = {} & \frac{t^2}{2(1-t)}(1+\frac{t}{4})\left[\frac{s}{1+p}\left(\frac{t^{1+p}}{2^{1+p}} + \frac{1}{2+p}\left(\frac{t^2}{2(1-t)}\left(1+\frac{t}{4}\right)\right)^{1+p}\right)\right. \\ & \left. + \frac{t}{2}\left(t + \frac{t^2}{2(1-t)}\left(1 + \frac{t}{4}\right)\right)\right].\end{aligned} \tag{3.6}$$

Remark 3.1. *In [14] the following conditions were used instead of (3), (4), respectively*

(3)' $\|F''(x)\| \le M_1$ for all $x \in D$

(4)' $\|F''(x) - F''(y)\| \le \omega_1(\|x - y\|)$ for all $x, y \in D$ *and* ω_1 *as* ω.

But, we have

$$D_0 \subseteq D,$$

so

$$M_0 \le M_1$$

$$M \leq M_1$$

and

$$\omega(\theta) \leq \omega_1(\theta).$$

Examples where the preceding items are strict can be found in [1, 2, 3, 4, 5, 6]. Notice that (3)' is used to determine D_0 *leading to* $M = M(D_0, x_))$. *Hence, the results in [13, 14] can be rewritten with* M *replacing* M_1. *So, if* $M < M_1$ *the new semi-local convergence analysis is finer. This is also done under the same computational effort because in practice finding* ω_1, M_1 *requires finding* ω, M_0, M *as special cases. This technique can be used to extend the applicability of other schemes involving inverses in an analogous fashion. Hence, the proof of the following semi-local convergence result for scheme (1.2) is omitted.*

Theorem 3.2. *Let* $r_0 = M\beta_0\eta_0 < \nu, s_0 = \beta_0\eta_0\omega(\eta_0)$ *and assumptions (1)-(4) hold. Then, for* $\bar{U}(x_0, R\eta_0) \subseteq D$, *where* $R = \frac{g(r_0)}{1-\delta\gamma}$, *the sequence* $\{x_k\}$ *generated by (1.2) converges to the solution* x_* *of* $F(x) = 0$. *Moreover,* $y_k, z_k, x_{k+1}, x_* \in \bar{U}(x_0, R\eta_0)$ *and* x_* *is the unique solution in* $U\left(x_0, \frac{2}{M_0\beta_0} - R\eta_0\right) \cap D$. *Furthermore, we have*

$$\|x_k - x_*\| \leq g(r_0)\delta^k \frac{\gamma^{\frac{(4+q)^k-1}{3+q}}}{1 - \delta\gamma^{(4+q)^k}}\eta_0.$$

4 Numerical Examples

Example 4.1. *Let us consider a system of differential equations governing the motion of an object and given by*

$$F_1'(x) = e^x,\ F_2'(y) = (e-1)y + 1,\ F_3'(z) = 1$$

with initial conditions $F_1(0) = F_2(0) = F_3(0) = 0$. *Let* $F = (F_1, F_2, F_3)$. *Let* $\mathcal{B}_1 = \mathcal{B}_2 = \mathbb{R}^3, D = \bar{U}(0,1), p = (0,0,0)^T$. *Define function* F *on* D *for* $w = (x, y, z)^T$ *by*

$$F(w) = \left(e^x - 1, \frac{e-1}{2}y^2 + y, z\right)^T.$$

The Fréchet-derivative is defined by

$$F'(v) = \begin{bmatrix} e^x & 0 & 0 \\ 0 & (e-1)y+1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Notice that using the ($\mathcal{A}$) *conditions, we get for* $\alpha = 1$, $w_0(t) = (e-1)t, w(t) = e^{\frac{1}{e-1}}t, v(t) = e^{\frac{1}{e-1}}$. *The radii are*

$$R_1 = 0.38269191223238574472986783803208, R_2 = 0.33841523581069998805048726353562,$$

$$R_3 = 0.32249343047238987480795913143083 \text{ and } R = R_3.$$

Example 4.2. *Let $\mathcal{B}_1 = \mathcal{B}_2 = C[0,1]$, the space of continuous functions defined on $[0,1]$ be equipped with the max norm. Let $D = \overline{U}(0,1)$. Define function F on D by*

$$F(\varphi)(x) = \varphi(x) - 5\int_0^1 x\theta\varphi(\theta)^3 d\theta. \tag{4.1}$$

We have that

$$F'(\varphi(\xi))(x) = \xi(x) - 15\int_0^1 x\theta\varphi(\theta)^2\xi(\theta)d\theta, \text{ for each } \xi \in D.$$

Then, we get that $x^* = 0$, so $w_0(t) = 7.5t, w(t) = 15t$ and $v(t) = 2$. Then the radii are

$$R_1 = 0.0666666666666666666666666666666667, R_2 = 0.059338915721683857529278327547217,$$

$$R_3 = 0.047722035514509826559237382070933 \text{ and } R = R_3.$$

Example 4.3. *Returning back to the motivational example at the introduction of this study, we have $w_0(t) = w(t) = 96.6629073t$ and $v_1(t) = 2$. The parameters for method (1.2) are*

$$R_1 = 0.006896819941465455287843422382 8208, R_2 = 0.0061008926455964288676492301988219,$$

$$R_3 = 0.0044632430213268044563723613293 86 \text{ and } R = R_3.$$

5 Conclusion

In general, the convergence domain of iterative schemes is small limiting their applications. Hence, any attempt to increase it is very important. This is achieved here by finding smaller $\omega-$ functions than before which are also specialization of the previous ones. Hence, the extensions are obtained under the same computational cost. Our idea can be used to extend the usage of other schemes in a similar way. Numerical experiments further demonstrate the superiority of our findings.

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Cubo
A Mathematical Journal

CUBO, A Mathematical Journal
Vol. 23, no. 01, pp. 109–119, April 2021
DOI: 10.4067/S0719-06462021000100109

Inequalities and sufficient conditions for exponential stability and instability for nonlinear Volterra difference equations with variable delay

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ABSTRACT

Inequalities and sufficient conditions that lead to exponential stability of the zero solution of the variable delay nonlinear Volterra difference equation

$$x(n+1) = a(n)h(x(n)) + \sum_{s=n-g(n)}^{n-1} b(n,s)h(x(s))$$

are obtained. Lyapunov functionals are constructed and employed in obtaining the main results. A criterion for the instability of the zero solution is also provided. The results generalizes some results in the literature.

RESUMEN

Se obtienen desigualdades y condiciones suficientes que implican la estabilidad exponencial de la solución cero de la ecuación en diferencias no lineal de Volterra con retardo variable

$$x(n+1) = a(n)h(x(n)) + \sum_{s=n-g(n)}^{n-1} b(n,s)h(x(s)).$$

Se construyen funcionales de Lyapunov y se utilizan para obtener los resultados principales. Se entrega también un criterio para la inestabilidad de la solución cero. Los resultados generalizan algunos resultados en la literatura.

Keywords and Phrases: Exponential stability, Lyapunov functional, Instability.

2020 AMS Mathematics Subject Classification: 34D20, 34D40, 34K20.

Accepted: 01 February, 2021
Received: 02 June, 2020

1 Introduction

Let $\mathbb{R}$ and $\mathbb{Z}^+$ denote the set of real numbers and the set of positive integers respectively. In recent times, research into the stability properties of solutions of difference equations have gained the attention of many Mathematicians, see [1], [2], [4], [6], [7], [8] and the references cited therein. We are mainly motivated by the work of Kublik and Raffoul in [6] in which the authors obtained inequalities that lead to the exponential stability of the zero solution of the linear Volterra difference equation with finite delay

$$x(n+1) = a(n)x(n) + \sum_{s=n-r}^{n-1} b(n,s)x(s), \tag{1.1}$$

for some positive constant r.

In this paper we consider the scalar nonlinear Volterra difference equation with variable delay

$$x(n+1) = a(n)h(x(n)) + \sum_{s=n-g(n)}^{n-1} b(n,s)h(x(s)), \tag{1.2}$$

where $a : \mathbb{Z}^+ \to \mathbb{R}$, $b : \mathbb{Z}^+ \times [-g_0, \infty) \to \mathbb{R}$, $h : \mathbb{R} \to \mathbb{R}$ and $0 < g(n) \leq g_0$, for all $n \in \mathbb{Z}^+$ for some positive constant g_0. We will obtain some inequalities regarding the solutions of (1.2) by employing Lyapunov functionals. These inequalities can be used to deduce exponential stability of the zero solution. Also, by means of a Lyapunov functional an instability criterion of the zero solution of equation (1.2) will be provided.

Let $\psi : [-g_0, 0] \to (-\infty, \infty)$ be a given bounded initial function with

$$||\psi|| = \max_{-g_0 \leq s \leq 0} |\psi(s)|.$$

We further denote the norm of a function $\varphi : [-g_0, \infty) \to (-\infty, \infty)$ by

$$||\varphi|| = \sup_{-g_0 \leq s \leq \infty} |\varphi(s)|.$$

Throughout this paper we let

$$h(x) = xh_1(x).$$

The notation x_n means that $x_n(\tau) = x(n+\tau), \tau \in [-g_0, 0]$ as long as $x(n+\tau)$ is defined. Thus, x_n is a function mapping an interval $[-g_0, 0]$ into $\mathbb{R}$. We say that $x(n) \equiv x(n, n_0, \psi)$ is a solution of (1.2) if $x(n)$ satisfies (1.2) for $n \geq n_0$ and $x_{n_0} = x(n_0 + s) = \psi(s)$, $s \in [-g_0, 0]$.

In this paper we use the convention that $\sum_{s=a}^{b} h(s) = 0$ if $a > b$. The following notation is introduced.

Let

$$A(n,s) = \sum_{u=n-s}^{\gamma} b(u+s, s), \quad \text{where } 0 < \gamma \leq g(n-1) \text{ for all } n \in \mathbb{Z}^+. \tag{1.3}$$

It follows from (1.3) that

$$A(n, n - g(n-1) - 1) = 0. \tag{1.4}$$

We assume throughout the paper that

$$\Delta_n A^2(n, z) \leq 0, \quad \text{for all } \; n + s + 1 \leq z \leq n - 1. \tag{1.5}$$

Due to (1.3) we can express (1.2) in the equivalent form

$$\begin{aligned} \Delta x(n) &= \Big(a(n)h_1(x(n)) + A(n+1,n)h_1(x(n)) - 1\Big)x(n) \\ &\quad - \Delta_n \sum_{s=n-g(n-1)-1}^{n-1} A(n,s)h(x(s)). \end{aligned} \tag{1.6}$$

Definition 1.1. *The zero solution of* (1.2) *is said to be exponentially stable if any solution* $x(n, n_0, \psi)$ *of* (1.2) *satisfies*

$$|x(n, n_0, \psi)| \leq C(||\psi||, n_0)\zeta^{\gamma(n-n_0)}, \text{ for all } n \geq n_0,$$

where ζ *is a constant with* $0 < \zeta < 1, C : \mathbb{R}^+ \times \mathbb{Z}^+ \to \mathbb{R}^+$, *and* γ *is a positive constant. The zero solution of* (1.2) *is said to be uniformly exponentially stable if* C *is independent of* n_0.

We end this section by stating a fact which will be used in the proof of Lemma 2.1, that is, if $u(n)$ is a sequence, then

$$\Delta u^2(n) = u(n+1)\Delta u(n) + u(n)\Delta u(n).$$

For more on the calculus of difference equations we refer to [3] and [5].

2 Exponential Stability

In this section we obtain inequalities that can be used to deduce the exponential stability of (1.2). To simplify notation we let

$$Q(n,x) = \Big(a(n) + A(n+1,n)\Big)h_1(x(n)) - 1,$$

and

$$Q_1(n) = \Big(a(n) + A(n+1,n)\Big) - 1.$$

Lemma 2.1. *Suppose that* (1.3), (1.5) *and for* $\delta > 0$,

$$-\frac{\delta}{\delta g_0 + g(n)} \leq Q(n,x) \leq -\delta g_0 A^2(n+1,n)h_1^2(x(n)) - Q^2(n,x), \tag{2.1}$$

holds. If $1 \le h_1(x)$, *and*

$$V(n) = \left[x(n) + \sum_{s=n-g(n-1)-1}^{n-1} A(n,s)h(x(s))\right]^2 + \delta \sum_{s=-g_0}^{-1} \sum_{z=n+s}^{n-1} A^2(n,z)h^2(x(z)), \tag{2.2}$$

then based on the solutions of (1.2) *we have*

$$\Delta V(n) \le Q_1(n)V(n). \tag{2.3}$$

Proof. Let $x(n, n_0, \psi)$ be a solution of (1.2) and let $V(n)$ be defined by (2.2). It must also be noted that in view of condition (2.1), $Q(n,x) < 0$ for all $n \ge 0$. This together with the fact that $1 \le h_1(x)$ also implies that $Q(n,x) \le Q_1(n) < 0$. Then based on the solutions of (1.2) we have

$$\begin{aligned}
\Delta V(n) \quad = \quad & \left[x(n+1) + \sum_{s=n-g(n)}^{n} A(n+1,s)h(x(s))\right] \\
& \times \Delta \left[x(n) + \sum_{s=n-g(n-1)-1}^{n-1} A(n,s)h(x(s))\right] \\
& + \left[x(n) + \sum_{s=n-g(n-1)-1}^{n-1} A(n,s)h(x(s))\right] \\
& \times \Delta \left[x(n) + \sum_{s=n-g(n-1)-1}^{n-1} A(n,s)h(x(s))\right] \\
& + \ \delta \Delta_n \sum_{s=-g_0}^{-1} \sum_{z=n+s}^{n-1} A^2(n,z)h^2(x(z)).
\end{aligned} \tag{2.4}$$

But

$$\begin{aligned}
& x(n+1) + \sum_{s=n-g(n)}^{n} A(n+1,s)h(x(s)) \\
= \quad & \Big(Q(n,x)+1\Big)x(n) - \Delta_n \sum_{s=n-g(n-1)-1}^{n-1} A(n,s)h(x(s)) + \sum_{s=n-g(n)}^{n} A(n+1,s)h(x(s)) \\
= \quad & \Big(Q(n,x)+1\Big)x(n) + \sum_{s=n-g(n-1)-1}^{n-1} A(n,s)h(x(s)) \\
= \quad & \Big(Q(n,x)+1\Big)x(n) + \sum_{s=n-g(n)}^{n-1} A(n,s)h(x(s))
\end{aligned} \tag{2.5}$$

where we have used the fact that $A(n, n-g(n-1)-1) = 0$. Using (2.5) in (2.4) we obtain

$$\begin{aligned}
\Delta V(n) &= \left[(Q(n,x)+1)\,x(n) + \sum_{s=n-g(n)}^{n-1} A(n,s)h(x(s))\right] Q(n,x)x(n) \\
&\quad + \left[x(n) + \sum_{s=n-g(n)}^{n-1} A(n,s)h(x(s))\right] Q(n,x)x(n) \\
&\quad + \delta\Delta_n \sum_{s=-g_0}^{-1}\sum_{z=n+s}^{n-1} A^2(n,z)h^2(x(z)) \\
&= Q(n,x)V(n) + (Q^2(n,x)+Q(n,x))x^2(n) + \delta\Delta_n \sum_{s=-g_0}^{-1}\sum_{z=n+s}^{n-1} A^2(n,z)h^2(x(z)) \\
&\quad - Q(n,x)\left(\sum_{s=n-g(n)}^{n-1} A(n,s)h(x(s))\right)^2 \\
&\quad - \delta Q(n,x)\sum_{s=-g_0}^{-1}\sum_{z=n+s}^{n-1} A^2(n,z)h^2(x(z))
\end{aligned} \tag{2.6}$$

Considering the third term on the right hand side of (2.6) we obtain

$$\begin{aligned}
&\Delta_n \sum_{s=-g_0}^{-1}\sum_{z=n+s}^{n-1} A^2(n,z)h^2(x(z)) \\
&= \sum_{s=-g_0}^{-1}\sum_{z=n+s+1}^{n} A^2(n+1,z)h^2(x(z)) - \sum_{s=-g_0}^{-1}\sum_{z=n+s}^{n-1} A^2(n,z)h^2(x(z)) \\
&= \sum_{s=-g_0}^{-1}\Bigg[A^2(n+1,n)h(x^2(n)) + \sum_{z=n+s+1}^{n-1} A^2(n+1,z)h^2(x(z)) \\
&\quad - \sum_{z=n+s+1}^{n-1} A^2(n,z)h^2(x(z)) - A^2(n,n+s)h^2(x(n+s))\Bigg] \\
&= \sum_{s=-g_0}^{-1}\Big(A^2(n+1,n)h_1^2(x(n))x^2(n) - A^2(n,n+s)h^2(x(n+s))\Big) \\
&\quad + \sum_{s=-g_0}^{-2}\sum_{z=n+s+1}^{n-1} \Delta_n A^2(n,z)h^2(x(z)) \\
&= g_0A^2(n+1,n)h_1^2(x(n))x^2(n) - \sum_{s=-g_0}^{-1} A^2(n,n+s)h^2(x(n+s)) \\
&\quad + \sum_{s=-g_0}^{-2}\sum_{z=n+s+1}^{n-1} \Delta_n A^2(n,z)h^2(x(z)) \\
&\leq g_0A^2(n+1,n)h_1^2(x(n))x^2(n) - \sum_{s=-g_0}^{-1} A^2(n,n+s)h^2(x(n+s)). \\
&= g_0A^2(n+1,n)h_1^2(x(n))x^2(n) - \sum_{z=n-g_0}^{n-1} A^2(n,z)h^2(x(z))
\end{aligned} \tag{2.7}$$

Applying the Holder's inequality to the squared term in the fourth term on the right hand side of (2.6) gives

$$\begin{aligned}\left(\sum_{s=n-g(n)}^{n-1} A(n,s)h(x(s))\right)^2 &\leq g(n)\sum_{s=n-g(n)}^{n-1} A^2(n,s)h^2(x(s)) \\ &\leq g(n)\sum_{s=n-g_0}^{n-1} A^2(n,s)h^2(x(s)). \qquad (2.8)\end{aligned}$$

Considering the last term on the right hand side of (2.6) we obtain

$$\sum_{s=-g_0}^{-1}\sum_{z=n+s}^{n-1} A^2(n,z)h^2(x(z)) \leq g_0\sum_{s=n-g_0}^{n-1} A^2(n,s)h^2(x(s)) \qquad (2.9)$$

Substituting (2.7), (2.8) and (2.9) in (2.6) we obtain

$$\begin{aligned}\Delta V(n) &\leq Q(n,x)V(n) + (Q^2(n,x) + Q(n,x) + \delta g_0 A^2(n+1,n)h_1^2(x(n)))x^2(n) \\ &\quad + [-(g(n)+\delta g_0)Q(n,x) - \delta]\sum_{s=n-g_0}^{n-1} A^2(n,s)h^2(x(s)) \\ &\leq Q(n,x)V(n) + (Q^2(n,x) + Q(n,x) + \delta g_0 A^2(n+1,n))x^2(n) \\ &\quad + [-(g(n)+\delta g_0)Q(n,x) - \delta]\sum_{s=n-g_0}^{n-1} A^2(n,s)h^2(x(s)) \\ &\leq Q(n,x)V(n) \\ &\leq Q_1(n)V(n).\end{aligned}$$

□

Theorem 2.2. *Suppose the hypothesis of Lemma 2.1 hold. Then any solution $x(n) = x(n, n_0, \psi)$ of* (1.2) *satisfies the exponential inequality*

$$|x(n)| \leq \sqrt{\frac{g_0+\delta}{\delta}V(n_0)\prod_{s=n_0}^{n-1}\Big(a(n) + A(n+1,n)\Big)} \qquad (2.10)$$

for $n \geq n_0$.

Proof. Let $V(n)$ be defined by (2.2). Changing the order of summation in the second term on the right hand side of (2.2) we obtain

$$\begin{aligned}\delta\sum_{s=-g_0}^{-1}\sum_{z=n+s}^{n-1} A^2(n,z)h^2(x(z)) &= \delta\sum_{z=n-g_0}^{n-1}\sum_{s=-g_0}^{z-n} A^2(n,z)h^2(x(z)) \\ &= \delta\sum_{z=n-g_0}^{n-1} A^2(n,z)h^2(x(z))(z-n+g_0+1) \\ &\geq \delta\sum_{z=n-g_0}^{n-1} A^2(n,z)h^2(x(z)) \\ &\geq \delta\sum_{z=n-g(n)}^{n-1} A^2(n,z)h^2(x(z)),\end{aligned}$$

where we have used the fact that if $n - g_0 \leq z \leq n-1$ then $1 \leq z - n + g_0 + 1 \leq g_0$ and $n - g_0 \leq n - g(n)$.
Also, we note that

$$\left(\sum_{z=n-g(n)}^{n-1} A(n,z)h(x(z))\right)^2 \leq g_0 \sum_{z=n-g(n)}^{n-1} A^2(n,z)h^2(x(z)).$$

Hence,

$$\delta \sum_{s=-g_0}^{-1} \sum_{z=n+s}^{n-1} A^2(n,z)h^2(x(z)) \quad \geq \quad \frac{\delta}{g_0}\left(\sum_{z=n-g(n)}^{n-1} A(n,z)h(x(z))\right)^2$$

Thus,

$$\begin{aligned} V(n) &\geq \left[x(n) + \sum_{s=n-g(n)}^{n-1} A^2(n,z)h^2(x(z))\right]^2 + \frac{\delta}{g_0}\left(\sum_{z=n-g(n)}^{n-1} A(n,z)h(x(z))\right)^2 \\ &= \frac{\delta}{g_0+\delta}x^2(n) + \left[\sqrt{\frac{g_0}{g_0+\delta}}x(n) + \sqrt{\frac{g_0+\delta}{g_0}} \sum_{z=n-g(n)}^{n-1} A(n,z)h(x(z))\right]^2 \\ &\geq \frac{\delta}{g_0+\delta}x^2(n). \end{aligned}$$

But

$$V(n) \quad \leq \quad V(n_0)\prod_{s=n_0}^{n-1}\Big((a(n) + A(n+1,n)\Big)$$

This implies that

$$\frac{\delta}{g_0+\delta}x^2(n) \quad \leq \quad V(n_0)\prod_{s=n_0}^{n-1}\Big((a(n) + A(n+1,n)\Big)$$

Hence,

$$|x(n)| \quad \leq \quad \sqrt{\frac{g_0+\delta}{\delta}V(n_0)\prod_{s=n_0}^{n-1}\Big(a(n) + A(n+1,n)\Big)}. \tag{2.11}$$

This completes the proof. □

Corollary 2.3. *Suppose that the hypotheses of Theorem 3.2 hold. Suppose that there exists a positive number $\alpha < 1$ such that*

$$0 < a(n) + A(n+1,n) \leq \alpha.$$

Then the zero solution of (1.2) *is exponentially stable.*

Proof. It follows from (2.10) that

$$\begin{aligned} |x(n)| &\leq \sqrt{\frac{g_0+\delta}{\delta}V(n_0)\prod_{s=n_0}^{n-1}(a(n)+A(n+1,n))} \\ &\leq \sqrt{\frac{g_0+\delta}{\delta}V(n_0)\alpha^{n-n_0}} \end{aligned}$$

for $n \geq n_0$. Since $\alpha \in (0,1)$ the proof is complete. □

3 Instability Criteria

In this section we consider the problem of finding a criteria for instability of the zero solution of (1.2). A suitable Lyapunov functional will be used to obtain the instability criteria.

Theorem 3.1. *Assume that* (1.3), (1.5) *hold and let* $\rho > g_0$ *be a constant. Assume that* $Q_1(n) > 0$ *and* $Q(n,x) > 0$ *such that*

$$Q^2(n,x) + Q(n,x) - \rho A^2(n+1,n)h_1^2(x(n)) \geq 0. \tag{3.1}$$

If $1 \leq h_1(x)$ *and*

$$V(n) = \left[x(n) + \sum_{s=n-g(n-1)-1}^{n-1} A(n,s)h(x(s))\right]^2 - \rho \sum_{s=n-g(n-1)-1}^{n-1} A^2(n,s)h^2(x(s)) \tag{3.2}$$

then, based on the solutions of (1.2) *we have*

$$\Delta V(n) \geq Q_1(n)V(n).$$

Proof. Let $x(n,n_0,\psi)$ be a solution of (1.2) and let $V(n)$ be defined by (3.2). Then based on the solutions of (1.2) we have

$$\begin{aligned} \Delta V(n) = &\left[x(n+1) + \sum_{s=n-g(n)}^{n-1} A(n,s)h(x(s))\right] \\ &\times \Delta\left[x(n) + \sum_{s=n-g(n-1)-1}^{n-1} A(n,s)h(x(s))\right] \\ &+ \left[x(n) + \sum_{s=n-g(n-1)-1}^{n-1} A(n,s)h(x(s))\right] \\ &\times \Delta\left[x(n) + \sum_{s=n-g(n-1)-1}^{n-1} A(n,s)h(x(s))\right] \\ &- \rho\left[A^2(n+1,n)h^2(x(n)) + \sum_{s=n-g(n)}^{n-1} \Delta_n A^2(n,s)h^2(x(s))\right] \end{aligned}$$

$$\begin{aligned}
&\geq \left[(Q(n,x)+1)x(n) + \sum_{s=n-g(n)}^{n-1} A(n,s)h(x(s))\right] Q(n,x)x(n) \\
&\quad + \left[x(n) + \sum_{s=n-g(n)}^{n-1} A(n,s)h(x(s))\right] Q(n,x)x(n) \\
&\quad - \rho A^2(n+1,n)h^2(x(n)) \\
&= Q(n,x)V(n) + (Q^2(n,x) + Q(n,x) - \rho A^2(n+1,n)h_1^2(x(n)))x^2(n) \\
&\quad - Q(n,x)\left(\sum_{s=n-g(n-1)-1}^{n-1} A(n,s)h(x(s))\right)^2 \\
&\quad + Q(n,x)\rho \sum_{s=n-g(n-1)-1}^{n-1} A^2(n,s)h^2(x(s)) \\
&\geq Q(n,x)V(n) + (Q^2(n,x) + Q(n,x) - \rho A^2(n+1,n)h_1^2(x(n)))x^2(n) \\
&\quad + Q(n,x)(\rho - g_0) \sum_{s=n-g(n-1)-1}^{n-1} A^2(n,s)h^2(x(s)) \\
&\geq Q(n,x)V(n) \\
&\geq Q_1(n)V(n).
\end{aligned}$$

This completes the proof. □

Theorem 3.2. *Suppose the hypothesis of Theorem 3.1 hold. Then the zero solution of* (1.2) *is unstable, provided that*

$$\prod_{s=0}^{\infty}(a(n) + A(n+1,n)) = \infty.$$

Proof. We have from Theorem 3.1 that

$$\Delta V(n) \geq Q_1(n)V(n),$$

which implies that

$$V(n) \geq V(n_0) \prod_{s=n_0}^{\infty} (a(s) + A(s+1,s)). \tag{3.3}$$

Using the definition of $V(n)$ in (3.2) we have that

$$\begin{aligned}
V(n) &= x^2(n) + 2x(n) \sum_{s=n-g(n-1)-1}^{n-1} A(n,s)h(x(s)) \\
&\quad + \left[\sum_{s=n-g(n-1)-1}^{n-1} A(n,s)h(x(s))\right]^2 - \rho \sum_{s=n-g(n-1)-1}^{n-1} A^2(n,s)h^2(x(s))
\end{aligned} \tag{3.4}$$

Now let $\beta = \rho - g_0$, then from

$$\Big(\frac{\sqrt{g_0}}{\sqrt{\beta}}a - \frac{\sqrt{\beta}}{\sqrt{g_0}}b\Big)^2 \geq 0,$$

we have

$$2ab \le \frac{g_0}{\beta}a^2 + \frac{\beta}{g_0}b^2.$$

It follows from this inequality that

$$\begin{aligned} 2x(n)\sum_{s=n-g(n-1)-1}^{n-1} A(n,s)h(x(s)) &\le 2|x(n)|\left|\sum_{s=n-g(n-1)-1}^{n-1} A(n,s)h(x(s))\right| \\ &\le \frac{g_0}{\beta}x^2(n) + \frac{\beta}{g_0}\left[\sum_{s=n-g(n-1)-1}^{n-1} A(n,s)h(x(s))\right]^2 \\ &\le \frac{g_0}{\beta}x^2(n) + \frac{\beta}{g_0}g_0\sum_{s=n-g(n-1)-1}^{n-1} A^2(n,s)h^2(x(s)). \end{aligned} \tag{3.5}$$

Substituting (3.5) into (3.4) we obtain

$$\begin{aligned} V(n) &\le x^2(n) + \frac{g_0}{\beta}x^2(n) + (\beta + g_0 - \rho)\sum_{s=n-g(n-1)-1}^{n-1} A^2(n,s)h^2(x(s)) \\ &= \frac{\beta + g_0}{\beta}x^2(n) \\ &\le \frac{\rho}{\rho - g_0}x^2(n). \end{aligned}$$

Using the last inequality and (3.3) we obtain

$$\begin{aligned} |x(n)|^2 &\ge \frac{\rho - g_0}{\rho}V(n) \\ &= \frac{\rho - g_0}{\rho}V(n_0)\prod_{s=n_0}^{\infty}[a(n) + A(n+1,n)]. \end{aligned}$$

This completes the proof. $\square$

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Cubo
A Mathematical Journal

CUBO, A Mathematical Journal
Vol. 23, no. 01, pp. 121–144, April 2021
DOI: 10.4067/S0719-06462021000100121

Energy transfer in open quantum systems weakly coupled with two reservoirs

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ABSTRACT

We show that the energy transfer through an open quantum system with non-degenerate Hamiltonian weakly coupled with two reservoirs in equilibrium is approximately proportional to the difference of their temperatures unless both temperatures are small.

RESUMEN

Mostramos que la transferencia de energía a través de un sistema cuántico abierto con Hamiltoniano no-degenerado débilmente acoplado con dos reservorios en equilibrio es aproximadamente proporcional a la diferencia de sus temperaturas a menos que ambas temperaturas sean pequeñas.

Keywords and Phrases: weak-coupling, quantum Markov semigroup, quantum transport, energy current.

2020 AMS Mathematics Subject Classification: 81S22, 82C10, 80A19.

Accepted: 09 February, 2021
Received: 21 September, 2020

1 Introduction

Energy transfer in classical and quantum systems and the validity of Fourier's law of heat conduction have been a hot topic for many years (see [3, 4, 6, 7, 12, 18, 25, 26] and the references therein). For quantum systems, in particular, after experimental evidence of effective quantum energy transfer in photosynthesis in some biological systems has been found (see [14, 24]), investigations have focused on understanding to what extent quantum mechanics contributes to transport efficiency.

Several models have been proposed involving open quantum systems (see e.g. [5, 6, 27]), mostly phenomenological, and also numerical simulations have been done showing different behaviours. The interaction of the open quantum system with reservoirs is described through interaction operators that appear in the dissipative part of the Gorini-Kossakowski-Sudharshan-Lindblad (GKSL) [17, 22] generator $\mathcal{L}$ of the dynamics, while the Hamiltonian part is given by the commutator with the system Hamiltonian H_S. However, when the GKSL generator is rigorously deduced from some scaling (weak coupling or low density limit) both the system Hamiltonian and the interaction operators appear in the GKSL generator $\mathcal{L}$ after non-trivial transformations (see [1, 2, 9, 10, 13, 19]).

In this paper we study models of open quantum systems rigorously deduced from the weak coupling limit. We consider a quantum system with non-degenerate Hamiltonian H_S coupled with two reservoirs in equilibrium at inverse temperatures $\beta_1 \leq \beta_2$ and study variation of energy due to couplings with each reservoir. It is well-known (see Lebowitz and Spohn [25] (V.28)) that, by the second law of thermodynamics, energy (heat) flows from the hotter to the cooler reservoir. The energy flow, in general, is not proportional to the difference of temperature because of the nonlinear dependence of susceptibilities on temperature, namely an exact Fourier's law does not hold.

However, we rigorously prove that it holds in an approximate way when the temperatures of reservoirs are not too small or, as an alternative, differences between nearest energy levels are small. More precisely, we show that the amount of energy flowing through the system, Theorem 4.2, formula (4.5), is approximately proportional to the product of the temperature differences and a constant (conductivity) which can be interpreted as the average energy needed to jump from a level to the following higher level.

The paper is organised as follows. In Section 2 we introduce quantum Markov semigroups (QMS) arising from the weak coupling limit of a non-degenerate system with two Boson reservoirs. The energy flow is computed explicitly in Section 3, Theorem 3.3, formula (3.7). The dependence of the energy flow on temperatures is studied in Section 4. Moreover, we also study (Theorem 4.3) the asymptotic behaviour of the invariant state when the eigenvalues of H_S increase in number and form a set more and more packed. It turns out that the invariant state converges towards a

Gibbs state with temperature equal to the mean temperatures of the two baths.

Finally, in Section 5, we consider as system the Ising model Hamiltonian and show that the energy flow in this case is zero. We have not been able to extend our analysis to quantum spin chains because their Hamiltonians are highly degenerate and the GKSL generator arising from the weak coupling limit, albeit explicit, is not easily treatable. In particular, we could not extract the relevant information on invariant states.

2 Semigroups of weak coupling limit type

We consider an open quantum system with Hamiltonian H_S acting on a complex separable Hilbert space h with discrete spectral decomposition

$$H_S = \sum_{m\geq 0} \varepsilon_m P_{\varepsilon_m} \tag{2.1}$$

where ε_m, with $\varepsilon_m < \varepsilon_n$ for $m < n$, are the eigenvalues of H_S and P_{ε_m} are the corresponding eigenprojectors. The system is coupled with two reservoirs each one in equilibrium with inverse temperatures $\beta_1 \leq \beta_2$ with interaction Hamiltonians

$$H_1 = D_1 \otimes A^+(\phi_1) + D_1^* \otimes A^-(\phi_1), \quad H_2 = D_2 \otimes A^+(\phi_2) + D_2^* \otimes A^-(\phi_2),$$

where D_1, D_2 are bounded operators on h and $A^+(\phi_j), A^-(\phi_j)$ creation and annihilation operators, in the Fock space of the reservoir j, with test function ϕ_j .

It is well-known (see [2, 9, 13, 25]) that, in the weak coupling limit, the evolution of the system observables is governed by a quantum Markov semigroup (QMS) on $\mathcal{B}(\mathsf{h})$, the algebra of all bounded operators in h, with generator of the form

$$\mathcal{L} = \sum_{j=1,2,\ \omega\in\mathsf{B}} \mathcal{L}_{j,\omega} \tag{2.2}$$

where B is the set of all Bohr frequencies

$$\mathsf{B} := \{\, \omega \mid \exists\, \varepsilon_n, \varepsilon_m \text{ s.t. } \omega = \varepsilon_n - \varepsilon_m > 0\}. \tag{2.3}$$

For every Bohr frequency ω, $\mathcal{L}_{j,\omega}$ is a generator with the Gorini-Kossakowski-Sudharshan-Lindblad (GKSL) structure (see [17, 22])

$$\begin{aligned} \mathcal{L}_{j,\omega}(x) \quad &= \quad \mathrm{i}[H_{j,\omega}, x] - \frac{\Gamma^-_{j,\omega}}{2}\left(D^*_{j,\omega}D_{j,\omega}x - 2D^*_{j,\omega}xD_{j,\omega} + xD_{j,\omega}D^*_{j,\omega}\right) \\ &- \quad \frac{\Gamma^+_{j,\omega}}{2}\left(D_{j,\omega}D^*_{j,\omega}x - 2D_{j,\omega}xD^*_{j,\omega} + xD_{j,\omega}D^*_{j,\omega}\right) \end{aligned} \tag{2.4}$$

for all $x \in \mathcal{B}(\mathsf{h})$, with Kraus operators $D_{j,\omega}$ defined by

$$D_{j,\omega} = \sum_{(\varepsilon_n,\varepsilon_m)\in\mathsf{B}_\omega} P_{\varepsilon_m} D_j P_{\varepsilon_n} \tag{2.5}$$

where $\mathsf{B}_\omega = \{\,(\varepsilon_n, \varepsilon_m) \mid \varepsilon_n - \varepsilon_m = \omega\,\}$, $\Gamma^{\pm}_{j,\omega} = f_{j,\omega}\gamma^{\pm}_{j,\omega}$

$$\gamma^{-}_{j,\omega} = \frac{\mathrm{e}^{\beta_j\omega}}{\mathrm{e}^{\beta_j\omega}-1}, \qquad \gamma^{+}_{j,\omega} = \frac{1}{\mathrm{e}^{\beta_j\omega}-1}, \qquad f_{j,\omega} = \int_{\{\,y\in\mathbb{R}^3\,|\,|y|=\omega\}} |\phi_j(y)|^2 \mathrm{d}_s y$$

(d_s denotes the surface integral) and $H_{j,\omega}$ are bounded self-adjoint operators on h commuting with H_S of the form

$$H_{j,\omega} = \kappa^{-}_{j,\omega} D^*_{j,\omega} D_{j,\omega} + \kappa^{+}_{j,\omega} D_{j,\omega} D^*_{j,\omega}$$

for some real constants $\kappa^{\pm}_{j,\omega}$.

In the sequel, following a customary convention to simplify the notation, we also denote $D^{-}_{j,\omega} := D_{j,\omega}$ and $D^{+}_{j,\omega} := D^*_{j,\omega}$ and write

$$\mathcal{Q}^{\pm}_{j,\omega}(x) = -\frac{1}{2} D^{\mp}_{j,\omega} D^{\pm}_{j,\omega} x + D^{\mp}_{j,\omega} x D^{\pm}_{j,\omega} - \frac{1}{2} x D^{\mp}_{j,\omega} D^{\pm}_{j,\omega} \tag{2.6}$$

the term of the GKSL generator arising from the interaction with the bath j due the Bohr frequency ω is

$$\mathcal{L}_{j,\omega} = \Gamma^{-}_{j,\omega}\mathcal{Q}^{-}_{j,\omega} + \Gamma^{+}_{j,\omega}\mathcal{Q}^{+}_{j,\omega} + \mathrm{i}[H_{j,\omega}, \cdot\,]$$

and the term arising from the interaction with the reservoir j is

$$\mathcal{L}_j = \sum_{\omega\in\mathsf{B}} \mathcal{L}_{j,\omega}.$$

We now make some assumptions on constants in such a way as to ensure boundedness of operators $\mathcal{L}_j$. First of all note that the series $\sum_\omega D^*_{j,\omega} D_{j,\omega}$ is strongly convergent. Indeed, for all vector $u = \sum_{n\geq 0} P_{\varepsilon_n} u$ in h, we have

$$\begin{aligned}
\sum_\omega \left\langle u, D^*_{j,\omega} D_{j,\omega} u \right\rangle &= \sum_\omega \sum_{n,m\geq 0} \langle P_{\varepsilon_m-\omega} D_j P_{\varepsilon_m} u, P_{\varepsilon_n-\omega} D_j P_{\varepsilon_n} u \rangle \\
&= \sum_\omega \sum_{n\geq 0} \langle D_j P_{\varepsilon_n} u, P_{\varepsilon_n-\omega} D_j P_{\varepsilon_n} u \rangle \\
&\leq \sum_{n\geq 0} \|D_j P_{\varepsilon_n} u\|^2 \\
&= \|D_j\|^2 \|u\|^2 .
\end{aligned}$$

As a consequence, if we assume

$$\sup_{\omega\in\mathsf{B}} \Gamma^{\pm}_{j,\omega} < +\infty, \qquad \sup_{\omega\in\mathsf{B}} \left|\kappa^{\pm}_{j,\omega}\right| < +\infty,$$

for $j = 1, 2$ GKSL generators $\mathcal{L}_j$ turn out to be bounded. The above condition will be assumed to be in force throughout the paper.

Remark. Note that $\mathcal{L}_j$ depends on the inverse temperature β_j only through the constants $\gamma^{\pm}_{j,\omega}$. The above notation follows that of [1].

For all normal linear operator $\mathcal{S}$ on $\mathcal{B}(\mathsf{h})$ we denote by $\mathcal{S}_*$ the predual operator acting on the Banach space of trace class operators on h. Therefore, we denote by $\mathcal{T} = (\mathcal{T}_t)_{t\geq 0}$ the QMS on $\mathcal{B}(\mathsf{h})$ generated by $\mathcal{L}$ and by $\mathcal{T}_* = (\mathcal{T}_{*t})_{t\geq 0}$ the predual semigroup acting on trace class operators. In the same way, $\mathcal{T}^j$ (resp. $\mathcal{T}^{j,\omega}$ and $\mathcal{T}_*^{j,\omega}$) stand for the QMS generated by $\mathcal{L}_j$ (resp. $\mathcal{L}_{j,\omega}$ and its predual semigroup). In this paper we are concerned with normal states, therefore we shall identify them with their densities which are positive operators on h with unit trace.

We end this section by checking that, if reservoirs have the same temperature $\beta_1 = \beta_2 = \beta$ and $Z_\beta := \operatorname{tr}\left(\mathrm{e}^{-\beta H_S}\right) < +\infty$, then the Gibbs state has density

$$\rho_\beta = Z_\beta^{-1}\mathrm{e}^{-\beta H_S} \tag{2.7}$$

and is stationary.

Proposition 2.1. *If $\beta_1 = \beta_2 = \beta$ and*

$$Z_\beta := \operatorname{tr}\left(\mathrm{e}^{-\beta H_S}\right) = \sum_{n\geq 0} e^{-\beta\varepsilon_n}\dim(P_{\varepsilon_n}) < +\infty$$

then the Gibbs state (2.7) *is invariant for all QMSs generated by $\mathcal{L}$, $\mathcal{L}_1$, $\mathcal{L}_2$.*

Proof. We begin by observing that for $(\varepsilon_n+\omega, \varepsilon_n), (\varepsilon_n, \varepsilon_n - \omega) \in \mathsf{B}_\omega$, we can compute directly

$$\begin{aligned}(\mathcal{L}_{j,\omega})_*(P_{\varepsilon_n}) =&\Gamma^-_{j,\omega}(P_{\varepsilon_n-\omega}D_jP_{\varepsilon_n}D_j^*P_{\varepsilon_n-\omega} - P_{\varepsilon_n}D_i^*P_{\varepsilon_n-\omega}D_jP_{\varepsilon_n})+\\ &\Gamma^+_{j,\omega}(P_{\varepsilon_n+\omega}D_j^*P_{\varepsilon_n}D_jP_{\varepsilon_n+\omega} - P_{\varepsilon_n}D_jP_{\varepsilon_n+\omega}D_j^*P_{\varepsilon_n}).\end{aligned}$$

A state of the form $\rho = \sum_n \rho_{\varepsilon_n}P_{\varepsilon_n}$, which is a function of the system Hamiltonian H_S (also called a *diagonal* state), satisfies

$$\begin{aligned}\mathcal{L}_{*j}(\rho) =& \sum_\omega\sum_n (\mathcal{L}_{j,\omega})_*(\rho_{\varepsilon_n}P_{\varepsilon_n})\\ =&\sum_\omega \sum_{(\varepsilon_n+\omega,\varepsilon_n)\in\mathsf{B}_\omega} (\rho_{\varepsilon_n+\omega}\Gamma^-_{j,\omega} - \rho_{\varepsilon_n}\Gamma^+_{j,\omega})P_{\varepsilon_n}D_jP_{\varepsilon_n+\omega}D_j^*P_{\varepsilon_n}+\\ &\sum_\omega \sum_{(\varepsilon_n,\varepsilon_n-\omega)\in\mathsf{B}_\omega} (\rho_{\varepsilon_n-\omega}\Gamma^+_{j,\omega} - \rho_{\varepsilon_n}\Gamma^-_{j,\omega})P_{\varepsilon_n}D_j^*P_{\varepsilon_n-\omega}D_jP_{\varepsilon_n}.\end{aligned}$$

Now if $\beta_1 = \beta_2 = \beta$ and $\rho_{\varepsilon_n} = \mathrm{e}^{-\beta\varepsilon_n}$ as in (2.7), we have

$$\frac{\Gamma^+_{j,\omega}}{\Gamma^-_{j,\omega}} = \frac{\gamma^+_{j,\omega}}{\gamma^-_{j,\omega}} = e^{-\beta\omega} = \frac{\rho_{\varepsilon_n+\omega}}{\rho_{\varepsilon_n}},$$

for all $j = 1, 2$, so that $\mathcal{L}_{*j}(\rho) = 0$ and $\rho = \mathrm{e}^{-\beta H_S}/Z_\beta$ is an invariant state for the QMS generated by $\mathcal{L}_j$. Since $\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2$ it is an invariant state also for the QMS generated by $\mathcal{L}$. □

3 Energy current

The rate of energy variation in the system, in a state ρ, due to interaction with the reservoir j is $\mathrm{tr}\,(\rho\mathcal{L}_j(H_S))$ (see [25] (V.28)). Therefore

$$\mathrm{tr}\,(\rho\mathcal{L}_1(H_S)) - \mathrm{tr}\,(\rho\mathcal{L}_2(H_S)) \tag{3.1}$$

is twice the rate at which the energy flows through the system from the hotter bath to the colder bath, namely, the energy current through the system.

Adapting a result by Lebowitz and Spohn [25] Theorem 2 and Corollary 1, it is possible to prove that the energy current is non-negative for finite dimensional systems.

Theorem 3.1. *Suppose that h is finite dimensional and let ρ be a faithful invariant state, then the energy current* (3.1) *is non-negative.*

Proof. If a system is weakly coupled to a *single* bath j at inverse temperature β_j, it is well-known that the Gibbs state $\rho_{\beta_j} = Z_{\beta_j}^{-1}\mathrm{e}^{-\beta_j H_S}$, with $Z_{\beta_j} = \mathrm{tr}\left(\mathrm{e}^{-\beta_j H_S}\right)$, is invariant.
Consider the relative entropy of ρ with respect to ρ_{β_j} defined by $S(\rho|\rho_{\beta_j}) = \mathrm{tr}\left(\rho(\log(\rho - \log\rho_{\beta_j})\right)$ which is a notoriously non-increasing function (see [23], Theorem 1.5), i.e.

$$S\left(\mathcal{T}_{*t}^{j}(\rho)|\mathcal{T}_{*t}^{j}(\rho_{\beta_j})\right) \le S(\rho|\rho_{\beta_j}),$$

for all ρ and $t \ge 0$. States $\mathcal{T}_{*t}^{j}(\rho)$, $j = 1, 2$ will still be faithful for small t, therefore no problem arises when considering logarithms. Since ρ_{β_j} is invariant, denoting $\rho_t := \mathcal{T}_{*t}^{j}(\rho)$, and differentiating we find

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}S(\rho_t|\rho_{\beta_j}) &= \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{tr}\left(\rho_t(\log\rho_t - \log\rho_{\beta_j})\right) \\ &= \mathrm{tr}\left(\rho_t'(\log\rho_t - \log\rho_{\beta_j})\right) + \mathrm{tr}\left(\rho_t\frac{\mathrm{d}}{\mathrm{d}t}\log\rho_t\right).\end{aligned}$$

Since for every $x > 0$, $\log x = \int_0^{+\infty}\left(\frac{1}{1+s} - \frac{1}{x+s}\right)\mathrm{d}s$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\log\rho_t = \int_0^{+\infty}(s+\rho_t)^{-1}\rho_t'(s+\rho_t)^{-1}\mathrm{d}s$$

so that

$$\mathrm{tr}\left(\rho_t\frac{\mathrm{d}}{\mathrm{d}t}\log\rho_t\right) = \mathrm{tr}\left(\rho_t'\int_0^{+\infty}\rho_t(s+\rho_t)^{-2}\mathrm{d}s\right) = \mathrm{tr}\,(\rho_t') = 0.$$

By imposing $\rho_{\beta_j} = Z_{\beta_j}^{-1}e^{-\beta_j H_S}$, and recalling that $\rho_t' = \mathcal{L}_{*j}(\rho_t)$, $\mathrm{tr}\,(\rho_t') = 0$ by trace preservation, we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}S(\rho_t|\rho_{\beta_j}) &= \mathrm{tr}\left(\rho_t'(\log\rho_t - \log\rho_{\beta_j})\right) \\ &= \mathrm{tr}\left(\rho_t'(\log\rho_t + \beta_j H_S - \log Z_{\beta_j}^{-1})\right) \\ &= \mathrm{tr}\,(\rho_t'\,\log\rho_t) + \beta_j\mathrm{tr}\,(\rho_t\mathcal{L}_j(H_S))\,.\end{aligned}$$

In particular $\mathrm{tr}\,(\rho_t'(\log\rho_t)) + \beta_j \mathrm{tr}\,(\rho_t \mathcal{L}(H_S)) \leq 0$ by monotonicity of the relative entropy, namely

$$-\mathrm{tr}\,(\mathcal{L}_{*j}(\rho_t)\,\log\rho_t) - \beta_j \mathrm{tr}\,(\rho_t \mathcal{L}_j(H_S)) \geq 0.$$

In our context, the entropy production of the system due to interaction with the bath at inverse temperature β_j is

$$-\,\mathrm{tr}\,(\mathcal{L}_{*j}(\rho_t)\,\log\rho_t) - \beta_j \mathrm{tr}\,(\rho_t \mathcal{L}_j(H_S)) \geq 0. \tag{3.2}$$

Now, for all β, β_1, β_2 and ρ stationary state for the system S interacting with both baths, By taking a sum over j of the inequality before (3.2), we obtain

$$\beta_1 \mathrm{tr}\,(\rho\mathcal{L}_1(H_S)) + \beta_2 \mathrm{tr}\,(\rho\mathcal{L}_2(H_S)) \leq 0.$$

Moreover, $\mathrm{tr}\,(\rho\mathcal{L}_1(H_S)) = -\mathrm{tr}\,(\rho\mathcal{L}_2(H_S))$ and so

$$(\beta_2 - \beta_1)\mathrm{tr}\,(\rho\mathcal{L}_2(H_S)) \geq 0$$

In view $\beta_1 \geq \beta_2$, we have $\mathrm{tr}\,(\rho\mathcal{L}_1(H_S)) = -\mathrm{tr}\,(\rho\mathcal{L}_2(H_S)) \geq 0$ and the proof is complete. □

In this section we prove a general explicit formula for the energy current in a stationary state ρ which is a function of the system Hamiltonian H_S. This not only confirms that it is positive also for possibly infinite dimensional systems if the eigenvalues of stationary state are a monotone system (i.e. there are no population inversions), but it allows us to establish proportionality to the difference of bath temperatures when they are not too small, namely an approximate Fourier law.

Lemma 3.2. *For all $\omega \in \mathsf{B}$ and $j = 1, 2$ we have*

$$\mathcal{Q}^-_{j,\omega}(H_S) = -\omega\, D^*_{j,\omega} D_{j,\omega} \qquad \mathcal{Q}^+_{j,\omega}(H_S) = \omega\, D_{j,\omega} D^*_{j,\omega} \tag{3.3}$$

and

$$\mathcal{L}_j(H_S) = \sum_{\omega\in\mathsf{B}} \omega \left(\Gamma^+_{j,\omega} D_{j,\omega} D^*_{j,\omega} - \Gamma^-_{j,\omega} D^*_{j,\omega} D_{j,\omega}\right). \tag{3.4}$$

Proof. Writing H_S as in (2.1) we compute

$$\begin{aligned}
\mathcal{Q}^-_{j,\omega}(H_S) &= -\frac{1}{2} D^*_{j,\omega} D_{j,\omega} H_S + D^*_{j,\omega} H_S D_{j,\omega} - \frac{1}{2} H_S D^*_{j,\omega} D_{j,\omega} \\
&= \sum_{(\varepsilon_n,\varepsilon_m)\in\mathsf{B}_\omega} \left(\varepsilon_m\, P_{\varepsilon_n} D^*_j P_{\varepsilon_m} D_j P_{\varepsilon_n} - \varepsilon_n\, P_{\varepsilon_n} D^*_j P_{\varepsilon_m} D_j P_{\varepsilon_n}\right) \\
&= -\sum_{(\varepsilon_n,\varepsilon_m)\in\mathsf{B}_\omega} \omega P_{\varepsilon_n} D^*_j P_{\varepsilon_m} D_j P_{\varepsilon_n} \\
&= -\omega\, D^*_{j,\omega} D_{j,\omega}.
\end{aligned}$$

The proof of the other identity (3.3) is similar. Since $[H_{j,\omega}, H_S] = 0$ for all j, ω, (3.4) follows immediately. □

We can now prove our formula for the energy current in a stationary state ρ which is a function of the system Hamiltonian H_S. We suppose that the interaction of the system with both reservoirs is similar; this property is reflected by the assumptions on $\mathrm{tr}\left(P_{\varepsilon_n} D_j^* P_{\varepsilon_m} D_j\right)$ and $f_{1,\omega}$. In the sequel, to simplify the notation we also write ρ_n instead of ρ_{ε_n}.

Theorem 3.3. *For any state ρ which is a function of the system Hamiltonian H_S, i.e.*

$$\rho = \sum_{n\geq 0} \rho_n P_{\varepsilon_n} \tag{3.5}$$

we have

$$\mathrm{tr}\left(\rho \mathcal{L}_j(H_S)\right) = \sum_{\omega\in \mathsf{B}} \omega \sum_{(\varepsilon_n,\varepsilon_m)\in \mathsf{B}_\omega} \left(\Gamma^+_{j,\omega}\rho_m - \Gamma^-_{j,\omega}\rho_n\right) \mathrm{tr}\left(P_{\varepsilon_n} D_j^* P_{\varepsilon_m} D_j\right). \tag{3.6}$$

If the state ρ is also stationary and, moreover,

(1) $\mathrm{tr}\left(P_{\varepsilon_n} D_1^* P_{\varepsilon_m} D_1\right) = \mathrm{tr}\left(P_{\varepsilon_n} D_2^* P_{\varepsilon_m} D_2\right)$ *for all n, m,*

(2) $f_{1,\omega} = f_{2,\omega}$ *for all ω,*

then

$$\mathrm{tr}\left(\rho \mathcal{L}_1(H_S)\right) = \frac{1}{2}\sum_{\omega\in \mathsf{B}} \omega\, f_{1,\omega}\left(\gamma^+_{1,\omega} - \gamma^+_{2,\omega}\right) \sum_{(\varepsilon_n,\varepsilon_m)\in \mathsf{B}_\omega} (\rho_m - \rho_n)\, \mathrm{tr}\left(P_{\varepsilon_n} D_1^* P_{\varepsilon_m} D_1\right). \tag{3.7}$$

Proof. The proof of (3.6) is immediate from (3.4) and the following identities (cyclic property of the trace)

$$\mathrm{tr}\left(P_{\varepsilon_m} D_{j,\omega} P_{\varepsilon_n} D^*_{j,\omega}\right) = \mathrm{tr}\left((P_{\varepsilon_m} D_{j,\omega}) P_{\varepsilon_m} D^*_{j,\omega}\right) = \mathrm{tr}\left(P_{\varepsilon_n} D^*_{j,\omega} P_{\varepsilon_m} D_{j,\omega}\right).$$

If the state ρ is stationary, then $\mathrm{tr}\left(\rho\mathcal{L}_1(H_S)\right) = \mathrm{tr}\left(\rho\mathcal{L}(H_S)\right) - \mathrm{tr}\left(\rho\mathcal{L}_2(H_S)\right) = -\mathrm{tr}\left(\rho\mathcal{L}_2(H_S)\right)$, so that $\mathrm{tr}\left(\rho\mathcal{L}_1(H_S)\right) = \left(\mathrm{tr}\left(\rho\mathcal{L}_1(H_S)\right) - \mathrm{tr}\left(\rho\mathcal{L}_2(H_S)\right)\right)/2$. Computing the right-hand side difference by means of (3.6) with $j = 1, 2$ we can write $2\mathrm{tr}\left(\rho\mathcal{L}_1(H_S)\right)$ as

$$\begin{aligned} &\sum_{\omega\in \mathsf{B}} \omega\, f_{1,\omega} \sum_{(\varepsilon_n,\varepsilon_m)\in \mathsf{B}_\omega} \left(\gamma^+_{1,\omega}\rho_m - \gamma^-_{1,\omega}\rho_n - \gamma^+_{2,\omega}\rho_m + \gamma^-_{2,\omega}\rho_n\right) \mathrm{tr}\left(P_{\varepsilon_n} D_1^* P_{\varepsilon_m} D_1\right) \\ = &\sum_{\omega\in \mathsf{B}} \omega\, f_{1,\omega} \sum_{(\varepsilon_n,\varepsilon_m)\in \mathsf{B}_\omega} \left((\gamma^+_{1,\omega} - \gamma^+_{2,\omega})\rho_m - (\gamma^-_{1,\omega} - \gamma^-_{2,\omega})\rho_n\right) \mathrm{tr}\left(P_{\varepsilon_n} D_1^* P_{\varepsilon_m} D_1\right). \end{aligned}$$

Since $\gamma^-_{j,\omega} = \gamma^+_{j,\omega} + 1$ for all j, ω, then $\gamma^+_{1,\omega} - \gamma^+_{2,\omega} = \gamma^-_{1,\omega} - \gamma^-_{2,\omega}$ and (3.7) follows. $\square$

Remark. Note that the above identity $\mathrm{tr}\left(P_{\varepsilon_n} D_1^* P_{\varepsilon_m} D_1\right) = \mathrm{tr}\left(P_{\varepsilon_n} D_2^* P_{\varepsilon_m} D_2\right)$ holds whenever there exists an isometry R on h, commuting with H_S, such that $D_2 = R D_1 R^*$. Indeed, in this case, R commutes with all spectral projections of H_S and

$$\begin{aligned} \mathrm{tr}\left(P_{\varepsilon_n} D_2^* P_{\varepsilon_m} D_2\right) &= \mathrm{tr}\left(P_{\varepsilon_n} R D_1^* R^* P_{\varepsilon_m} R D_1 R^*\right) \\ &= \mathrm{tr}\left(P_{\varepsilon_n} D_1^* P_{\varepsilon_m} D_1 R^* R\right) \\ &= \mathrm{tr}\left(P_{\varepsilon_n} D_1^* P_{\varepsilon_m} D_1\right). \end{aligned}$$

We will see later (Section 5) that this happens when the system interacts in the same way with the two baths.

Formula (3.7) can be applied to effectively compute the energy current in several models highlighting the dependence on the difference of temperatures. Indeed, one readily sees that, for β_1, β_2 very close the term $\omega\left(\gamma^+_{1,\omega} - \gamma^+_{2,\omega}\right)$ is an infinitesimum of order $\beta_1^{-1} - \beta_2^{-1}$ while the other terms are close to some nonzero values. Moreover, it is also clear from (3.7) that the energy current is non-negative whenever the invariant state satisfies $\rho_m > \rho_n$ for all n, m such that $\varepsilon_m < \varepsilon_n$ i.e. population inversion does not occur.

However, in order to find more explicit formulae we need additional information on the invariant state. This problem will be studied in the next section. We end this section by the following example

Example 3.4. Let $\mathsf{h} = \mathbb{C}^{n+1}$ with orthonormal basis $(e_k)_{0\le k\le n}$. Consider an n-level system with Hamiltonian

$$H_S = \sum_{k=0}^{n} k|e_k\rangle\langle e_k|$$

and interaction operators D_1, D_2 acting as

$$D_j e_k \quad = \quad e_{k-1} \quad \text{for } k = 1, \dots, n \qquad D_j e_0 = 0.$$

Clearly $\mathsf{B} = \{1, 2, \dots, n\}$ but the only nonzero $D_{j,\omega}$ are those corresponding to the frequency $\omega = 1$ and $D_{1,1} = D_1$, $D_{2,1} = D_2$. Moreover, since $\varepsilon_k = k$,

$$\operatorname{tr}\left(P_{\varepsilon_k} D_1^* P_{\varepsilon_{k-1}} D_1\right) = \operatorname{tr}\left(P_{\varepsilon_k} D_2^* P_{\varepsilon_{k-1}} D_2\right) = 1$$

for $k = 1, \dots, n$. By Theorem 3.3 formula (3.6) we have

$$\operatorname{tr}\left(\rho\, \mathcal{L}_j(H_S)\right) = \sum_{k=0}^{n-1} \left(\Gamma^+_{j,1}\rho_k - \Gamma^-_{j,1}\rho_{k+1}\right).$$

If all $\Gamma^{\pm}_{j,1}$ $(j = 1, 2)$ are nonzero, a straightforward computation shows that the unique stationary state is

$$\rho = \frac{1-\nu}{1-\nu^{n+1}} \sum_{k=0}^{n} \nu^k |e_k\rangle\langle e_k|, \qquad \nu := \frac{\Gamma^+_{1,1} + \Gamma^+_{2,1}}{\Gamma^-_{1,1} + \Gamma^-_{2,1}}$$

and the energy current due to interaction with reservoir j is

$$\begin{aligned}
\operatorname{tr}\left(\rho\, \mathcal{L}_j(H_S)\right) \quad &= \quad \frac{1-\nu}{1-\nu^{n+1}} \sum_{k=0}^{n-1} \left(\Gamma^+_{j,1}\nu^k - \Gamma^-_{j,1}\nu^{k+1}\right) \\
&= \quad \frac{1-\nu^n}{1-\nu^{n+1}} \left(\Gamma^+_{j,1} - \nu\Gamma^-_{j,1}\right).
\end{aligned}$$

Note that, dropping the index 1 corresponding to the unique effective frequency ω to simplify the notation, we have

$$
\begin{aligned}
\Gamma_j^+ - \nu\Gamma_j^- &= \Gamma_j^- \left(\frac{\Gamma_j^+}{\Gamma_j^-} - \frac{\Gamma_1^+ + \Gamma_2^+}{\Gamma_1^- + \Gamma_2^-} \right) \\
&= \Gamma_j^- \left(\frac{\gamma_j^+}{\gamma_j^-} - \frac{f_1\,\gamma_1^+ + f_2\,\gamma_2^+}{f_1\,\gamma_1^- + f_2\,\gamma_2^-} \right) \\
&= \Gamma_j^- \left(\mathrm{e}^{-\beta_j} - \frac{f_1\,(\mathrm{e}^{\beta_2}-1) + f_2\,(\mathrm{e}^{\beta_1}-1)}{f_1\,\mathrm{e}^{\beta_1}(\mathrm{e}^{\beta_2}-1) + f_2\,\mathrm{e}^{\beta_2}(\mathrm{e}^{\beta_1}-1)} \right) \\
&= \Gamma_j^- \left(\mathrm{e}^{-\beta_j} - \frac{f_1\,\mathrm{e}^{-\beta_1}(1-\mathrm{e}^{-\beta_2}) + f_2\mathrm{e}^{-\beta_2}(1-\mathrm{e}^{-\beta_1})}{f_1(1-\mathrm{e}^{-\beta_2}) + f_2(1-\mathrm{e}^{-\beta_1})} \right).
\end{aligned}
$$

For $j = 1$ we find

$$
\Gamma_{j,1}^+ - \nu\Gamma_{j,1}^- = \Gamma_j^-\, f_2(1-\mathrm{e}^{-\beta_1}) \frac{\mathrm{e}^{-\beta_1} - \mathrm{e}^{-\beta_2}}{f_1(1-\mathrm{e}^{-\beta_2}) + f_2(1-\mathrm{e}^{-\beta_1})}
$$

and so

$$
\mathrm{tr}\,(\rho\,\mathcal{L}_1(H_S)) = \frac{1-((\Gamma_1^+ + \Gamma_2^+)/(\Gamma_1^- + \Gamma_2^-))^n}{1-((\Gamma_1^+ + \Gamma_2^+)/(\Gamma_1^- + \Gamma_2^-))^{n+1}} \frac{\Gamma_1^-\, f_2(1-\mathrm{e}^{-\beta_1})(\mathrm{e}^{-\beta_1} - \mathrm{e}^{-\beta_2})}{f_1(1-\mathrm{e}^{-\beta_2}) + f_2(1-\mathrm{e}^{-\beta_1})}
$$

Since $\Gamma_j^+ < \Gamma_j^-$, this formula, for n big and β_1, β_2 small becomes

$$
\begin{aligned}
\mathrm{tr}\,(\rho\,\mathcal{L}_1(H_S)) &\approx \frac{f_1 f_2(\mathrm{e}^{-\beta_1} - \mathrm{e}^{-\beta_2})}{f_1(1-\mathrm{e}^{-\beta_2}) + f_2(1-\mathrm{e}^{-\beta_1})} \\
&\approx \frac{f_1 f_2(\beta_2 - \beta_1)}{f_2\beta_1 + f_1\beta_2} \\
&= \frac{f_1 f_2 \left(\frac{1}{\beta_1} - \frac{1}{\beta_2} \right)}{\frac{f_1}{\beta_1} + \frac{f_2}{\beta_2}}
\end{aligned}
$$

showing that, in a certain regime of high temperature a Fourier law holds for all choices f_1, f_2 of the interactions strength.

4 Dependence of the energy current from temperature difference and conductivity

In this section we consider systems whose Hamiltonian H_S has simple spectrum, namely each spectral projection P_{ε_n} is one-dimensional, and make explicit the dependence of the energy current on the difference of temperatures $1/\beta_1$ and $1/\beta_2$.

We begin by noting that, if spectral projections P_{ε_n} are one-dimensional one can associate with the open quantum system a classical (time continuous) Markov chain with state space V the spectrum $\mathrm{sp}(H_S)$ of H_S in a canonical way. Indeed, for every bounded function f on V, we have

$$\begin{aligned}\mathcal{L}(f(H_S)) &= \sum_{n\geq 0} f(\varepsilon_n)\mathcal{L}(P_{\varepsilon_n})\\ &= \sum_{\omega\in\mathsf{B},\,(\varepsilon_n,\varepsilon_m)\in\mathsf{B}_\omega}\left(\sum_j \Gamma^-_{j,\omega}P_{\varepsilon_n}D_j^*P_{\varepsilon_m}D_jP_{\varepsilon_n}\right)(f(\varepsilon_m)-f(\varepsilon_n))\\ &+\sum_{\omega\in\mathsf{B},\,(\varepsilon_n,\varepsilon_m)\in\mathsf{B}_\omega}\left(\sum_j \Gamma^+_{j,\omega}P_{\varepsilon_m}D_jP_{\varepsilon_n}D_j^*P_{\varepsilon_m}\right)(f(\varepsilon_n)-f(\varepsilon_m))\end{aligned}$$

and we find a classical Markov chain with transition rate matrix $Q=(q_{nm})$

$$q_{nm}=\begin{cases}\sum_j \Gamma^-_{j,\varepsilon_n-\varepsilon_m}\mathrm{tr}\left(D_j^*P_{\varepsilon_m}D_jP_{\varepsilon_n}\right), & \text{if } \varepsilon_n>\varepsilon_m,\\ \sum_j \Gamma^+_{j,\varepsilon_m-\varepsilon_n}\mathrm{tr}\left(D_jP_{\varepsilon_m}D_j^*P_{\varepsilon_n}\right), & \text{if } \varepsilon_n<\varepsilon_m,\\ -\sum_{m\neq n} q_{nm}, & \text{if } n=m.\end{cases}$$

Now, if we consider the conditional expectation

$$\mathcal{E}:\mathcal{B}(\mathsf{h})\to\ell^\infty(V;\mathbb{C}),\qquad \mathcal{E}(x)=\sum_{m\geq 0}P_{\varepsilon_m}xP_{\varepsilon_m},$$

where $\ell^\infty(V;\mathbb{C})$ is the abelian algebra of bounded functions on V, we have that

$$\mathcal{E}\circ\mathcal{L}=\mathcal{L}\circ\mathcal{E}. \tag{4.1}$$

Therefore, by defining the predual map $\mathcal{E}_*$ such that $\mathrm{tr}\left(\mathcal{E}_*(\rho)x\right)=\mathrm{tr}\left(\rho\mathcal{E}(x)\right)$, if ρ is an invariant state, we have also $0=\mathcal{E}_*(\mathcal{L}_*(\rho))=\mathcal{L}_*(\mathcal{E}_*(\rho))$ and

$$(\pi_n)\mapsto\sum_{n\geq 0}\pi_nP_{\varepsilon_n}$$

gives a one-to-one correspondence between diagonal invariant states of the open quantum system and invariant measures of the associated Markov chain.

In the following, in order to have at hand an explicit formula for the invariant measure, we suppose, for simplicity, that the graph associated with the above Markov chain is a path graph and *jumps can occur only to nearest neighbour levels*, namely $q_{nm}=0$ for $|n-m|\geq 2$. This assumption may hold, for instance, if the Hamiltonian H_S is generic in the sense of [8], namely it is not only non-degenerate but also if $\varepsilon_n-\varepsilon_m=\varepsilon_{n'}-\varepsilon_{m'}$ then $\varepsilon_n=\varepsilon_{n'}$ and $\varepsilon_m=\varepsilon_{m'}$. Moreover, we assume that $q_{nm}\neq 0$ for $|n-m|\leq 1$. In this case the associated classical Markov chain has a simpler structure allowing one to make explicit computations and describe explicitly the structure of invariant states (see also [11] in a more general situation).

The explicit expression for the invariant state is $\rho=\sum_n\rho_nP_{\varepsilon_n}$ where

$$\rho_n=\prod_{0\leq k<n}\frac{q_{k,k+1}}{q_{k+1,k}}\rho_0 \tag{4.2}$$

with

$$
\begin{aligned}
q_{k,k+1} &= \sum_{j=1}^{2} \Gamma^{+}_{j,\varepsilon_{k+1}-\varepsilon_k} \mathrm{tr}\left(D_j P_{\varepsilon_{k+1}} D_j^* P_{\varepsilon_k}\right), \\
q_{k+1,k} &= \sum_{j=1}^{2} \Gamma^{-}_{j,\varepsilon_{k+1}-\varepsilon_k} \mathrm{tr}\left(D_j^* P_{\varepsilon_k} D_j P_{\varepsilon_{k+1}}\right)
\end{aligned}
$$

provided that the normalization condition

$$
\sum_{n\geq 1} \prod_{0\leq k<n} \frac{q_{k,k+1}}{q_{k+1,k}} < +\infty \tag{4.3}
$$

holds, in which case ρ_0 is the inverse of the sum of the above series increased by 1.

With the explicit formula for the invariant state we can find a Fourier's law for the energy current through the system. We begin by a technical lemma

Lemma 4.1. *The following inequalities hold*

$$
\mathrm{e}^{-(\beta_1+\beta_2)\omega/2} \frac{\dfrac{1}{\beta_1} - \dfrac{1}{\beta_2}}{\dfrac{1}{\beta_1} + \dfrac{1}{\beta_2}} \leq \frac{\dfrac{1}{\mathrm{e}^{\beta_1\omega}-1} - \dfrac{1}{\mathrm{e}^{\beta_2\omega}-1}}{\dfrac{\mathrm{e}^{\beta_1\omega}}{\mathrm{e}^{\beta_1\omega}-1} + \dfrac{\mathrm{e}^{\beta_2\omega}}{\mathrm{e}^{\beta_2\omega}-1}} \leq \frac{\dfrac{1}{\beta_1} - \dfrac{1}{\beta_2}}{\dfrac{1}{\beta_1} + \dfrac{1}{\beta_2}}, \tag{4.4}
$$

for all $0 < \beta_1 \leq \beta_2$ *and* $\omega > 0$.

Proof. Note that $1/(\mathrm{e}^{\beta_1\omega}-1) - 1/(\mathrm{e}^{\beta_2\omega}-1) \leq 1/(\beta_1\omega) - 1/(\beta_2\omega)$ because the function $x \mapsto 1/(\mathrm{e}^{x\omega}-1) - 1/(\omega x)$ is increasing on $]0,+\infty[$ since

$$
\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{\mathrm{e}^{x\omega}-1} - \frac{1}{\omega x}\right) = \frac{1}{\omega x^2} - \frac{\omega}{\left(\mathrm{e}^{\omega x/2} - \mathrm{e}^{-\omega x/2}\right)^2} \geq 0
$$

by the elementary inequality $\mathrm{e}^{\omega x/2} - \mathrm{e}^{-\omega x/2} \geq \omega x$. Moreover, by another elementary inequality $1 - \mathrm{e}^{-\beta_j\omega} \leq \beta_j\omega$, we have

$$
\frac{\mathrm{e}^{\beta_1\omega}}{\mathrm{e}^{\beta_1\omega}-1} + \frac{\mathrm{e}^{\beta_2\omega}}{\mathrm{e}^{\beta_2\omega}-1} = \frac{1}{1-\mathrm{e}^{-\beta_1\omega}} + \frac{1}{1-\mathrm{e}^{-\beta_2\omega}} \geq \frac{1}{\beta_1\omega} + \frac{1}{\beta_2\omega}
$$

and the second inequality (4.4) follows.

In order to prove the first inequality we first write the right-hand side as

$$
\begin{aligned}
& \frac{(\mathrm{e}^{\beta_1\omega}-1)^{-1} - (\mathrm{e}^{\beta_2\omega}-1)^{-1}}{\mathrm{e}^{\beta_1\omega}(\mathrm{e}^{\beta_1\omega}-1)^{-1} + \mathrm{e}^{\beta_2\omega}(\mathrm{e}^{\beta_2\omega}-1)^{-1}} \\
= \quad & \frac{\mathrm{e}^{\beta_2\omega} - \mathrm{e}^{\beta_1\omega}}{\mathrm{e}^{\beta_1\omega}\mathrm{e}^{\beta_2\omega/2}\left(\mathrm{e}^{\beta_2\omega/2} - \mathrm{e}^{-\beta_2\omega/2}\right) + \mathrm{e}^{\beta_2\omega}\mathrm{e}^{\beta_1\omega/2}\left(\mathrm{e}^{\beta_1\omega/2} - \mathrm{e}^{-\beta_1\omega/2}\right)} \\
= \quad & \mathrm{e}^{-(\beta_1+\beta_2)\omega/2} \frac{\mathrm{e}^{(\beta_2-\beta_1)\omega/2} - \mathrm{e}^{-(\beta_2-\beta_1)\omega/2}}{(1-\mathrm{e}^{-\beta_2\omega}) + (1-\mathrm{e}^{-\beta_1\omega})}
\end{aligned}
$$

Noting that

$$\begin{aligned} \mathrm{e}^{(\beta_2-\beta_1)\omega/2} - \mathrm{e}^{-(\beta_2-\beta_1)\omega/2} &\geq 1 + \frac{(\beta_2-\beta_1)\omega}{2} - \left(1 - \frac{(\beta_2-\beta_1)\omega}{2}\right) \\ (1-\mathrm{e}^{-\beta_2\omega}) + (1-\mathrm{e}^{-\beta_1\omega}) &\leq (\beta_1+\beta_2)\,\omega \end{aligned}$$

we find

$$\frac{(\mathrm{e}^{\beta_1\omega}-1)^{-1} - (\mathrm{e}^{\beta_2\omega}-1)^{-1}}{\mathrm{e}^{\beta_1\omega}(\mathrm{e}^{\beta_1\omega}-1)^{-1} + \mathrm{e}^{\beta_2\omega}(\mathrm{e}^{\beta_2\omega}-1)^{-1}} \geq \mathrm{e}^{-(\beta_1+\beta_2)\omega/2}\,\frac{(\beta_2-\beta_1)\,\omega}{(\beta_1+\beta_2)\,\omega}.$$

This completes the proof. □

Remark. Note that the inequalities of Lemma 4.1 provide a sharp estimate in terms of the inverse temperature difference $\beta_2 - \beta_1$ for small β_1, β_2, i.e. when the average of temperatures T_1, T_2 is big. Indeed, the difference of the right-hand side and left-hand side is equal to

$$\left(1 - \mathrm{e}^{-(\beta_1+\beta_2)\omega/2}\right)\frac{\beta_2-\beta_1}{\beta_1+\beta_2}$$

and for temperatures $T_j > k_B \cdot 180\,\mathrm{K} = 2.49 \cdot 10^{-21}\,\mathsf{J}$ (approximately the lowest natural temperature ever recorded at ground level) we have $\beta_j < 1/(k_B \cdot 180\,\mathrm{K}) = 4.02 \cdot 10^{20}\,\mathsf{J}^{-1}$ so that the quantity that multiplies $\beta_2 - \beta_1$ is

$$\frac{1}{\beta_1+\beta_2} < 1.24 \cdot 10^{-21}\,\mathsf{J}.$$

Theorem 4.2. *Suppose that*

(1) $\operatorname{tr}\left(P_{\varepsilon_n} D_j^* P_{\varepsilon_m} D_j\right) = 1$ *for all* n, m *and all* $j = 1, 2$,

(2) $f_{j,\omega} = 1$ *for all* ω *and all* $j = 1, 2$,

(3) *Jumps can occur only to nearest neighbour levels,*

(4) *Formula (4.3) holds so that the state* ρ *defined by (4.2) with* ρ_0 *determined by the normalization condition is invariant.*

Then

$$\kappa_{\mathrm{m}}\frac{\dfrac{1}{\beta_1}-\dfrac{1}{\beta_2}}{\dfrac{1}{\beta_1}+\dfrac{1}{\beta_2}}\,\kappa(\rho, H_S) \leq \operatorname{tr}\left(\rho\mathcal{L}_1(\mathcal{H}_S)\right) \leq \frac{\dfrac{1}{\beta_1}-\dfrac{1}{\beta_2}}{\dfrac{1}{\beta_1}+\dfrac{1}{\beta_2}}\,\kappa(\rho, H_S) \tag{4.5}$$

where $\kappa_{\mathrm{m}} = \inf_{m\geq 0} \mathrm{e}^{-(\beta_1+\beta_2)(\varepsilon_{m+1}-\varepsilon_m)/2}$ *and*

$$\widehat{H}_S = \sum_{m\geq 0} \varepsilon_{m+1} P_{\varepsilon_m}, \qquad \kappa(\rho, H_S) = \operatorname{tr}\left(\rho(\widehat{H}_S - H_S)\right).$$

Proof. By applying (3.7) in this context, we have

$$\begin{aligned}\operatorname{tr}(\rho\mathcal{L}_1(H_S)) &= \frac{1}{2}\sum_{n\geq 0}(\varepsilon_{n+1}-\varepsilon_n)(\rho_n-\rho_{n+1})\left(\Gamma^+_{1,\varepsilon_{n+1}-\varepsilon_n}-\Gamma^+_{2,\varepsilon_{n+1}-\varepsilon_n}\right)\\ &= \frac{1}{2}\sum_{n\geq 0}(\varepsilon_{n+1}-\varepsilon_n)\left(1-\frac{q_{n,n+1}}{q_{n+1,n}}\right)\rho_n\left(\Gamma^+_{1,\varepsilon_{n+1}-\varepsilon_n}-\Gamma^+_{2,\varepsilon_{n+1}-\varepsilon_n}\right)\\ &= \sum_{n\geq 0}(\varepsilon_{n+1}-\varepsilon_n)\rho_n\frac{\Gamma^+_{1,\varepsilon_{n+1}-\varepsilon_n}-\Gamma^+_{2,\varepsilon_{n+1}-\varepsilon_n}}{\Gamma^-_{1,\varepsilon_{n+1}-\varepsilon_n}+\Gamma^-_{2,\varepsilon_{n+1}-\varepsilon_n}}.\end{aligned}$$

Now the proof follows applying Lemma 4.1 with $\omega = \varepsilon_{n+1}-\varepsilon_n$ to estimate the right-hand side ratio. $\square$

Remark. Formula (4.5) shows that the energy current $\operatorname{tr}(\rho\mathcal{L}_1(H_S))$ has an explicit dependence on the difference $\beta_1^{-1}-\beta_2^{-1}$ of the reservoirs' temperatures. This dependence holds only through two inequalities, but it suggests the existence of an "approximate" Fourier law (see [4, 21]) for the current. Clearly there can be further dependecies through the term $\kappa(\rho, H_S)$, however it holds

$$\inf_k(\varepsilon_{k+1}-\varepsilon_k) \leq \kappa(\rho, H_S) \leq \sup_k(\varepsilon_{k+1}-\varepsilon_k).$$

Therefore the energy current depends on the temperature difference mainly through the explicit term and one could say that there really is an "approximate" Fourier Law. Furthermore it is worth noticing that, for β_1, β_2 fixed, the inequality (4.5) is better the smaller is $\sup_{m\geq 0}(\varepsilon_{m+1}-\varepsilon_m)$ so that κ_{m} is close to 1 and the inequalities are approximately equalities. However, it should also be noted that, in this case, $\kappa(\rho, H_S)$ becomes small as well. Eventually note that, due to the nature of our system, we cannot investigate spatial properties of energy flow. Therefore our discussion of the Fourier's law is concerned with proportionality to temperature difference and not with dependency on size.

Remark. Since the above QMS are of weak coupling limit type, one can write explicitly the entropy production (in the sense of [15, 16]).

It is tempting to study in detail what happens when $\sup_{m\geq 0}(\varepsilon_{m+1}-\varepsilon_m)$ tends to 0 so that the eigenvalues of H_S increase in number and form a set more and more packed. In a more precise way, for all $n \geq 1$ we assume that the system Hamiltonian is a self-adjoint operator $H_S^{(n)}$ on an $(n+1)$-dimensional Hilbert space h with simple pure point spectrum $\left(\varepsilon_k^{(n)}\right)_{0\leq k\leq n}$ with $\varepsilon_0 = 0$ and, for all a, b with $0 \leq a < b \leq +\infty$, we have

$$\lim_{n\to\infty}\frac{\operatorname{card}\left\{k \mid a < \varepsilon_k^{(n)} \leq b\right\}}{n} = \mu(]a,b]) \tag{4.6}$$

for some continuous probability density μ on $[0,+\infty[$. In other words, the empirical distribution of eigenvalues of $H_S^{(n)}$ converges weakly to a probability distribution on $[0,+\infty[$. Suppose, for simplicity, that μ has no atoms, i.e. $\mu(\{r\}) = 0$ for all $r \geq 0$.

We can now prove the following result on the distribution of eigenvalues of the stationary state and energy in stationary conditions.

Theorem 4.3. *Under the assumptions of Theorem 4.2, for all $n \geq 1$, let $H_S^{(n)}$ be as above and suppose that (4.6) holds. Let $\rho^{(n)}$ be the invariant state (4.2) and let*

$$\widetilde{\beta} = 2\,(1/\beta_1 + 1/\beta_2)^{-1}$$

be the harmonic mean of the inverse temperatures (i.e. $\widetilde{\beta}^{-1}$ arithmetic mean of $\beta_1^{-1}, \beta_2^{-1}$).

(i) Eigenvectors $\rho_k^{(n)}$ of $\rho^{(n)}$ satisfy

$$\lim_{n\to\infty} \sum_{\{\,k \mid a<\varepsilon_k\leq b\}} \rho_k^{(n)} = \frac{\displaystyle\int_a^b \mathrm{e}^{-\widetilde{\beta}r}\mathrm{d}\mu(r)}{\displaystyle\int_0^\infty \mathrm{e}^{-\widetilde{\beta}r}\mathrm{d}\mu(r)}$$

(ii) The average energy in the system satisfies

$$\lim_{n\to\infty} \mathrm{tr}\left(\rho^{(n)} H_S^{(n)}\right) = \frac{\displaystyle\int_0^\infty \mathrm{e}^{-\widetilde{\beta}r}\, r\mathrm{d}\mu(r)}{\displaystyle\int_0^\infty \mathrm{e}^{-\widetilde{\beta}r}\mathrm{d}\mu(r)}.$$

This result reminds the one in [20] where the steady state can be described by a generalized Gibbs state and the steady-state current is proportional to the difference in the reservoirs' magnetizations.

In the proof we need the following Lemma.

Lemma 4.4. *Let $\widetilde{\beta} = 2/\left(\beta_1^{-1} + \beta_2^{-1}\right)$ be the harmonic mean of inverse temperatures (i.e. $\widetilde{\beta}^{-1}$ is the arithmetic mean of β_1^{-1} and β_2^{-1}). For all $1 \leq k \leq n$ and for $\sup_j \omega_j < 1/(3\beta_2)$,*

$$1 - \widetilde{\beta}\,\omega_k \leq \frac{q_{k,k+1}}{q_{k+1,k}} \leq 1 - \widetilde{\beta}\,\omega_k + \left(\widetilde{\beta}\,\omega_k\right)^2 \tag{4.7}$$

where $\omega_k = \varepsilon_{k+1} - \varepsilon_k$ and

$$\mathrm{e}^{-\widetilde{\beta}\varepsilon_k\left(1+\widetilde{\beta}\sup_j \omega_j\right)} \leq \prod_{j=0}^{k-1} \frac{q_{j,j+1}}{q_{j+1,j}} \leq \mathrm{e}^{-\widetilde{\beta}\varepsilon_k\left(1-\widetilde{\beta}\sup_j \omega_j\right)} \tag{4.8}$$

Proof. By the elementary inequality $1 - \mathrm{e}^{-\beta_j\omega_k} \leq \beta_j\omega_k$ we have

$$\begin{aligned}
\frac{q_{k,k+1}}{q_{k+1,k}} &= \frac{\dfrac{1}{\mathrm{e}^{\beta_1\omega_k}-1} + \dfrac{1}{\mathrm{e}^{\beta_2\omega_k}-1}}{\dfrac{\mathrm{e}^{\beta_1\omega_k}}{\mathrm{e}^{\beta_1\omega_k}-1} + \dfrac{\mathrm{e}^{\beta_2\omega_k}}{\mathrm{e}^{\beta_2\omega_k}-1}} \\
&= 1 - \frac{2}{\left(1-\mathrm{e}^{-\beta_1\omega_k}\right)^{-1} + \left(1-\mathrm{e}^{-\beta_2\omega_k}\right)^{-1}} \\
&\geq 1 - \frac{2\omega_k}{1/\beta_1 + 1/\beta_2}
\end{aligned}$$

In the same way, by the elementary inequalities $1-\mathrm{e}^{-\beta_j\omega_k} \geq \beta_j\omega_k - (\beta_j\omega_k)^2/2$ and $1/(1-\beta_j\omega_k/2) \leq 1+\beta_j\omega_k$, we find for $\beta_j\omega_k < 1$

$$\begin{aligned}
\frac{q_{k,k+1}}{q_{k+1,k}} &\leq 1 - \frac{2\omega_k}{1/\left(\beta_1\left(1-\beta_1\omega_k/2\right)\right) + 1/\left(\beta_2\left(1-\beta_2\omega_k/2\right)\right)} \\
&\leq 1 - \frac{2\omega_k}{1/\beta_1\left(1+\beta_1\omega_k/2\right) + 1/\beta_2\left(1+\beta_2\omega_k/2\right)} \\
&\leq 1 - \frac{2\omega_k}{1/\beta_1 + 1/\beta_2 + 2\omega_k} \\
&= 1 - \frac{\widetilde{\beta}\,\omega_k}{1+\widetilde{\beta}\,\omega_k}
\end{aligned}$$

and so (4.7) follows.

In order to prove the upper bound in (4.8), note that, since $\log(1-x) \leq -x$

$$\log\left(\prod_{j=0}^{k-1}\frac{q_{j,j+1}}{q_{j+1,j}}\right) \leq \sum_{j=0}^{k-1}\log\left(1-\widetilde{\beta}\,\omega_j\left(1-\widetilde{\beta}\,\omega_j\right)\right) \leq -\sum_{j=0}^{k-1}\widetilde{\beta}\omega_j\left(1-\widetilde{\beta}\omega_j\right),$$

as a consequence

$$\log\left(\prod_{j=0}^{k-1}\frac{q_{j,j+1}}{q_{j+1,j}}\right) \leq -\sum_{j=0}^{k-1}\widetilde{\beta}\omega_j\left(1-\widetilde{\beta}\sup_l\omega_l\right) = -\widetilde{\beta}\varepsilon_k\left(1-\widetilde{\beta}\sup_l\omega_l\right).$$

For the lower bound, we begin by the inequality

$$\log\left(\prod_{j=0}^{k-1}\frac{q_{j,j+1}}{q_{j+1,j}}\right) = \sum_{j=0}^{k-1}\log\left(\frac{q_{j,j+1}}{q_{j+1,j}}\right) \geq \sum_{j=0}^{k-1}\log\left(1-\widetilde{\beta}\omega_j\right).$$

Note that $\log(1-x)+x+x^2 \geq 0$ for $0 \leq x \leq 2/3$ and, since $\widetilde{\beta}\omega_j < 2/3$ by our assumption, we have

$$\log\left(\prod_{j=0}^{k-1}\frac{q_{j,j+1}}{q_{j+1,j}}\right) \geq -\sum_{j=0}^{k-1}\widetilde{\beta}\omega_j\left(1+\widetilde{\beta}\sup_l\omega_l\right) = -\widetilde{\beta}\epsilon_k\left(1+\widetilde{\beta}\sup_l\omega_l\right).$$

This completes the proof. $\square$

Proof of Theorem 4.3. Let μ_n be the empirical distribution of the eigenvalues of $H_S^{(n)}$ i.e.

$$\mu_n = \frac{1}{n+1}\sum_{k=0}^{n}\delta_{\varepsilon_k}$$

and note that

$$\sum_{\{\,k \mid a<\varepsilon_k\leq b\}}\rho_k^{(n)} = \frac{\dfrac{1}{n+1}\displaystyle\sum_{\{\,k \mid a<\varepsilon_k\leq b\}}\prod_{j=0}^{k-1}\frac{q_{j,j+1}}{q_{j+1,j}}}{\dfrac{1}{n+1}\displaystyle\sum_{k=0}^{n}\prod_{j=0}^{k-1}\frac{q_{j,j+1}}{q_{j+1,j}}}. \tag{4.9}$$

Clearly, by Lemma 4.4,

$$
\begin{aligned}
\frac{1}{n+1}\sum_{\{\,k\,|\,a<\varepsilon_k\leq b\}}\prod_{j=0}^{k-1}\frac{q_{j,j+1}}{q_{j+1,j}} &\leq \frac{1}{n+1}\sum_{\{\,k\,|\,a<\varepsilon_k\leq b\}} \mathrm{e}^{-\widetilde{\beta}\varepsilon_k\left(1-\widetilde{\beta}\sup_j\omega_j\right)}\\
&\leq \mathrm{e}^{\widetilde{\beta}^2 b\sup_j\omega_j}\int_{]a,b]}\mathrm{e}^{-\widetilde{\beta}\varepsilon_k}\,\mathrm{d}\mu_n(r)
\end{aligned}
$$

and also

$$
\begin{aligned}
\frac{1}{n+1}\sum_{\{\,k\,|\,a<\varepsilon_k\leq b\}}\prod_{j=0}^{k-1}\frac{q_{j,j+1}}{q_{j+1,j}} &\geq \frac{\mathrm{e}^{-\widetilde{\beta}^2 a\sup_j\omega_j}}{n+1}\sum_{\{\,k\,|\,a<\varepsilon_k\leq b\}} \mathrm{e}^{-\widetilde{\beta}\varepsilon_k}\\
&= \mathrm{e}^{-\widetilde{\beta}^2 a\sup_j\omega_j}\int_{]a,b]}\mathrm{e}^{-\widetilde{\beta}\varepsilon_k}\,\mathrm{d}\mu_n(r).
\end{aligned}
$$

Since $\sup_j\omega_j$ goes to 0, probability measures μ_n converge weakly to μ and the function $r\to\mathrm{e}^{-\widetilde{\beta}r}$ is bounded continuous on $[0,+\infty[$, taking the limit as $n\to\infty$, we have

$$
\lim_{n\to\infty}\frac{1}{n+1}\sum_{\{\,k\,|\,a<\varepsilon_k\leq b\}}\prod_{j=0}^{k-1}\frac{q_{j,j+1}}{q_{j+1,j}}=\int_{]a,b]}\mathrm{e}^{-\widetilde{\beta}\varepsilon_k}\,\mathrm{d}\mu(r).
$$

In the same way, taking $a=0$ and $b=+\infty$, we see that the denominator of (4.9) converges to

$$
\int_0^{+\infty}\mathrm{e}^{-\tilde{\beta}r}\,\mathrm{d}\mu(r)
$$

and the proof of (i) is complete. The proof of (ii) is similar. □

Remark. Theorem 4.3 (i) shows that, if μ has density μ', then the asymptotic distribution of eigenvalues of the stationary state is

$$
\lambda\mapsto\frac{\mathrm{e}^{-\tilde{\beta}\lambda}\mu'(\lambda)}{\displaystyle\int_0^{+\infty}\mathrm{e}^{-\tilde{\beta}r}\mu'(r)\mathrm{d}r}.
$$

The asymptotic average energy in the system can be easily computed in some remarkable cases noting that the integral of $\mathrm{e}^{-\tilde{\beta}r}$ with respect to μ is the moment generating function ϕ of μ evaluated at $-\tilde{\beta}$ and so the asymptotic average energy in the system is

$$
-\frac{\frac{\mathrm{d}}{\mathrm{d}\tilde{\beta}}\phi(-\tilde{\beta})}{\phi(-\tilde{\beta})}=-\frac{\mathrm{d}}{\mathrm{d}\tilde{\beta}}\log\left(\phi(-\tilde{\beta})\right).
$$

We can easily find an explicit result in two cases:

$$
\begin{array}{rll}
\mu \text{ normal distribution } N(m,\sigma^2) & \text{average energy} & m-\tilde{\beta}\sigma\\
\mu \text{ gamma distribution } \Gamma(\alpha,\theta) & \text{average energy} & \alpha/(\tilde{\beta}+\theta)
\end{array}
$$

The asymptotic average energy in the system is decreasing in $\tilde{\beta}$, i.e. increasing in the average temperature as expected, for all probability measure μ because the moment generating function of a probability distribution is log-convex and the derivative of a convex function is increasing.

Remark. Note that, by choosing a suitable spacing of eigenvalues ε_n we can control the rate of convergence to 0 of $\kappa\left(\rho^{(n)},H_S^{(n)}\right)$ at will, as n tends to $+\infty$.

5 One dimensional Ising chain

In this section we consider a one-dimensional Ising chain with nearest neighbour interaction. We will show that, in this case, if the heat baths interact locally at both ends of the chain, then the energy current is zero. Spin interaction (see 5.1) occurs only in the z component. In the case where also the other components interact the derivation of the GKSL generator turns out to be really difficult (see [5]). Indeed, starting from the diagonalized H_S, one finds a cumbersome expression for the operators D_ω.

In spite of the simple system Hamitonian H_S (5.1) Theorems 4.2 and 4.3 do not apply to this model because its spectrum is degenerate.

The system space is $\mathsf{h} = \mathbb{C}^{2^{\otimes N}}$ with $N > 2$. Define Pauli matrices

$$\sigma^x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \sigma^y = \begin{bmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{bmatrix} \qquad \sigma^z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

with respect to the orthonormal basis $e_+ = [1,0]^{\mathrm{T}}$, $e_- = [0,1]^{\mathrm{T}}$ of $\mathbb{C}^2$.

Consider the one dimensional Ising chain with Hamiltonian

$$H_S = J_z \sum_{j=1}^{N-1} \sigma_j^z \sigma_{j+1}^z, \qquad J_z > 0,\ N > 2 \tag{5.1}$$

Subsequently let us define

$$e_\alpha := \otimes_{j=1}^N e_{\alpha(j)}, \quad \alpha \in \{-1,1\}^N,$$

as a basis of h, where $e_{-1} := e_-$ and $e_{+1} := e_+$. Vectors $\{e_\alpha\}_\alpha$ form an eigenbasis for H_S and the spectrum is

$$\mathrm{sp}(H_S) = \{\, J_z\,(2k - (N-1)) \mid k = 0, \ldots, N-1 \,\}.$$

The eigenspace associated with the eigenvalue $\varepsilon_k = J_z(2k-(N-1))$ is the linear span of the elements e_α such that exactly k neighbouring elements in α have the same sign. Thus one can define the sets

$$A_k := \left\{ \alpha \in \{-1,1\}^N \;\middle|\; \textstyle\sum_{j=1}^{N-1} \alpha(j)\alpha(j+1) = 2k - (N-1) \right\},$$

and the spectral projection associated with the eigenvalue ε_k is given by

$$P_k := \textstyle\sum_{\alpha \in A_k} |e_\alpha\rangle\langle e_\alpha|.$$

The system is coupled with two heat reservoirs at inverse temperature β_1, β_2 with $\beta_1 \le \beta_2$ through the interaction

$$H_1 = \sigma_1^u \otimes (A^-(\phi_1) + A^+(\phi_1)), \quad H_2 = \sigma_N^v \otimes (A^-(\phi_2) + A^+(\phi_2)), \tag{5.2}$$

where $u, v \in \mathbb{R}^3$ and σ_i^u is defined as

$$\sigma_i^u = u_1\sigma_i^x + u_2\sigma_i^y + u_3\sigma_i^z.$$

The set of positive Bohr frequencies is given by

$$\mathsf{B} := \{\, 2J_z(n-m) = \varepsilon_n - \varepsilon_m \mid n, m \in \{0, \dots, N-1\},\, n > m \,\},$$

while the operators $D_{j,\omega}$ are given by (2.5). Thus one has

$$D_{1,2J_z} = (u_1 - \mathrm{i}u_2) \sum_{\alpha \in C^l_{++}} \sigma_1^x |e_\alpha\rangle\langle e_\alpha| + (u_1 + \mathrm{i}u_2) \sum_{\alpha \in C^l_{--}} \sigma_1^x |e_\alpha\rangle\langle e_\alpha|$$

where C^l_{++} (resp. C^l_{--}) denotes the set of configurations $\alpha \in \{-1, +1\}^N$ with $++$ (resp. $--$) in the first two sites (l stands for left). While $D_{1,\omega} = 0$ for every $\omega \in \mathsf{B} - \{2J_z\}$ because the Pauli matrices act only on the first site and so the number of neighbouring sites with the same sign can vary of at most one after the action of σ_1^u and for $\omega = 2J_z$ one has

$$D_{1,2J_z} = \sum_{n=1}^{N-1} \sum_{\alpha \in A_n} \sum_{\beta \in A_{n+1}} \langle e_\alpha, \sigma_1^x e_\beta\rangle\, |e_\alpha\rangle\langle e_\beta|.$$

With similar arguments one can see that $D_{2,\omega} = 0$ for every $\omega \in \mathsf{B} - \{2J_z\}$, while

$$D_{2,2J_z} = (v_1 - \mathrm{i}v_2) \sum_{\alpha \in C^r_{++}} \sigma_N^x |e_\alpha\rangle\langle e_\alpha| + (v_1 + \mathrm{i}v_2) \sum_{\alpha \in C^r_{--}} \sigma_N^x |e_\alpha\rangle\langle e_\alpha|$$

where C^r_{++} (resp. C^r_{--}) denotes the set of configurations with $++$ (resp. $--$) in the last two sites (r stands for right).

From now on we will drop the subscript $2J_z$ and only deal with operators related to that Bohr frequency, as the others vanish.
Recalling the definition of linear maps (2.6) and the constants

$$\gamma_i^+ = 1/(\mathrm{e}^{2J_z\beta_i} - 1), \quad \gamma_i^- = \mathrm{e}^{2J_z\beta_i}/(\mathrm{e}^{2J_z\beta_i} - 1),$$

we can write the GKSL generator of the evolution as follows

$$\mathcal{L} = \sum_{i \in \{1,N\}} \gamma_i^-\,\mathcal{Q}_i^- + \gamma_i^+\,\mathcal{Q}_i^+.$$

A close scrutiny at the operators D_i, D_i^* shows that, for each fixed configuration $\overline{\alpha} \in \{-1, +1\}^{N-2}$ of the $N-2$ inner sites of the chain the 4-dimensional projections $p_{\overline{\alpha}}$ on subspaces

$$\mathsf{h}_{\overline{\alpha}} := \mathrm{span}\,\{\, e_\alpha \;\mid\; \alpha(j) = \overline{\alpha}(j) \text{ for all } 2 \le j \le N-1;\, \alpha(1), \alpha(N) \in \{-1, 1\}\,\}$$

commute with both D_i and D_i^* for $i \in \{1, N\}$, then subalgebras $p_{\alpha_1}\mathcal{B}(\mathsf{h})p_{\alpha_2}$ are invariant for the semigroup $\mathcal{T}$ generated by $\mathcal{L}$. This commutation allows us to restrict our study only to cases where the invariant state is of the form

$$\rho = \sum_{\overline{\alpha}\in\{-1,1\}^{N-2}} p_{\overline{\alpha}}\rho p_{\overline{\alpha}} = \sum_{\overline{\alpha}\in\{-1,1\}^{N-2}} \lambda_{\overline{\alpha}}\rho_{\overline{\alpha}}, \tag{5.3}$$

where $\rho_{\overline{\alpha}}$ is an invariant state supported only on $\mathsf{h}_{\overline{\alpha}}$ and $\lambda_{\overline{\alpha}}$ are real constants that sum up to 1. Indeed the off diagonal terms, $p_{\overline{\alpha_1}}\rho p_{\overline{\alpha_2}}$ with $\alpha_1 \neq \alpha_2$, do not contribute to current flow, since

$$\mathrm{tr}\left(p_{\overline{\alpha_1}}\rho p_{\overline{\alpha_2}}\mathcal{L}_1(H_S)\right) = \mathrm{tr}\left(p_{\overline{\alpha_1}}\rho\mathcal{L}_1(H_S)p_{\overline{\alpha_2}}\right) = 0.$$

Moreover all the conditional expectations $\mathcal{E}_{\overline{\alpha}}(x) := p_{\overline{\alpha}}xp_{\overline{\alpha}}$ commute with $\mathcal{L}$, ensuring that both $\sum_{\overline{\alpha}}\mathcal{E}_{\overline{\alpha},*}(\rho)$ and every $\mathcal{E}_{\overline{\alpha},*}(\rho)$ must also be invariant states on their own. As a further refinement we can repeat the same argument using the conditional expectation $\mathcal{E}(x) := \sum_{k=0}^{N-1} P_k x P_k$. Indeed $\mathcal{E}$ commutes with the Lindbladian $\mathcal{L}$ and

$$\mathrm{tr}\left(P_{k_1}\rho P_{k_2}\mathcal{L}_1(H_S)\right) = \mathrm{tr}\left(P_{k_1}\rho\mathcal{L}_1(H_S)P_{k_2}\right) = 0$$

for $k_1 \neq k_2$, since the spectral projections commute with $D_jD_j^*$, $D_j^*D_j$ and $\mathcal{L}_1(H_S)$ is a linear combination of these operators by Lemma 3.2, equation (3.3). In this way we can focus our study on invariant states of the form (5.3) with

$$p_{\overline{\alpha}}\rho p_{\overline{\alpha}} = \rho_{\overline{\alpha}} = \begin{bmatrix} \rho_{11}^{\overline{\alpha}} & 0 & 0 & 0 \\ 0 & \rho_{22}^{\overline{\alpha}} & \rho_{23}^{\overline{\alpha}} & 0 \\ 0 & \rho_{32}^{\overline{\alpha}} & \rho_{33}^{\overline{\alpha}} & 0 \\ 0 & 0 & 0 & \rho_{44}^{\overline{\alpha}} \end{bmatrix},$$

where we expanded the state with respect to the basis of four vectors $e_{c\,\overline{\alpha}\,c}, e_{d\,\overline{\alpha}\,c}, e_{c\overline{\alpha}d}, e_{d\overline{\alpha}d}$ defined as follows: $e_{c\overline{\alpha}c}$ is the vector $e_{\overline{\alpha}(2),\overline{\alpha}(2),\ldots,\overline{\alpha}(N-1),\overline{\alpha}(N-1)}$, $e_{c\overline{\alpha}d} = e_{\overline{\alpha}(2),\overline{\alpha}(2),\ldots,\overline{\alpha}(N-1),-\overline{\alpha}(N-1)}$ and vectors $e_{d\overline{\alpha}c}$, $e_{d\overline{\alpha}d}$ are defined in a similar way.

Now we have reduced and simplified the class of states we want to use when looking for a invariant state, without, however, losing any contribution to the current flow. In order to find the invariant state, first of all it is not too difficult to show that $\mathcal{L}_*$ leaves invariant the subspace of diagonal elements. Then compute

$$\mathcal{L}_*(\rho_{23}^{\overline{\alpha}}|e_{d\,\alpha\,c}\rangle\langle e_{c\,\alpha\,d}|) = -\frac{1}{2}\left[\Gamma_1^+ + \Gamma_1^- + \Gamma_N^+ + \Gamma_N^-\right]\rho_{23}^{\overline{\alpha}}|e_{d\,\alpha\,c}\rangle\langle e_{c\,\alpha\,d}|,$$

and similarly

$$\mathcal{L}_*(\rho_{32}^{\overline{\alpha}}|e_{c\,\alpha\,d}\rangle\langle e_{d\,\alpha\,c}|) = -\frac{1}{2}\left[\Gamma_1^+ + \Gamma_1^- + \Gamma_N^+ + \Gamma_N^-\right]\rho_{32}^{\overline{\alpha}}|e_{c\,\alpha\,d}\rangle\langle e_{d\,\alpha\,c}|,$$

where $\Gamma_1^\pm = \|u_1 + \mathrm{i}u_2\|^2\gamma_1^\pm$ and $\Gamma_N^\pm = \|v_1 + \mathrm{i}v_2\|^2\gamma_N^\pm$. (The above $\Gamma_i^\pm$ slightly differ from the constants in Section 2). Therefore the invariant state condition $\mathcal{L}_*(\rho) = 0$ implies $\rho_{23}^{\overline{\alpha}} = \rho_{32}^{\overline{\alpha}} = 0$.

We can now just consider the reduced dynamics on diagonal elements of $p_{\overline{\alpha}}\mathcal{B}(\mathsf{h})p_{\overline{\alpha}}$, given by

$$\mathcal{L}_* = \begin{bmatrix} -(\Gamma_1^- + \Gamma_N^-) & \Gamma_1^- & \Gamma_N^- & 0 \\ \Gamma_1^+ & -(\Gamma_1^+ + \Gamma_N^-) & 0 & \Gamma_N^- \\ \Gamma_N^+ & 0 & -(\Gamma_N^+ + \Gamma_1^-) & \Gamma_1^- \\ 0 & \Gamma_N^+ & \Gamma_1^+ & -(\Gamma_N^+ + \Gamma_1^+) \end{bmatrix},$$

The unique invariant law for the time-continuous Markov chain generated by the above matrix is

$$\pi = Z^{-1}\left[1, \mathrm{e}^{2J_z\beta_1}, \mathrm{e}^{2J_z\beta_2}, \mathrm{e}^{2J_z(\beta_1+\beta_2)}\right],$$

where Z^{-1} is a normalization constant that is independent of u, v and is the same for all $\overline{\alpha}$. Therefore the unique $\mathcal{T}$-invariant state supported on h_α is

$$\rho_\alpha = Z^{-1}\Big(|e_{c\overline{\alpha}c}\rangle\langle e_{c\overline{\alpha}c}| + \mathrm{e}^{2J_z\beta_1}|e_{d\,\overline{\alpha}c}\rangle\langle e_{d\,\overline{\alpha}c}| \\ + \mathrm{e}^{2J_z\beta_2}|e_{c\overline{\alpha}\,d}\rangle\langle e_{c\overline{\alpha}\,d}| + \mathrm{e}^{2J_z(\beta_1+\beta_2)}|e_{d\,\overline{\alpha}\,d}\rangle\langle e_{d\,\overline{\alpha}\,d}|\Big).$$

Recalling (5.3) we can now write any invariant state for the semigroup $\mathcal{T}$.

We can now evaluate the energy flow $\mathrm{tr}\,(\rho\mathcal{L}_1(H_S))$ via the expression

$$\mathcal{L}_1(H_S) = \sum_{\omega\in B^+}\omega\left(\gamma_{1,\omega}^+ D_1D_1^* - \gamma_{1,\omega}^- D_1^*D_1\right) = 2J_z\left(\gamma_1^+ D_1D_1^* - \gamma_1^- D_1^*D_1\right)$$

that, together with the formula for ρ_α, yields

$$\begin{aligned} Z\,\mathrm{tr}\,(\rho\mathcal{L}_1(H_S)) &= Z\,\mathrm{tr}\left(\sum_{\overline{\alpha}\in\{-1,1\}^{N-2}}\lambda_{\overline{\alpha}}\rho_{\overline{\alpha}}\mathcal{L}_1(H_S)\right) \\ &= \sum_{\overline{\alpha}\in\{-1,1\}^{N-2}} 2J_z\lambda_{\overline{\alpha}}\left(\gamma_1^+\mathrm{e}^{\beta_1\omega} + \gamma_1^+\mathrm{e}^{(\beta_1+\beta_2)\omega} - \gamma_1^-\mathrm{e}^{\beta_2\omega} - \gamma_1^-\right) \\ &= 0 \end{aligned}$$

Remark. For $N = 2$, it can be shown by direct computation that the energy current is strictly positive. Indeed, because of low dimensionality the ends of the chain can interact directly.

Acknowledgement.

The authors would like to thank Stefano Olla for drawing their attention to the problem of energy transport in quantum systems and fruitful discussions at the workshop "Quantum Transport Equations and Applications" at Casa Matemática Oaxaca (México) in September 2018. They also gratefully thank CUBO's anonymous referees whose remarks and comments to improve an earlier version of this paper. The financial support of GNAMPA INdAM 2020 projects "Processi stocastici quantistici e applicazioni" and "Evoluzioni markoviane quantistiche" is gratefully acknowledged.

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CUBO, A Mathematical Journal
Vol. 23, no. 01, pp. 145–159, April 2021
DOI: 10.4067/S0719-06462021000100145

Existence and attractivity results for ψ-Hilfer hybrid fractional differential equations

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ABSTRACT

In this work, we present some results on the existence of attractive solutions of fractional differential equations of the ψ-Hilfer hybrid type. The results on the existence of solutions are a consequence of the Schauder fixed point theorem. Next, we prove that all solutions are uniformly locally attractive.

RESUMEN

En este trabajo, presentamos algunos resultados sobre la existencia de soluciones atractivas de ecuaciones diferenciales fraccionarias de tipo ψ-Hilfer híbridas. Los resultados de existencia de soluciones son consecuencia del teorema de punto fijo de Schauder. A continuación, probamos que todas las soluciones son uniformemente localmente atractivas.

Keywords and Phrases: ψ-Hilfer fractional derivative; Schauder fixed-point Theorem; uniformly locally attractive.

2020 AMS Mathematics Subject Classification: 26A33, 34A08.

Accepted: 01 March, 2021
Received: 12 November, 2020

1 Introduction

The theory of derivatives and integrals to a real or complex order is none other than the fractional theory which began in 1695 between G.A. de L'Hospital and G.W. Leibniz. The fractional integration and differentiation go back to Leibniz, Riemann, Liouville, Abel, Weyl, and Riesz [27]. Many monographs to which the reader can refer such as Abbas *et al.* [1, 5, 6], Diethelm [13], Kilbas *et al.* [17], Oldham *et al.* [22], Podlubny [23], Samko *et al.* [24], Zhou [32, 33], Zhou *et al.* [34] and the works by Abbas and Benchohra [2], Lakshmikantham *et al.* [19, 20, 21]. Recently several works have been done concerning hybrid fractional differential equations see [9, 12, 14, 26, 31], and the references therein.

Functional $\psi-$ fractional differential equations received a great importance in applied mathematics and other sciences, see [8, 16, 18, 25, 28, 29, 30], and the references therein.

Some interesting results on existence and attractivity have been obtained in [3, 4, 7]. In this work, we are interested in the existence and attractivity of solutions for the following problem

$$\begin{cases} D_{0^+}^{\lambda,\sigma;\psi} \frac{u(t)}{v(t,u(t))} = w\left(t,u(t)\right); \ a.e. \quad t \in \mathbb{R}_+, \\ (\psi(t)-\psi(0))^{1-\varsigma} u(t) \mid_{t=0} = u_0; \qquad u_0 \in \mathbb{R}, \end{cases} \tag{1.1}$$

where $\mathbb{R}_+ := [0,+\infty),\ 0<\lambda<1,\ 0\leq\sigma\leq 1,\ \varsigma=\lambda+\sigma(1-\lambda),\ {}^H D_{0^+}^{\lambda,\sigma;\psi}$ is the ψ -Hilfer fractional derivative of order λ and type σ, $v:\mathbb{R}_+\times\mathbb{R}\to\mathbb{R}^*$ and $w:\mathbb{R}_+\times\mathbb{R}\to\mathbb{R}$, are given functions.

Special cases:

• For $\sigma=0,\psi(t)=t,u_0=0$, we will get nonlinear hybrid FDEs of the form

$$\begin{cases} {}^{RL}D_{0^+}^{\lambda}\left[\frac{u(t)}{v(t,u(t))}\right] = w(t,u(t)), \text{ a.e. } t\in\mathbb{R}_+, \\ u(0)=0. \end{cases}$$

• For $\lambda=1,\sigma=1,\psi(t)=t$, we will get nonlinear integer order hybrid differential equations of the form

$$\begin{cases} \frac{d}{dt}\left[\frac{u(t)}{v(t,u(t))}\right] = w(t,y(t)), \text{ a.e. } t\in\mathbb{R}_+, \\ u(0)=u_0\in\mathbb{R}. \end{cases}$$

For $v=1$, we will get nonlinear ψ-Hilfer FDEs of the form

$$\begin{cases} {}^{H}D_{0^+}^{\lambda,\sigma;\psi}u(t) = w(t,y(t)), \text{ a.e. } t\in\mathbb{R}_+, \\ (\psi(t)-\psi(0))^{1-\varsigma}u(t)\big|_{t=0} = u_0\in\mathbb{R}. \end{cases}$$

• For $v=1,\sigma=0$ (in this case $\varsigma=\lambda$), $\psi(t)=t$, we will get nonlinear FDEs involving Riemann-Liouville fractional derivative

$${}^{RL}D_{0^+}^{\lambda}u(t) = w(t,y(t)), \text{ a.e. } t\in\mathbb{R}_+.$$

2 Preliminaries

Let $\psi : [a,b] \to \mathbb{R}$ be an increasing differentiable function such that $\psi'(t) \neq 0$, for all $t \in [a,b]$, $(-\infty \leq a < b \leq +\infty)$. Define on $[a,b]$, $(0 < a < b < \infty)$ the weighted space

$$C_{\varsigma\psi[a,b]} = \{\tau : (a,b] \to \mathbb{R} : (\psi(t) - \psi(a))^{\varsigma}\tau(t) \in C[a,b]\}, \quad 0 \leq \varsigma < 1,$$

with the norm

$$\|\tau\|_{C_{\varsigma;\psi}[a,b]} = \|(\psi(t) - \psi(a))^{\varsigma}\tau(t)\|_{C[a,b]} = \max\left\{|(\psi(t) - \psi(a))^{\varsigma}\tau(t)| : t \in [a,b]\right\},$$

where $C([a,b])$ denotes the Banach space of all real continuous functions on $[a,b]$.

Let $BC := BC(\mathbb{R}_+)$ be the Banach space of all bounded and continuous functions from $\mathbb{R}_+$ into $\mathbb{R}$. By $BC_{\varsigma} := BC_{\varsigma}(\mathbb{R}_+)$, we denote the weighted space of all bounded and continuous functions defined by $BC_{\varsigma} = \{\phi : \mathbb{R}_+ \to \mathbb{R} : (\psi(t) - \psi(0))^{1-\varsigma}\phi(t) \in BC\}$, with the norm

$$\|\phi\|_{BC_{\varsigma}} := \sup_{t \in \mathbb{R}_+} \left|(\psi(t) - \psi(0))^{1-\varsigma}\phi(t)\right|.$$

Let us recall some definitions and properties of fractional calculus.

Definition 2.1. *[17] The left-sided ψ-Riemann-Liouville fractional integral and fractional derivative of order λ, $(n-1 < \lambda < n)$ for an integrable function $\phi : [a,b] \to \mathbb{R}$ with respect to another function $\psi : [a,b] \to \mathbb{R}$, that is an increasing differentiable function such that $\psi'(t) \neq 0$, for all $t \in [a,b], (-\infty \leq a < b \leq +\infty)$, are respectively defined as follows:*

$$I_{a^+}^{\lambda;\psi}\phi(t) = \frac{1}{\Gamma(\lambda)}\int_a^t \psi'(s)(\psi(t) - \psi(s))^{\lambda-1}\phi(s)ds,$$

and

$$D_{a+}^{\lambda;\psi}\phi(t) = \left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n I_{a^+}^{n-\alpha;\psi}\phi(t) = \frac{1}{\Gamma(n-\lambda)}\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n \int_a^t \psi'(s)(\psi(t) - \psi(s))^{n-\lambda-1}\phi(s)ds,$$

where $\Gamma(\cdot)$ is the Euler gamma function defined by

$$\Gamma(\delta) = \int_0^{\infty} e^{-t}t^{\delta-1}dt, \quad \delta > 0.$$

Definition 2.2. *[10] The left-sided ψ -Caputo fractional derivative of function $\chi \in C^n[a,b]$, $(n-1 < \lambda < n)$ $n = [\alpha] + 1$ with respect to another function ψ is defined by*

$${}^{c}D_{a^+}^{\lambda;\psi}\phi(t) = I_{a^+}^{n-\lambda;\psi}\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n \phi(t) = \frac{1}{\Gamma(n-\lambda)}\int_a^t \psi'(s)(\psi(t) - \psi(s))^{n-\lambda-1}\phi_{\psi}^{[n]}(s)ds,$$

where $\phi_{\psi}^{[n]}(t) = \left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n \phi(t)$ and ψ defined as in Definition $\mathbb{Q}$. Moreover, the $\psi-$ Caputo fractional derivative of function $\phi \in AC^n[a,b]$ is determined as

$${}^{c}D_{a^+}^{\lambda;\psi}\phi(t) = D_{a^+}^{\lambda;\psi}\left[\phi(t) - \sum_{k=0}^{n-1}\frac{\left[\frac{1}{\psi'(t)}\frac{d}{dt}\right]^k \phi(a)}{k!}(\psi(t) - \psi(a))^k\right].$$

Definition 2.3. *[29] Let* $n-1<\lambda<n, n\in\mathbb{N}$, *with* $[a,b], -\infty\leq a<b\leq+\infty$, *and* $\psi\in C^n([a,b],\mathbb{R})$ *a function such that* $\psi(t)$ *is increasing and* $\psi'(t)\neq 0$, *for all* $t\in[a,b]$. *The* ψ *-Hilfer fractional derivative (left-sided) of function* $\phi\in C^n([a,b],\mathbb{R})$ *of order* λ *and type* $\sigma\in[0,1]$ *is determined as*

$$D_{a^+}^{\lambda,\sigma;\psi}\phi(t)=I_{a^+}^{\sigma(n-\lambda);\psi}\left[\frac{1}{\psi'(t)}\frac{d}{dt}\right]^n I_{a^+}^{(1-\sigma)(n-\lambda);\psi}\phi(t), t>a.$$

In other way

$$D_{a^+}^{\lambda,\sigma;\psi}\phi(t)=I_{a^+}^{\sigma(n-\lambda);\psi}D_{a^+}^{\gamma;\psi}\phi(t), t>a,$$

where

$$D_{a^+}^{\gamma;\psi}\phi(t)=\left[\frac{1}{\psi'(t)}\frac{d}{dt}\right]^n I_{a^+}^{(1-\sigma)(n-\lambda);\psi}\phi(t).$$

In particular, the ψ *-Hilfer fractional derivative of order* $\lambda\in(0,1)$ *and type* $\sigma\in[0,1]$, *can be written in the following form*

$$\begin{aligned}D_{a^+}^{\lambda,\sigma;\psi}\phi(t)&=\frac{1}{\Gamma(\varsigma-\lambda)}\int_a^t(\psi(t)-\psi(s))^{\varsigma-\lambda-1}D_{a^+}^{\gamma;\psi}\phi(s)ds\\&=I_{a^+}^{\varsigma-\lambda;\psi}D_{a^+}^{\varsigma;\psi}\phi(t),\end{aligned}$$

where $\varsigma=\lambda+\sigma-\lambda\sigma$, *and* $D_{a^+}^{\varsigma;\psi}\phi(t)=\left[\frac{1}{\psi'(t)}\frac{d}{dt}\right]I_{a^+}^{1-\varsigma;\psi}\phi(t)$.

Lemma 2.4. *[29] Let* $\lambda>0, 0\leq\varsigma<1$ *and* $\phi\in L^1(a,b)$. *Then*

$$I_{a^+}^{\lambda;\psi}I_{a^+}^{\sigma;\psi}\phi(t)=I_{a^+}^{\lambda+\sigma;\psi}\phi(t),\ \ a.e.\ t\in[a,b].$$

In particular (i) if $\phi\in C_{\varsigma;\psi}[a,b]$, *then* $I_{a+}^{\lambda;\psi}I_{a+}^{\sigma;\psi}\phi(t)=I_{a+}^{\lambda+\sigma;\psi}\phi(t), t\in(a,b]$.
(ii) If $\phi\in C[a,b]$, *then* $I_{a^+}^{\lambda;\psi}I_{a^+}^{\sigma;\psi}\phi(t)=I_{a^+}^{\lambda+\sigma;\psi}\phi(t), t\in[a,b]$.

Lemma 2.5. *[29] Let* $\lambda>0$, $0\leq\sigma\leq 1$ *and* $0\leq\varsigma<1$. *If* $h\in C_{\varsigma;\psi}[a,b]$ *then*

$$D_{a^+}^{\lambda,\sigma;\psi}I_{a^+}^{\lambda;\psi}\phi(t)=\phi(t),\ \ t\in(a,b].$$

If $\phi\in C^1[a,b]$ *then*

$$D_{a+}^{\lambda,\sigma;\psi}I_{a+}^{\alpha;\psi}\phi(t)=\phi(t),\ \ t\in[a,b].$$

Lemma 2.6. *Let* $\lambda>0, 0\leq\varsigma<1$ *and* $\phi\in C_{\varsigma;\psi}[a,b]$. *If* $\lambda>\varsigma$, *then* $I_{a^+}^{\lambda;\psi}\phi\in C[a,b]$ *and*

$$I_{a^+}^{\lambda;\psi}\phi(a)=\lim_{t\to a^+}I_{a^+}^{\lambda;\psi}\phi(t)=0.$$

Lemma 2.7. *[29] Let* $\phi\in C^n[a,b], n-1<\lambda<n, 0\leq\sigma\leq 1$, *and* $\varsigma=\lambda+\sigma-\lambda\sigma$. *Then for all* $t\in(a,b]$

$$I_{a^+}^{\lambda;\psi}D_{a^+}^{\lambda,\sigma;\psi}\phi(t)=\phi(t)-\sum_{k=1}^n\frac{[\psi(t)-\psi(a)]^{\varsigma-k}}{\Gamma(\varsigma-k+1)}\phi_\psi^{(n-k)}I_{a^+}^{(1-\sigma)(n-\lambda);\psi}\phi(a).$$

In particular, if $0<\lambda<1$, *we have*

$$I_{a^+}^{\lambda;\psi}D_{a^+}^{\lambda,\sigma;\psi}\phi(t)=\phi(t)-\frac{[\psi(t)-\psi(a)]^{\varsigma-1}}{\Gamma(\varsigma)}I_{a^+}^{(1-\sigma)(1-\lambda);\psi}\phi(a).$$

Moreover, if $\phi \in C_{1-\varsigma;\psi}[a,b]$ *and* $I_{a^+}^{1-\varsigma;\psi}\phi \in C^1_{1-\varsigma;\psi}[a,b]$ *such that* $0 < \varsigma < 1$. *Then for all* $t \in (a,b]$

$$I_{a^+}^{\varsigma;\psi} D_{a^+}^{\varsigma;\psi}\phi(t) = \phi(t) - \frac{[\psi(t)-\psi(a)]^{\gamma-1}}{\Gamma(\varsigma)} I_{a^+}^{1-\varsigma;\psi}\phi(a).$$

We deduce from the above lemma the following lemmas:

Lemma 2.8. *[18] Let* $v \in C(\Upsilon \times \mathbb{R}, \mathbb{R}^*)$; $\Upsilon := [0,d]$, $d > 0$, $\kappa \in C_{1-\zeta,\psi}(\Upsilon)$. *Then the problem*

$$\begin{cases} D_{0^+}^{\lambda,\sigma;\psi} \frac{u(t)}{v(t,u(t))} = \kappa(t), a.e. \quad t \in \Upsilon. \\ (\psi(t)-\psi(0))^{1-\varsigma} u(t) \mid_{t=0} = u_0, \qquad u_0 \in \mathbb{R}, \end{cases}$$

has a unique solution given by

$$u(t) = v(t,u(t)) \left\{ \frac{u_0}{v(0,u(0))} (\psi(t)-\psi(0))^{\varsigma-1} + I_{0^+}^{\lambda;\psi}\kappa(t) \right\}.$$

Lemma 2.9. *Let* $v \in C(\Upsilon \times \mathbb{R}, \mathbb{R}^*)$, $w : \Upsilon \times \mathbb{R} \to \mathbb{R}$ *be a function such that* $w(\cdot, u(\cdot)) \in BC_\varsigma$ *for any* $u \in BC_\varsigma$. *Then the problem (1.1) is equivalent to the problem of obtaining the solutions of the integral equation*

$$u(t) = v(t,u(t)) \left\{ \frac{u_0}{v(0,u(0))} (\psi(t)-\psi(0))^{\varsigma-1} + I_{0^+}^{\lambda;\psi} w(\cdot,u(\cdot))(t) \right\}.$$

Let $\varnothing \neq \Lambda \subset BC$ and let $K : \Lambda \to \Lambda$. We consider the solution of the equation

$$(Ku)(t) = u(t). \tag{2.1}$$

We introduce the concept of attractivity of solutions for equation (2.1).

Definition 2.10. *Solutions of equation (2.1) are locally attractive if there exists a ball* $B(u_0,\mu)$ *in the space* BC *such that, for any solutions* $\tau = \tau(t)$ *and* $\xi = \xi(t)$ *of equations (2.1) that belong to* $B(u_0,\mu) \cap \Lambda$, *we can write*

$$\lim_{t\to\infty} (\tau(t) - \xi(t)) = 0. \tag{2.2}$$

If the limit (2.2) is uniform with respect to $B(u_0,\mu) \cap \Lambda$, *then the solutions of equation (2.1) are said to be uniformly locally attractive (or, equivalently, that the solutions of (2.1) are locally asymptotically stable).*

Lemma 2.11. *[11] Let* $M \subset BC$. *Then* M *is relatively compact in* BC *if the following conditions are satisfied:*

(a) M *is uniformly bounded in* BC*;*

(b) the functions belonging to M *are almost equicontinuous in* $\mathbb{R}_+$, *i.e., equicontinuous on every compact set in* $\mathbb{R}_+$*;*

(c) *the functions from M are equiconvergent, i.e., given $\varepsilon > 0$, there exists $L(\varepsilon) > 0$ such that*

$$\left|u(t) - \lim_{t\to\infty} u(t)\right| < \varepsilon,$$

for any $t \geq L(\varepsilon)$ and $u \in M$.

Theorem 2.12. *(Schauder Fixed-Point Theorem [15]). Let F be a Banach space, let U be a nonempty bounded convex and closed subset of F, and let $K : U \to U$ be a compact and continuous map. Then K has at least one fixed point in U.*

3 Existence and Attractivity Results

Definition 3.1. *A measurable function $u \in BC_\varsigma$ is a solution of problem (1.1) if it verifies the initial condition $(\psi(t)-\psi(0))^{1-\varsigma}u(t)\mid_{t=0}= u_0$ and the equation $D^{\lambda,\sigma;\psi}_{0^+}\frac{u(t)}{v(t,u(t))} = w\,(t, u(t))$ on $\mathbb{R}_+$.*

We will give the following hypotheses:

(H_1) The function $t \mapsto w(t, u)$ is measurable on $\mathbb{R}_+$ for each $u \in BC_\varsigma$, the function $u \mapsto w(t, u)$ is continuous on BC_ς for a.e. $t \in \mathbb{R}_+$, and the function v is bounded such that $u \mapsto v(t, u)$ is continuous.

(H_2) There exists a continuous function $T : \mathbb{R}_+ \to \mathbb{R}_+$ such that for a.e. $t \in \mathbb{R}_+$ and each $u \in \mathbb{R}$,

$$|w(t, u)| \leq \frac{T(t)}{1 + |u|},$$

and

$$\lim_{t\to\infty} (\psi(t) - \psi(0))^{1-\varsigma} \left(I^{\lambda;\psi}_{0^+} T\right)(t) = 0.$$

Set

$$T^* = \sup_{t\in\mathbb{R}_+} (\psi(t) - \psi(0))^{1-\varsigma} \left(I^{\lambda;\psi}_{0^+} T\right)(t) < \infty.$$

Now we present a theorem on the existence and attractivity of solutions of the problem (1.1).

Theorem 3.2. *Assume that the hypotheses (H_1) and (H_2) hold. Then the problem (1.1) has at least one solution defined on $\mathbb{R}_+$ and the solutions of problem (1.1) are uniformly locally attractive.*

Proof. Consider the operator K such that, for any $u \in BC_\varsigma$,

$$(Ku)(t) = v(t, u(t))\left\{\frac{u_0}{v(0, u(0))}(\psi(t) - \psi(0))^{\varsigma-1} + \frac{1}{\Gamma(\lambda)}\int_0^t \psi'(s)(\psi(t) - \psi(s))^{\lambda-1}w(s, u(s))ds\right\}.$$

Let L be a bound of the function v. For any $u \in BC_\varsigma$, and for each $t \in \mathbb{R}_+$, we have

$$\begin{aligned}
&\left|(\psi(t)-\psi(0))^{1-\varsigma}(Ku)(t)\right| \\
\leq & |v(t,u(t))| \left\{ \left| \frac{u_0}{v(0,u(0))} \right| + \frac{(\psi(t)-\psi(0))^{1-\varsigma}}{\Gamma(\lambda)} \int_0^t \psi'(s)(\psi(t)-\psi(s))^{\lambda-1} |w(s,u(s))| ds \right\} \\
\leq & |v(t,u(t))| \left\{ \left| \frac{u_0}{v(0,u(0))} \right| + \frac{(\psi(t)-\psi(0))^{1-\varsigma}}{\Gamma(\lambda)} \int_0^t \psi'(s)(\psi(t)-\psi(s))^{\lambda-1} T(s) ds \right\} \\
\leq & L \left\{ \left| \frac{u_0}{v(0,u(0))} \right| + T^* \right\} \\
:= & R_*.
\end{aligned}$$

So

$$|K(u)\|_{BC} \leq R_*. \tag{3.1}$$

Therefore, $K(u) \in BC_\varsigma$. Since, the map $K(u)$ is continuous on $\mathbb{R}_+$; for any $u \in BC_\varsigma$, and $K(BC_\varsigma) \subset BC_\varsigma$, then the operator K maps BC_ς into itself. Furthermore, equation (3.1) implies that the operator K transforms the ball

$$B_{R_*} := B(0, R_*) = \{v \in BC_\varsigma : \|v\|_{BC_\varsigma} \leq R_*\}$$

into itself. From Lemma 2.9 the solution of problem (1.1) is reduced to finding the solutions of the operator equation $K(u) = u$. We show that the operator $K : BC_\varsigma \to BC_\varsigma$ satisfies all assumptions of Theorem 2.12. The proof is divided into several steps:

Step 1. K is continuous.

Let $\{u_n\}_{n\in N}$ be a sequence such that $u_n \to u$ in B_{R_*}.

Then, for each $t \in \mathbb{R}_+$, we have

$$\begin{aligned}
&\left|((\psi(t)-\psi(0))^{1-\varsigma}(Ku_n)(t) - ((\psi(t)-\psi(0))^{1-\varsigma}(Ku)(t)\right| \\
\leq & \Bigg| v(t,u_n(t)) \left\{ \frac{u_0}{v(0,u(0))} + \frac{(\psi(t)-\psi(0))^{1-\varsigma}}{\Gamma(\lambda)} \int_0^t \psi'(s)(\psi(t)-\psi(s))^{\lambda-1} w(s,u_n(s)) ds \right\} \\
& - v(t,u(t)) \left\{ \frac{u_0}{v(0,u(0))} + \frac{(\psi(t)-\psi(0))^{1-\varsigma}}{\Gamma(\lambda)} \int_0^t \psi'(s)(\psi(t)-\psi(s))^{\lambda-1} w(s,u(s)) ds \right\} \Bigg| \\
\leq & \Bigg| v(t,u_n(t)) \left\{ \frac{u_0}{v(0,u(0))} + \frac{(\psi(t)-\psi(0))^{1-\varsigma}}{\Gamma(\lambda)} \int_0^t \psi'(s)(\psi(t)-\psi(s))^{\lambda-1} w(s,u_n(s)) ds \right\} \\
& - v(t,u(t)) \left\{ \frac{u_0}{v(0,u(0))} + \frac{(\psi(t)-\psi(0))^{1-\varsigma}}{\Gamma(\lambda)} \int_0^t \psi'(s)(\psi(t)-\psi(s))^{\lambda-1} w(s,u_n(s)) ds \right\} \\
& + v(t,u(t)) \left\{ \frac{u_0}{v(0,u(0))} + \frac{(\psi(t)-\psi(0))^{1-\varsigma}}{\Gamma(\lambda)} \int_0^t \psi'(s)(\psi(t)-\psi(s))^{\lambda-1} w(s,u_n(s)) ds \right\} \\
& - v(t,u(t)) \left\{ \frac{u_0}{v(0,u(0))} + \frac{(\psi(t)-\psi(0))^{1-\varsigma}}{\Gamma(\lambda)} \int_0^t \psi'(s)(\psi(t)-\psi(s))^{\lambda-1} w(s,u(s)) ds \right\} \Bigg|
\end{aligned}$$

$$
\begin{aligned}
&\leq \Big|v(t,u_n(t))-v(t,u(t))\Big|\Big|\frac{u_0}{v(0,u(0))}+\frac{(\psi(t)-\psi(0))^{1-\varsigma}}{\Gamma(\lambda)}\int_0^t\psi'(s)(\psi(t)-\psi(s))^{\lambda-1}\\
&\times w(s,u_n(s))ds\Big|+|v(t,u(t))|\frac{(\psi(t)-\psi(0))^{1-\varsigma}}{\Gamma(\lambda)}\\
&\times\int_0^t\psi'(s)(\psi(t)-\psi(s))^{\lambda-1}|w(s,u_n(s))-w(s,u(s))|ds.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\left|(\psi(t)-\psi(0))^{1-\varsigma}\left(Ku_n\right)(t)-(\psi(t)-\psi(0))^{1-\varsigma}(Ku)(t)\right|\\
&\leq\Big|v(t,u_n(t))-v(t,u(t))\Big|\Bigg\{\Big|\frac{u_0}{v(0,u(0))}\Big|\\
&+\frac{(\psi(t)-\psi(0))^{1-\varsigma}}{\Gamma(\lambda)}\int_0^t\psi'(s)(\psi(t)-\psi(s))^{\lambda-1}|w(s,u_n(s))|ds\Bigg\}\\
&+L\frac{(\psi(t)-\psi(0))^{1-\varsigma}}{\Gamma(\lambda)}\int_0^t\psi'(s)(\psi(t)-\psi(s))^{\lambda-1}|w(s,u_n(s))-w(s,u(s))|ds.
\end{aligned}\tag{3.2}
$$

Case 1. If $t\in[0,d]$, then, in view of the facts that $u_n\to u$ as $n\to\infty$, v and w are continuous, by the Lebesgue dominated convergence theorem, from the equation (3.2), we have

$$\|K\left(u_n\right)-K\left(u\right)\|_{BC_\varsigma}\to 0\quad\text{ as }\quad n\to\infty.$$

Case 2. If $t\in(d,\infty)$, then, from the hypotheses and (3.2), we have

$$
\begin{aligned}
&\left|(\psi(t)-\psi(0))^{1-\varsigma}\left(Ku_n\right)(t)-(\psi(t)-\psi(0))^{1-\varsigma}(Ku)(t)\right|\\
&\leq\Big|v(t,u_n(t))-v(t,u(t))\Big|\Bigg\{\Big|\frac{u_0}{v(0,u(0))}\Big|\\
&+\frac{(\psi(t)-\psi(0))^{1-\varsigma}}{\Gamma(\lambda)}\int_0^t\psi'(s)(\psi(t)-\psi(s))^{\lambda-1}T(s)ds\Bigg\}\\
&+2L\frac{(\psi(t)-\psi(0))^{1-\varsigma}}{\Gamma(\lambda)}\int_0^t\psi'(s)(\psi(t)-\psi(s))^{\lambda-1}T(s)ds.
\end{aligned}
$$

Then

$$
\begin{aligned}
&\left|(\psi(t)-\psi(0))^{1-\varsigma}\left(Ku_n\right)(t)-(\psi(t)-\psi(0))^{1-\varsigma}(Ku)(t)\right|\\
&\leq\Big|v(t,u_n(t))-v(t,u(t))\Big|\Bigg\{\Big|\frac{u_0}{v(0,u(0))}\Big|+((\psi(t)-\psi(0))^{1-\varsigma}\left(I_{0^+}^{\lambda;\psi}T\right)(t)\Bigg\}\\
&+2L((\psi(t)-\psi(0))^{1-\varsigma}\left(I_{0^+}^{\lambda;\psi}T\right)(t).
\end{aligned}\tag{3.3}
$$

Since $u_n\to u$ as $n\to\infty$, v is continuous and $(\psi(t)-\psi(0))^{1-\varsigma}\left(I_{0^+}^{\lambda;\psi}T\right)(t)\to 0$ as $t\to\infty$, it follows from (3.3) that

$$\|K\left(u_n\right)-K(u)\|_{BC_\varsigma}\to 0\quad\text{ as }\quad n\to\infty.$$

Step 2. $L\left(B_{R_*}\right)$ is uniformly bounded, and equicontinuous on every compact subset $[0,d]$ of $\mathbb{R}_+$, $d>0$.

We have $L(B_{R_*}) \subset B_{R_*}$ and B_{R_*} is bounded, so $L(B_{R_*})$ is uniformly bounded.

Next, for each $t_1, t_2 \in [0,d], t_1 < t_2$, and $u \in B_{R_*}$, we have

$$
\begin{aligned}
&\left|(\psi(t_2)-\psi(0))^{1-\varsigma}(Ku)(t_2)-(\psi(t_1)-\psi(0))^{1-\varsigma}(Ku)(t_1)\right| \\
&\le \Bigg| v(t_2,u(t_2))\bigg\{\frac{u_0}{v(0,u(0))}+\frac{(\psi(t_2)-\psi(0))^{1-\varsigma}}{\Gamma(\lambda)}\int_0^{t_2}\psi'(s)(\psi(t_2)-\psi(s))^{\lambda-1}w(s,u(s))ds\bigg\} \\
&-v(t_1,u(t_1))\bigg\{\frac{u_0}{v(0,u(0))}+\frac{(\psi(t_1)-\psi(0))^{1-\varsigma}}{\Gamma(\lambda)}\int_0^{t_1}\psi'(s)(\psi(t_1)-\psi(s))^{\lambda-1}w(s,u(s))ds\bigg\}\Bigg| \\
&\le \Bigg| v(t_2,u(t_2))\bigg\{\frac{u_0}{v(0,u(0))}+\frac{(\psi(t_2)-\psi(0))^{1-\varsigma}}{\Gamma(\lambda)}\int_0^{t_2}\psi'(s)(\psi(t_2)-\psi(s))^{\lambda-1}w(s,u(s))ds\bigg\} \\
&-v(t_1,u(t_1))\bigg\{\frac{u_0}{v(0,u(0))}+\frac{(\psi(t_2)-\psi(0))^{1-\varsigma}}{\Gamma(\lambda)}\int_0^{t_2}\psi'(s)(\psi(t_2)-\psi(s))^{\lambda-1}w(s,u(s))ds\bigg\} \\
&+v(t_1,u(t_1))\bigg\{\frac{u_0}{v(0,u(0))}+\frac{(\psi(t_2)-\psi(0))^{1-\varsigma}}{\Gamma(\lambda)}\int_0^{t_2}\psi'(s)(\psi(t_2)-\psi(s))^{\lambda-1}w(s,u(s))ds\bigg\} \\
&-v(t_1,u(t_1))\bigg\{\frac{u_0}{v(0,u(0))}+\frac{(\psi(t_1)-\psi(0))^{1-\varsigma}}{\Gamma(\lambda)}\int_0^{t_1}\psi'(s)(\psi(t_1)-\psi(s))^{\lambda-1}w(s,u(s))ds\bigg\}\Bigg|.
\end{aligned}
$$

Thus

$$
\begin{aligned}
&\left|(\psi(t_2)-\psi(0))^{1-\varsigma}(Ku)(t_2)-(\psi(t_1)-\psi(0))^{1-\varsigma}(Ku)(t_1)\right| \\
&\le |v(t_2,u(t_2))-v(t_1,u(t_1))|\Bigg|\frac{u_0}{v(0,u(0))} \\
&+\frac{(\psi(t_2)-\psi(0))^{1-\varsigma}}{\Gamma(\lambda)}\int_0^{t_2}\psi'(s)(\psi(t_2)-\psi(s))^{\lambda-1}w(s,u(s))ds\Bigg| \\
&+|v(t_1,u(t_1))|\Bigg|\frac{(\psi(t_2)-\psi(0))^{1-\varsigma}}{\Gamma(\lambda)}\int_0^{t_1}\psi'(s)(\psi(t_2)-\psi(s))^{\lambda-1}w(s,u(s))ds \\
&+\frac{(\psi(t_2)-\psi(0))^{1-\varsigma}}{\Gamma(\lambda)}\int_{t_1}^{t_2}\psi'(s)(\psi(t_2)-\psi(s))^{\lambda-1}w(s,u(s))ds \\
&-\frac{(\psi(t_1)-\psi(0))^{1-\varsigma}}{\Gamma(\lambda)}\int_0^{t_1}\psi'(s)(\psi(t_1)-\psi(s))^{\lambda-1}w(s,u(s))ds\Bigg|.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\left|(\psi(t_2)-\psi(0))^{1-\varsigma}(Ku)(t_2)-(\psi(t_1)-\psi(0))^{1-\varsigma}(Ku)(t_1)\right| \\
&\le |v(t_2,u(t_2))-v(t_1,u(t_1))|\Bigg(\left|\frac{u_0}{v(0,u(0))}\right| \\
&+\frac{(\psi(t_2)-\psi(0))^{1-\varsigma}}{\Gamma(\lambda)}\int_0^{t_2}\psi'(s)(\psi(t_2)-\psi(s))^{\lambda-1}|w(s,u(s))|ds\Bigg) \\
&+L\Bigg(\int_0^{t_1}\Bigg|\frac{(\psi(t_2)-\psi(0))^{1-\varsigma}}{\Gamma(\lambda)}\psi'(s)(\psi(t_2)-\psi(s))^{\lambda-1} \\
&-\frac{(\psi(t_1)-\psi(0))^{1-\varsigma}}{\Gamma(\lambda)}\psi'(s)(\psi(t_1)-\psi(s))^{\lambda-1}\Bigg| \\
&|w(s,u(s))|ds+\frac{(\psi(t_2)-\psi(0))^{1-\varsigma}}{\Gamma(\lambda)}\int_{t_1}^{t_2}\psi'(s)(\psi(t_2)-\psi(s))^{\lambda-1}|w(s,u(s))|ds\Bigg)
\end{aligned}
$$

$$\begin{aligned}
&\leq |v(t_2,u(t_2))-v(t_1,u(t_1))|\left(\left|\frac{u_0}{v(0,u(0))}\right|\right.\\
&+\left.\frac{(\psi(t_2)-\psi(0))^{1-\varsigma}}{\Gamma(\lambda)}\int_0^{t_2}\psi'(s)(\psi(t_2)-\psi(s))^{\lambda-1}T(s)ds\right)\\
&+L\Bigg(\int_0^{t_1}\left|\frac{(\psi(t_2)-\psi(0))^{1-\varsigma}}{\Gamma(\lambda)}\psi'(s)(\psi(t_2)-\psi(s))^{\lambda-1}\right.\\
&-\left.\frac{(\psi(t_1)-\psi(0))^{1-\varsigma}}{\Gamma(\lambda)}\psi'(s)(\psi(t_1)-\psi(s))^{\lambda-1}\right|\\
&T(s)ds+\frac{(\psi(t_2)-\psi(0))^{1-\varsigma}}{\Gamma(\lambda)}\int_{t_1}^{t_2}\psi'(s)(\psi(t_2)-\psi(s))^{\lambda-1}T(s)ds\Bigg).
\end{aligned}$$

From the continuity of the functions T and v, by setting $T_*=\sup_{t\in[0,d]}T(t)$, we obtain

$$\begin{aligned}
&\left|(\psi(t_2)-\psi(0))^{1-\varsigma}\left(Ku\right)(t_2)-(\psi(t_1)-\psi(0))^{1-\varsigma}(Ku)(t_1)\right|\\
&\leq |v(t_2,u(t_2))-v(t_1,u(t_1))|\left(\left|\frac{u_0}{v(0,u(0))}\right|+\frac{T_*(\psi(t_2)-\psi(0))^{1-\varsigma}}{\Gamma(\lambda)}\int_0^{t_2}\psi'(s)(\psi(t_2)-\psi(s))^{\lambda-1}ds\right)\\
&+LT_*\Bigg(\int_0^{t_1}\left|\frac{(\psi(t_2)-\psi(0))^{1-\varsigma}}{\Gamma(\lambda)}\psi'(s)(\psi(t_2)-\psi(s))^{\lambda-1}\right.\\
&-\left.\frac{(\psi(t_1)-\psi(0))^{1-\varsigma}}{\Gamma(\lambda)}\psi'(s)(\psi(t_1)-\psi(s))^{\lambda-1}\right|ds\\
&+\frac{(\psi(t_2)-\psi(0))^{1-\varsigma}}{\Gamma(\lambda)}\int_{t_1}^{t_2}\psi'(s)(\psi(t_2)-\psi(s))^{\lambda-1}ds\Bigg)\\
&\leq |v(t_2,u(t_2))-v(t_1,u(t_1))|\left(\left|\frac{u_0}{v(0,u(0))}\right|+\frac{T_*(\psi(t_2)-\psi(0))^{1-\varsigma+\lambda}}{\Gamma(\lambda+1)}\right)\\
&+LT_*\Bigg(\int_0^{t_1}\left|\frac{(\psi(t_2)-\psi(0))^{1-\varsigma}}{\Gamma(\lambda)}\psi'(s)(\psi(t_2)-\psi(s))^{\lambda-1}\right.\\
&-\left.\frac{(\psi(t_1)-\psi(0))^{1-\varsigma}}{\Gamma(\lambda)}\psi'(s)(\psi(t_1)-\psi(s))^{\lambda-1}\right|ds+\frac{(\psi(t_2)-\psi(0))^{1-\varsigma}}{\Gamma(\lambda+1)}(\psi(t_2)-\psi(t_1))^{\lambda}\Bigg).
\end{aligned}$$

As $t_1\to t_2$, the right-hand side of the inequality tends to zero.

Step 3. $L\left(B_R\right)$ is equiconvergent.
Let $u\in B_{R*}$. Then, for each $t\in\mathbb{R}_+$ we have

$$\begin{aligned}
&\left|(\psi(t)-\psi(0))^{1-\varsigma}(Ku)(t)\right|\leq|v(t,u(t))|\Bigg\{\left|\frac{u_0}{v(0,u(0))}\right|\\
&+\left|\frac{(\psi(t)-\psi(0))^{1-\varsigma}}{\Gamma(\lambda)}\int_0^t\psi'(s)(\psi(t)-\psi(s))^{\lambda-1}w(s,u(s))ds\right|\Bigg\}\\
&\leq|v(t,u(t))|\left\{\left|\frac{u_0}{v(0,u(0))}\right|+\left|\frac{(\psi(t)-\psi(0))^{1-\varsigma}}{\Gamma(\lambda)}\int_0^t\psi'(s)(\psi(t)-\psi(s))^{\lambda-1}T(s)ds\right|\right\}\\
&\leq L\left\{\left|\frac{u_0}{v(0,u(0))}\right|+(\psi(t)-\psi(0))^{1-\varsigma}\left(I_{0^+}^{\lambda;\psi}T\right)(t)\right\}.
\end{aligned}$$

Since

$$(\psi(t)-\psi(0))^{1-\varsigma}\left(I_{0^+}^{\lambda;\psi}T\right)(t)\to 0\, as\;\; t\to+\infty,$$

we find

$$|(Ku)(t)| \leq L\left\{\left|\frac{u_0}{(\psi(t)-\psi(0))^{1-\varsigma}v(0,u(0))}\right| + \frac{(\psi(t)-\psi(0))^{1-\varsigma}\left(I_{0+}^{\lambda;\psi}T\right)(t)}{(\psi(t)-\psi(0))^{1-\varsigma}}\right\}.$$

Hence,

$$|(Lu)(t)-(Lu)(+\infty)| \to 0 \quad \text{as} \quad t \to +\infty,$$

in view of Lemma 2.11 as a consequence of Steps $1-4$, we conclude that $K : B_{R_*} \to B_{R_*}$ is compact and continuous. Applying the Theorem 2.12, we have that K has a fixed point u, which is a solution of problem (1.1) on $\mathbb{R}_+$.

Step 4. The uniform local attractivity of solutions.
We assume that u_* is a solution of problem (1.1) under the conditions of this theorem.
Set $u \in B\left(u_*, 2L\left\{\left|\frac{u_0}{v(0,u(0))}\right| + 2T^*\right\}\right)$, we have

$$\begin{aligned}
&\left|(\psi(t)-\psi(0))^{1-\varsigma}(Ku)(t) - (\psi(t)-\psi(0))^{1-\varsigma}(u_*)(t)\right| \\
&\leq \left|(\psi(t)-\psi(0))^{1-\varsigma}(Ku)(t) - (\psi(t)-\psi(0))^{1-\varsigma}(Ku_*)(t)\right| \\
&\leq |v(t,u(t)) - v(t,u_*(t))|\left\{\left|\frac{u_0}{v(0,u(0))}\right|\right. \\
&\left. + \frac{(\psi(t)-\psi(0))^{1-\varsigma}}{\Gamma(\lambda)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\lambda-1}|w(s,u(s))|ds\right\} \\
&+ L\frac{(\psi(t)-\psi(0))^{1-\varsigma}}{\Gamma(\lambda)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\lambda-1}|w(s,u(s))-w(s,u_*(s))|ds \\
&\leq 2L\left\{\left|\frac{u_0}{v(0,u(0))}\right| + \frac{(\psi(t)-\psi(0))^{1-\varsigma}}{\Gamma(\lambda)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\lambda-1}T(s)ds\right\} \\
&+ 2L\frac{(\psi(t)-\psi(0))^{1-\varsigma}}{\Gamma(\lambda)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\lambda-1}T(s)ds \\
&\leq 2L\left\{\left|\frac{u_0}{v(0,u(0))}\right| + 2T^*\right\}.
\end{aligned}$$

Thus, we get

$$\|K(u)-u_*\|_{BC_\varsigma} \leq 2L\left\{\left|\frac{u_0}{v(0,u(0))}\right| + 2T^*\right\}.$$

So, we conclude that K is a continuous function such that

$$K\left(B\left(u_*, 2L\left\{\left|\frac{u_0}{v(0,u(0))}\right| + 2T^*\right\}\right)\right) \subset B\left(u_*, 2L\left\{\left|\frac{u_0}{v(0,u(0))}\right| + 2T^*\right\}\right).$$

Moreover, if u is a solution of problem (1.1), then

$$\begin{aligned}|u(t)-u_*(t)| &= |(Ku)(t)-(Ku_*)(t)|\\
&\leq |v(t,u(t))-v(t,u_*(t))|\Big\{(\psi(t)-\psi(0))^{\varsigma-1}\Big|\frac{u_0}{v(0,u(0))}\Big|\\
&+\frac{1}{\Gamma(\lambda)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\lambda-1}|w(s,u(s))|ds\Big\}\\
&+\frac{L}{\Gamma(\lambda)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\lambda-1}|w(s,u(s))-w(s,u_*(s))|ds\\
&\leq 2L\Big\{(\psi(t)-\psi(0))^{\varsigma-1}\Big|\frac{u_0}{v(0,u(0))}\Big|\\
&+\frac{1}{\Gamma(\lambda)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\lambda-1}|w(s,u(s))|ds\Big\}\\
&+\frac{L}{\Gamma(\lambda)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\lambda-1}|w(s,u(s))-w(s,u_*(s))|ds\\
&\leq 2L\Big\{(\psi(t)-\psi(0))^{\varsigma-1}\Big|\frac{u_0}{v(0,u(0))}\Big|+2(I_{0^+}^{\lambda;\psi}T)(t)\Big\}.\end{aligned}$$

Therefore,

$$|u(t)-u_*(t)|\leq 2L\Big\{(\psi(t)-\psi(0))^{\varsigma-1}\Big|\frac{u_0}{v(0,u(0))}\Big|+2\frac{(\psi(t)-\psi(0))^{1-\varsigma}(I_{0^+}^{\lambda;\psi}T)(t)}{(\psi(t)-\psi(0))^{1-\varsigma}}\Big\}. \tag{3.4}$$

By using (3.4) and the fact that

$$\lim_{t\to\infty}(\psi(t)-\psi(0))^{1-\varsigma}(I_{0^+}^{\lambda;\psi}T)(t)=0,$$

we conclude

$$\lim_{t\to\infty}|u(t)-u_*(t)|=0.$$

Consequently, all solutions of problem (1.1) are uniformly locally attractive. □

4 An Example

As an application of our results, we consider the following problem for a ψ-Hilfer fractional differential equation

$$\begin{cases}D_{0^+}^{\frac{1}{2},\frac{1}{2};\psi}\dfrac{u(t)}{v(t,u(t))}=w\left(t,u(t)\right), a.e. \quad t\in\mathbb{R}_+,\\ (\psi(t)-\psi(0))^{\frac{1}{4}}u(t)\mid_{t=0}=1,\end{cases} \tag{4.1}$$

where $\psi:[0,1]\to\mathbb{R}$ with $\psi(t)=\sqrt{t+3}$,

$$v(t,u)=\frac{1}{(1+t)(1+|u|)},$$

$$\begin{cases}w(t,u)=\dfrac{\beta(\psi(t)-\psi(0))^{\frac{-1}{4}}\sin t}{64(1+\sqrt{t})(1+|u|)}, \quad t\in(0,\infty), \quad u\in\mathbb{R},\\ w(0,u)=0, \ u\in\mathbb{R},\end{cases}$$

and

$$\beta = \frac{9\sqrt{\pi}}{16}.$$

Clearly, the function w is continuous. The hypothesis (H_2) is satisfied with

$$\begin{cases} T(t) = \dfrac{\beta(\psi(t)-\psi(0))^{\frac{-1}{4}}|\sin t|}{64(1+\sqrt{t})}, & t \in (0,\infty), \\ T(0) = 0. \end{cases}$$

In addition, we have

$$\begin{aligned}(\psi(t)-\psi(0))^{\frac{1}{4}}\left(I_{0^+}^{\frac{1}{2};\psi}T\right)(t) &= \frac{(\psi(t)-\psi(0))^{\frac{1}{4}}}{\Gamma\left(\frac{1}{2}\right)}\int_0^t \psi'(\tau)(\psi(t)-\psi(\tau))^{\frac{-1}{2}}T(\tau)d\tau \\ &\leq \frac{1}{4}(\psi(t)-\psi(0))^{\frac{-1}{4}} \to 0 \quad \text{as} \quad t \to \infty.\end{aligned}$$

Simple computations show that all conditions of Theorem 3.2 are satisfied. Consequently, our problem (4.1) has at least one solution defined on $\mathbb{R}_+$, and all solutions of this problem are uniformly locally attractive.

5 Conclusion

In this paper, we provided some sufficient conditions ensuring the existence and the uniform locally attractivity of solutions of some ψ-Hilfer fractional differential equations. The technique used is based on Schauder's fixed point theory theorem.

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CUBO, A Mathematical Journal
Vol. 23, no. 01, pp. 161–170, April 2021
DOI: 10.4067/S0719-06462021000100161

Idempotents in an ultrametric Banach algebra

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ABSTRACT

Let $\mathbb{K}$ be a complete ultrametric field and let A be a unital commutative ultrametric Banach $\mathbb{K}$-algebra. Suppose that the multiplicative spectrum admits a partition in two open closed subsets. Then there exist unique idempotents $u,\ v \in A$ such that $\phi(u) = 1,\ \phi(v) = 0\ \forall \phi \in U,\ \phi(u) = 0\ \phi(v) = 1\ \forall \phi \in V$. Suppose that $\mathbb{K}$ is algebraically closed. If an element $x \in A$ has an empty annulus $r < |\xi - a| < s$ in its spectrum $sp(x)$, then there exist unique idempotents $u,\ v$ such that $\phi(u) = 1,\ \phi(v) = 0$ whenever $\phi(x - a) \leq r$ and $\phi(u) = 0,\ \phi(v) = 1$ whenever $\phi(x - a) \geq s$.

RESUMEN

Sea $\mathbb{K}$ un cuerpo ultramétrico completo y sea A una $\mathbb{K}$-algebra de Banach ultramétrica unital conmutativa. Suponga que el espectro multiplicativo admite una partición en dos conjuntos abiertos y cerrados. Luego, existen idempotentes únicos $u,\ v \in A$ tales que $\phi(u) = 1,\ \phi(v) = 0\ \forall \phi \in U,\ \phi(u) = 0\ \phi(v) = 1\ \forall \phi \in V$. Suponga que $\mathbb{K}$ es algebraicamente cerrado. Si un elemento $x \in A$ tiene un anillo vacío $r < |\xi - a| < s$ en su espectro $sp(x)$, entonces existen idempotentes únicos $u,\ v$ tales que $\phi(u) = 1,\ \phi(v) = 0$ cada vez que $\phi(x - a) \leq r$ y $\phi(u) = 0,\ \phi(v) = 1$ cada vez que $\phi(x - a) \geq s$.

Keywords and Phrases: ultrametric Banach algebras, multiplicative semi-norms, idempotents, affinoid algebras.

2020 AMS Mathematics Subject Classification: 12J25, 30D35, 30G06.

Accepted: 17 March, 2021
Received: 05 November, 2020

1 Introduction and main Theorem

Ultrametric Banach algebras have been a topic of many resarch along the last years [1], [3], [4], [5],[6], [10], [11], [12]. The following Theorem 1.1 (stated in [14]) corresponds in ultrametric Banach algebras to a well known theorem in complex Banach algebra: if the spectrum of maximal ideals admits a partition in two open closed subsets U and V with respect to the Gelfand topology, there exist idempotents u and v such that $\chi(u) = 1,\ \chi(v) = 0\ \forall \chi \in U$ and $\chi(u) = 0,\ \chi(v) = 1\ \forall \chi \in V$.

In an ultrametric Banach algebra, it is impossible to have a similar result because a partition in two open closed subsets for the Gelfand topology on the spectrum of maximal ideals then makes no sense, due to the total disconnection of the spectrum. B. Guennebaud first had the idea to consider the set of continuous multiplicative semi-norms of an ultrametric Banach algebra, denoted by $Mult(A, \|\ .\ \|)$ instead of the spectrum of maximal ideals [14], an idea that later suggested Berkovich theory [2]. Recall that $Mult(A, \|\ .\ \|)$ is compact with respect to the topology of pointwise convergence (Theorem 1.11 in [7]).

The proof of Theorem 1.1, stated in [14], was heavy and involved many particular notions in a chapter of over 40 pages that was never published. We will use Propositions 2.10, 2.11, 2.12 in order to assure the unicity. Finally, we will show that if the theorem is proven for affinoid algebras, that may be generalised to all ultrametric Banach algebras (Proposition 2.12).

Notations: We denote by $\mathbb{K}$ a complete ultrametric field. Given a $\mathbb{K}$-algebra A, we denote by $Mult(A)$ the set of multplicative semi-norms of A and if A is a normed $\mathbb{K}$-algebra, we denote by $Mult(A, \|\ .\ \|)$ the set of continuous multplicative semi-norms of A provided with the topology of pointwise convergence. Next, we denote by $Mult_m(A, \|\ .\ \|)$ the set of continuous multplicative semi-norms of A whose kernel is a maximal ideal of A. Given $\phi \in Mult(A, \|\ .\ \|)$, we denote by $Ker(\phi)$ the closed prime ideal of the $x \in A$ such that $\phi(x) = 0$.

It is well known that every maximal ideal is the kernel of at least one multiplicative semi-norm on A (see for example [9]). The algebra A is said to be *multbijective* if for every maximal ideal $\mathcal{M}$, $\dfrac{A}{\mathcal{M}}$ admits only one absolute value that is an expansion of this of $\mathbb{K}$. It is easily seen that if every maximal ideal is of finite codimension, then the algebra A is multbijective.

Consider then a multbijective unital commutative ultrametric $\mathbb{K}$-Banach algebra A. We denote by $\mathcal{X}(A)$ the set of algebra homomorphisms from A onto a field extension of $\mathbb{K}$ of the form $\dfrac{A}{\mathcal{M}}$ where $\mathcal{M}$ is a maximal ideal of A. So, for every $\chi \in \mathcal{X}(A)$, the mapping $|\chi|$ defined on A by $|\chi|(x) = |\chi(x)|$ belongs to $Mult_m(A, \|\ .\ \|)$ and this is the unique $\phi \in Mult_m(A, \|\ .\|)$ such that $Ker(\phi) = Ker(\chi)$.

Theorem 1.1. *Let A be a unital commutative ultrametric $\mathbb{K}$-Banach algebra such that $Mult(A, \|\ .\ \|)$ admits a partition in two compact subsets U, V. There exist unique idempotents $u, v \in A$ such that $\phi(u) = 1,\ \phi(v) = 0,\ \forall \phi \in U$ and $\phi(u) = 0,\ \phi(v) = 1,\ \forall \phi \in V$.*

Corollary 1.2. *Let A be a unital commutative ultrametric $\mathbb{K}$-Banach algebra such that $Mult(A, \| \, . \, \|)$ admits a partition in two compact subsets U, V. Then A is isomorphic to a direct product of two $\mathbb{K}$-Banach algebras $A_U \times A_V$ such that $Mult(A_U, \| \, . \, \|) = U$ and $Mult(A_V, \| \, . \, \|) = V$. Given the idempotent $u \in A$ such that $\phi(u) = 1 \ \forall \phi \in U$, $\phi(u) = 0 \ \forall \phi \in V$, then $A_U = uA$, $A_V = (1-u)A$.*

As an easy consequence, we have Theorem 1.3. A few definitions are necessary:

Definitions and notations: Suppose that $\mathbb{K}$ is algebraically closed. Let $a \in \mathbb{K}$ and $r, \ s \in \mathbb{R}_+$ with $0 < r < s$. We denote by $\Gamma(a, r, s)$ the set $\{x \in \mathbb{K} | \ r < |x - a| < s\}$. Let D be a subset of $\mathbb{K}$, let $a \in D$ be such that $D \cap \Gamma(a, r, s) = \emptyset$ and that $r = \sup\{|a - x|, \ x \in D, |a - x| \leq r\}$ and $s = \inf\{|a - x|, \ x \in D, |a - x| \geq s\}$. The annulus $\Gamma(a, r, s)$ is called *an empty-annulus of D.*

Let A be a unital commutative $\mathbb{K}$-algebra and let $x \in A$. We denote by $sp(x)$ the set of all $\lambda \in \mathbb{K}$ such that $x - \lambda$ is not invertible.

Theorem 1.3. *Suppose that $\mathbb{K}$ is algebraically closed. Let A be a unital commutative ultrametric $\mathbb{K}$-Banach algebra such that $Mult_m(A, \| \, . \, \|)$ is dense in $Mult(A, \| \, . \, \|)$ and let $x \in A$ be such that $sp(x)$ admits an empty-annulus $\Gamma(a, r, s)$. Then there exist a unique idempotent $u \in A$ and a unique idempotent $v \in A$ such that $\chi(u) = 1, \ \chi(v) = 0 \ \forall \chi \in \mathcal{X}(A)$ satisfying $|\chi(x) - a| \leq r$ and $\chi(u) = 0, \ \chi(v) = 1 \ \forall \chi \in \mathcal{X}(A)$ satisfying $|\chi(x) - a| \geq s$.*

2 The proofs

Proving theorem 1.1 requires some preparation. We will use Propositions 2.10, 2.11 and 2.12 and mainly Theorem 2.7.

Definitions and notations: Let A be a unital commutative ultrametric $\mathbb{K}$-Banach algebra whose norm is $\| \, . \, \|$. We define the spectral semi-norm $\| \, . \, \|_{sp}$ as $\|f\|_{sp} = \lim_{n \to +\infty} \|f^n\|^{\frac{1}{n}}$. By [13] we have Theorem 2.1 (see also [9], theorem 6.19).

Theorem 2.1. $\|f\|_{sp} = \sup\{\phi(f) \mid \phi \in Mult(A, \| \, . \, \|)\}$.

Affinoid algebras were introduced by John Tate in [17] who called them *algebras topologically of finite type* and are now usually called *affinoid algebras*. As this first name suggests, such an algebra is the completion of an algebra of finite type for a certain norm.

Definitions and notation: The $\mathbb{K}$-algebra of polynomials in n variables

$\mathbb{K}[X_1, \ldots, X_n]$ is equipped with the *Gauss norm* $\| \, . \, \|$ defined as

$$\left\| \sum_{i_1, \ldots, i_n} a_{i_1, \ldots, i_n} X_1^{i_1} \cdots X_n^{i_n} \right\| = \sup_{i_1, \ldots, i_n} |a_{i_1, \ldots, i_n}|.$$

We denote by $\mathbb{K}\{X_1, \dots, X_n\}$ the set of power series in n variables $\sum_{i_1,\dots,i_n} a_{i_1,\dots,i_n} X_1^{i_1} \cdots X_n^{i_n}$ such that $\lim_{i_1+\dots+i_n \to \infty} a_{i_1,\dots,i_n} = 0$. The elements of such an algebra are called *the restricted power series in n variables, with coefficients in* $\mathbb{K}$. Hence, by definition, $\mathbb{K}[X_1, \dots, X_n]$ is dense in $\mathbb{K}\{X_1, \dots, X_n\}$. Then $\mathbb{K}\{X_1, \dots, X_n\}$ is a $\mathbb{K}$-Banach algebra which is just the completion of $\mathbb{K}[X_1, \dots, X_n]$ and is denoted by T_n.

By [16] (see also [9]): we have Theorem 2.2:

Theorem 2.2. *Every algebra $\mathbb{K}\{X_1, \dots, X_n\}$ is factorial and all ideals are closed.*

A $\mathbb{K}$- *affinoid algebra* corresponds to a quotient of any algebra of the form $\mathbb{K}\{X_1, \dots, X_n\}$ by one of its ideals equipped with its quotient norm of Banach $\mathbb{K}$-algebra.

By Theorems 31.1 and 32.7 of [9] (see also [17] and [14]):

Theorem 2.3. *Let A be a $\mathbb{K}$-affinoid algebra. Then A is noetherian and all its ideals are closed. Each maximal ideal is of finite codimension. Moreover the nilradical of A is equal to its Jacobson radical. Further, A has finitely many minimal prime ideals.*

By Theorems 35.4 in [9] or Proposition 2.8 of III in [14], we have Theorem 2.4:

Theorem 2.4. *Let A be a $\mathbb{K}$-affinoid algebra. Then $Mult_m(A, \| \,.\, \|)$ is dense in $Mult(A, \| \,.\, \|)$ for the topology of pointwise convergence.*

By Theorems 35.4 in [9] we can state Theorem 2.5:

Theorem 2.5. *Let A be a reduced $\mathbb{K}$-affinoid algebra. Then the spectral norm $\| \,.\, \|$ of A is a norm and is equivalent to the norm of affinoid algebra.*

Remark 2.6. *The proofs given in [9] for Theorems 2.2, 2.3, 2.4, 2.5 are given for algebraically closed complete ultrametric field but they hold on any complete ultrametric field.*

By Corollary 2.2.7 in [2] we have Theorem 2.7:

Theorem 2.7. *Let A be a reduced $\mathbb{K}$-affinoid algebra such that $Mult(A, \| \,.\, \|)$ admits a partition in two compact subsets U_1 and U_2. Then A is isomorphic to a direct product $A_1 \times A_2$ where A_j is a $\mathbb{K}$-affinoid algebra such that $Mult(A_j, \| \,.\, \|)$ is homeomorphic to U_j, $j = 1, 2$.*

Proposition 2.8. *Let A be a $\mathbb{K}$-affinoid algebra of Jacobson radical $\mathcal{R}$ and let $w \in \mathcal{R}$. The equation $x^2 - x + w = 0$ has a solution in $\mathcal{R}$.*

Proof. Since A is affinoid, by Theorem 2.3, w is nilpotent, hence we can consider the element

$$u = -\frac{1}{2} \sum_{n=1}^{+\infty} \binom{\frac{1}{2}}{n} (-4w)^n.$$

Now we can check that $(2u-1)^2 = 1-4w$ and then $u^2 - u - w = 0$. $\square$

Proposition 2.9. *Let A be a $\mathbb{K}$-affinoid algebra of Jacobson radical $\mathcal{R}$ and let $w \in A$ be such that $w^2 - w \in \mathcal{R}$. There exists an idempotent $u \in A$ such that $w - u \in \mathcal{R}$.*

Proof. We will roughly follow the proof known in complex algebra [15]. Let $r = w^2 - w$. We first notice that $1 + 4r = (2w-1)^2$. Next, $\frac{r}{1+4r}$ belongs to $\mathcal{R}$ hence by Proposition 2.8, there exists $x \in \mathcal{R}$ such that $x^2 - x + \frac{r}{1+4r} = 0$, and hence

$$((2w-1)x)^2 - (2w-1)^2 x + r = 0.$$

Now set $s = (2w-1)x$. Then s belongs to $\mathcal{R}$, as x. Then we obtain

$$s^2 - (2w-1)s + r = 0.$$

Let us now put $u = w - s$ and compute u^2:

$$(w-s)^2 = w^2 - 2ws + s^2 = w + r - 2ws + s^2.$$

But $s^2 = -r + (2w-1)s$, hence finally:

$$(w+s)^2 = w - r + 2ws + r - (2w-1)s = w + s.$$

Thus u is an idempotent such that $u - w \in \mathcal{R}$. $\square$

Proposition 2.10. *[14] Let A be a commutative unital ultrametric $\mathbb{K}$-Banach algebra and assume that $Mult(A, \| \, . \, \|)$ admits a partition in two compact subsets U, V. Suppose that there exist two idempotents u and e such that $\forall \phi \in U,\ \phi(u) = \phi(e) = 1$ and $\forall \phi \in V,\ \phi(u) = \phi(e) = 0$. Then $u = e$.*

Proof. Put $e = u + r$. Since $e^2 = e$, we have $(u+r)^2 = u + 2ur + r^2$ hence $u + r = u + 2ur + r^2$ and hence $r = 2ur + r^2$, therefore $r(2u + r - 1) = 0$.

Suppose $r \neq 0$. Then $2u + r - 1$ is a divisor of zero. Now, when $\phi \in U$, we have $\phi(1-u) = 0$, hence $\phi(-1 + 2u + r) = \phi(u + r) = \phi(e) = 1$, and when $\phi \in V$, we have $\phi(u) = \phi(e) = 0$, hence $\phi(1 - 2u - r) = \phi(1 - u - r) = \phi(1 - e) = 1$. Hence, $\forall \phi \in Mult(A, \| \, . \, \|)$, we have $\phi(1 - 2u - r) = 1$. Consequently, $1 - 2u - r$ does not belong to any maximal ideal of A and hence is invertible. But then $1 - 2u - r$ is not a divisor of zero, which proves that $r = 0$ and hence $e = u$. $\square$

Proposition 2.11. *[14] Let A be a $\mathbb{K}$-affinoid algebra such that $Mult(A, \| \, . \, \|)$ admits a partition in two compact subsets U_1, U_2. There exist unique idempotents $e_1, e_2 \in A$ such that $\phi(e_1) = 1, \phi(e_2) = 0\ \forall \phi \in U_1$ and $\phi(e_1) = 0,\ \phi(e_2) = 1 \forall \phi \in U_2$.*

Proof. Suppose first that A is reduced. By Theorem 2.7, A is isomorphic to the direct product $A_1 \times A_2$ where A_j is a $\mathbb{K}$-affinoid algebra such that $Mult(A_j, \| \, . \, \|) = U_j, \ j = 1, \ 2$. Let Φ be the isomorphism from $A_1 \times A_2$ onto A, let u_j be the unity of $A_j, \ j = 1, \ 2$ and let $e_1 = \Phi(u_1, 0), \ e_2 = \Phi(0, u_2)$. So $e_1, \ e_2$ are idempotents of A. Let $A'_1 = \{\Phi(x, 0) \mid x \in A_1\}$ and let $A'_2 = \{\Phi(0, x) \mid x \in A_2\}$.

Then, given $\varphi \in U_j$, it factorizes in the form $\psi \circ \Phi^{-1}$ with $\psi \in Mult(A_j, \ \| \, . \, \|), \ (j = 1, \ 2)$ and for $\varphi \in U_1$, we have $\varphi(e_1) = 1, \ \varphi(e_2) = 0$, and given $\varphi \in U_2$, we have $\varphi(e_1) = 0, \ \varphi(e_2) = 1$. By Proposition 2.10, the idempotents $e_1, \ e_2$ are unique. $\square$

We can easily greneralize when A is no longer supposed to be reduced. Let $\mathcal{R}$ be the Jacobson radical of A and let $B = \dfrac{A}{\mathcal{R}}$. Let θ be the canonical surjection from A onto B. Every $\phi \in Mult(A, \| \, . \, \|)$ is of the form $\varphi \circ \theta$ with $\varphi \in Mult(B, \| \, . \, \|)$. Let $U'_1 = \{\varphi \in Mult(B, \| \, . \, \|)\}$ be such that $\varphi \circ \theta \in U_1$ and let $U'_2 = \{\varphi \in Mult(B, \| \, . \, \|)\}$ be such that $\varphi \circ \theta \in U_2$. Then U'_1 and U'_2 are two compact subsets making a partition of $Mult(B, \| \, . \, \|)$. Therefore, B has an idempotent u_1 such that $\varphi(u_1) = 1 \ \forall \varphi \in U'_1$ and $\varphi(u_1) = 0 \ \forall \varphi \in U'_2$. Let $w \in A$ be such that $\theta(w) = u_1$. Then we can check that $\phi(w) = 1 \ \forall \phi \in U_1$ and $\phi(w) = 0 \ \forall \phi \in U_2$. But by Proposition 2.9, there exists an idempotent $e_1 \in A$ such that $e_1 - w \in \mathcal{R}$. Then $\chi(e_1) = \chi(w) \ \forall \chi \in X(A)$ and hence $\phi(e_1) = \phi(w) \ \forall \phi \in Mult(A, \| \, . \, \|)$ because, by Theorem 2.4 $Mult_m(A, \| \, . \, \|)$ is dense in $Mult(A, \| \, . \, \|)$. The unicity of e_1 follows from Proposition 2.10. Similarly, there exists a unique idempotent $e_2 \in A$ such that $\phi(e_2) = 1 \ \forall \phi \in U_2$ and $\phi(e_2) = 0 \ \forall \phi \in U_1$.

Definition and notations: We will denote by $| \, . \, |_\infty$ the Archimedean absolute value of $\mathbb{R}$. Given a unital commutative ultrametric $\mathbb{K}$-normed algebra A and $\phi \in Mult(A, \| \, . \, \|)$, $y_1, \dots y_q \in A$ and $\epsilon > 0$, we will denote by $W(\phi, y_1, \dots, y_q, \epsilon)$ the set of $\theta \in Mult(A, \| \, . \, \|)$ such that $|\phi(y_j) - \theta(y_j)|_\infty \leq \epsilon \ \forall j = 1, \dots, q$. Given a unital commutative ultrametric $\mathbb{K}$-normed algebra A and a subalgebra B, we call *canonical mapping from* $Mult(A, \| \, . \, \|)$ *to* $Mult(B, \| \, . \, \|)$ the mapping Φ defined by $\Phi(\varphi)(x) = \varphi(x) \ \forall x \in B, \ \varphi \in Mult(A, \| \, . \, \|)$.

Proposition 2.12. *[14] Let A be a unital commutative ultrametric $\mathbb{K}$-Banach algebra and assume that $Mult(A, \| \, . \, \|)$ admits a partition in two compact subsets U, V. There exists a $\mathbb{K}$-affinoid algebra B included in A, admitting for norm this of A, such that $Mult(B, \| \, . \, \|)$ admits a partition in two open subsets U', V' where the canonical mapping Φ from $Mult(A, \| \, . \, \|)$ to $Mult(B, \| \, . \, \|)$ satisfies $\Phi(U) \subset U'$, $\Phi(V) \subset V'$.*

Proof. Since U and V are compact sets, we can easily define a covering of open sets $(O_j)_{j \in J}$ such that $O_j \cap V = \emptyset \ \forall j \in J$. From this, we can extract a finite covering $(U_i)_{1 \leq i \leq n}$ of U where the U_i are of the form $W(f_i, x_{i,1}, \dots, x_{i,m_i}, \epsilon_i)$ with $x_{i,j} \in A$, such that $U_i \cap V = \emptyset \ \forall i = 1, \dots, n$. Let $\widetilde{A}$ be the finite type $\mathbb{K}$-subalgebra generated by all the $x_{i,j}, 1 \leq j \leq m_i, 1 \leq i \leq n$. Consider the image of $Mult(A, \| \, . \, \|)$ in $Mult(\widetilde{A}, \| \, . \, \|)$ through the mapping Φ that associates to each

$\phi \in Mult(A, \| \, . \, \|)$ its restriction to $\widetilde{A}$ and let $\widetilde{U} = \Phi(U)$, $\widetilde{V} = \Phi(V)$. Then both $\widetilde{U}$, $\widetilde{V}$ are compact with respect to the topology of $Mult(\widetilde{A}, \| \, . \, \|)$ and hence there exist open neighborhoods U' of $\widetilde{U}$ and V' of $\widetilde{V}$ in $Mult(\widetilde{A}, \| \, . \, \|)$ such that $U' \cap V' = \emptyset$. Let $Y = U' \cup V'$. By construction we have $\Phi(U) \subset U'$, $\Phi(V) \subset V'$.

Let $\phi \in Mult(\widetilde{A}, \| \, . \, \|) \setminus Y$. There exists a finite type algebra $\widetilde{A}_\phi$ containing $\widetilde{A}$, such that the canonical image H_φ of $Mult(\widetilde{A}_\phi, \| \, . \, \|)$ in $Mult(\widetilde{A}, \| \, . \, \|)$ does not contain ϕ. Since this image H_ϕ is compact, there exists a neighborhood $G(\phi)$ of ϕ such that $G(\phi) \cap H_\phi = \emptyset$. Next, we notice that $Mult(\widetilde{A}, \| \, . \, \|) \setminus Y$ is compact, hence we can find $\phi_1, \ldots, \phi_n \in Mult(\widetilde{A}, \| \, . \, \|) \setminus Y$ and neighborhoods $Z(\phi_1), \ldots, Z(\phi_n)$ making a covering of $Mult(\widetilde{A}, \| \, . \, \|) \setminus Y$. Let E be the finite type algebra generated by the $\widetilde{A}_{\phi_i}, 1 \le i \le n$. Then E is a $\mathbb{K}$-subalgebra of A of finite type which contains $\widetilde{A}$ and hence is equipped with the $\mathbb{K}$-algebra norm $\| \, . \, \|$ of A. Moreover, by construction, $Mult(E, \| \, . \, \|)$ is equal to $Y = U' \cup V'$.

Let $\{x_1, \ldots, x_N\}$ be a finite subset of the unit ball of E such that $\mathbb{K}[x_1, \ldots, x_N] = E$. Let T be the topologically pure extension $\mathbb{K}\{X_1, \ldots, X_N\}$ and consider the canonical morphism Θ from $\mathbb{K}[X_1, \ldots, X_N]$ equipped with the Gauss norm, into E, equipped with the norm $\| \, . \, \|$ of A, defined as $\Theta(F(X_1, \ldots, X_N)) = F(x_1, \ldots, x_N)$. Since by hypotheses, $\|x_j\| \le 1 \; \forall j = 1, \ldots, N$, Θ is continuous and has expansion to a continuous morphism $\overline{\Theta}$ from T into A. Let $\mathcal{I}$ be the closed ideal of the elements $F \in T$ such that $\overline{\Theta}(F) = 0$. Then $\overline{\Theta}(T)$ is the $\mathbb{K}$-affinoid algebra $B = \dfrac{T}{\mathcal{I}}$ containing E and included in A. By construction, the $\mathbb{K}$-affinoid norm of B is the restriction of the norm $\| \, . \, \|$ of A. Since by construction E is dense in B, we have $Mult(B, \| \, . \, \|) = Mult(E, \| \, . \, \|) = U' \cup V'$. Consequently, $\Phi(U) \subset U'$, $\Phi(V) \subset V'$, which ends the proof. $\square$

Remark 2.13. *Proposition 2.12 was roughly stated in [14]. However, its proof was confusing about subsets containing U and V and norms defined on an affinoid subalgebra B, which then puts in doubt the conclusion.*

We can now conclude.

Proof of Theorem 1.1. By Proposition 2.12, there exists a $\mathbb{K}$-affinoid algebra B included in A such that $Mult(B, \| \, . \, \|)$ admits a partition in two open disjoint subsets U', V' and such that the canonical mapping Φ from $Mult(A, \| \, .\|)$ to $Mult(B, \| \, . \, \|)$ satisfies $\Phi(U) \subset U'$, $\Phi(V) \subset V'$. Now, by Proposition 2.11, there exist idempotents u', $v' \in B$ such that $\phi(u') = 1 \; \forall \phi \in U'$ and $\phi(u') = 0 \; \forall \phi \in V'$. Consequently, we have $\phi(u) = 1 \; \forall \phi \in U$, $\phi(u) = 0 \; \forall \phi \in V$ and $\phi(v) = 0 \; \forall \phi \in U$, $\phi(v) = 1 \; \forall \phi \in V$. The unicity follows from Proposition 2.11. That ends the proof.

Proof of Theorem 1.3. Without loss of generality, we can suppose $a = 0$. Let $U = \{\phi \in Mult(A, \| \, . \, \|)\}$ such that $\phi(x) \le r$, and let $V = \{\phi \in Mult(A, \| \, . \, \|)\}$ such that $\phi(x) \ge s$. Since $Mult_m(A, \| \, . \, \|)$ is dense in $Mult(A, \| \, . \, \|)$, it is clear that no $\phi \in Mult(A, \| \, . \, \|)$ can satisfy $r < \phi(x) < s$. Consequently, U, V make a partition of $Mult(A, \| \, . \, \|)$. Next, one can easily check that U and V are open and closed with respect to the pointwise convergence. Indeed, given $\phi \in Mult(A, \| \ . \ \|)$, $g_1, \dots, g_t \in A$ and $\epsilon > 0$, we denote by $W(\phi, g_1, \dots, g_t, \epsilon)$ the neighborhood of ϕ defined as $\{\theta \in Mult(A, \| \ . \ \|) \ |\phi(g_j) - \theta(g_j)|_\infty \leq \epsilon \ \forall j = 1, \dots, t\}$. So, let $\epsilon \in \left]0, \dfrac{s-r}{2}\right[$ and consider the families of neighborhoods of U and V of the form $W(\phi, x, f_1, \dots, f_m, \epsilon)_{\phi \in U}$ and $W(\psi, x, g_1, \dots, g_n, \epsilon)_{\psi \in V}$ respectively. Then given any $W(\phi, x, f_1, \dots, f_m, \epsilon)$, $\phi \in U$ and $W(\psi, x, g_1, \dots, g_n, \epsilon)$, $\psi \in V$ we have $W(\phi, x, f_1, \dots, f_m, \epsilon) \cap W(\psi, x, g_1, \dots, g_n, \epsilon) = \emptyset$ hence U and V are two open subsets such that $U \cap V = \emptyset$. By construction $Mult(A, \| \ . \ \|) = U \cup V$. Consequently, U and V are two open subsets making a partition of $Mult(A, \| \ . \ \|)$, which by Theorem 1.1, ends the proof.

Remark 2.14. *The proof of Theorem 1.3 consists of injecting the Krasner-Tae algebra [8] $H(\Gamma(a, r, s))$ into A.*

Acknowledgement: I am grateful to the Referee for useful remarks.

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Cubo
A Mathematical Journal

CUBO, A Mathematical Journal
Vol. 23, no. 01, pp. 171–190, April 2021
DOI: 10.4067/S0719-06462021000100171

Existence, well-posedness of coupled fixed points and application to nonlinear integral equations

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ABSTRACT

We investigate a fixed point problem for coupled Geraghty type contraction in a metric space with a binary relation. The role of the binary relation is to restrict the scope of the contraction to smaller number of ordered pairs. Such possibilities have been explored for different types of contractions in recent times which has led to the emergence of relational fixed point theory. Geraghty type contractions arose in the literatures as a part of research seeking the replacement contraction constants by appropriate functions. Also coupled fixed point problems have evoked much interest in recent times. Combining the above trends we formulate and solve the fixed point problem mentioned above. Further we show that with some additional conditions such solution is unique. Well-posedness of the problem is investigated. An illustrative example is discussed. The consequences of the results are discussed considering α-dominated mappings and graphs on the metric space. Finally we apply our result to show the existence of solution of some system of nonlinear integral equations.

RESUMEN

Investigamos un problema de punto fijo para contracciones acopladas de tipo Geraghty en un espacio métrico con una relación binaria. El rol de la relación binaria es restringir el alcance de la contracción a un número menor de pares ordenados. Tales posibilidades han sido exploradas para diferentes tipos de contracciones recientemente, lo que ha conllevado el nacimiento de la teoría de punto fijo relacional. Las contracciones de tipo Geraghty aparecen en la literatura como parte de la investigación buscando reemplazar las constantes de contracción por funciones apropiadas. También problemas de puntos fijos acoplados han sido de mucho interés recientemente. Combinando las ideas anteriores, formulamos y resolvemos el problema de punto fijo mencionado anteriormente. Más aún, mostramos que bajo condiciones adicionales tal solución es única. Se investiga la bien-definición del problema. Se discute un ejemplo ilustrativo. Las consecuencias de los resultados se discuten considerando aplicaciones α-dominadas y grafos en espacios métricos. Finalmente aplicamos nuestros resultados para mostrar la existencia de soluciones de algunos sistemas de ecuaciones integrales no lineales.

Keywords and Phrases: Metric space; coupled fixed point; well-posedness; application.

2020 AMS Mathematics Subject Classification: 54H10, 54H25, 47H10.

Accepted: 29 March, 2021
Received: 14 June, 2020

1 Introduction

Coupled fixed point results constitute a domain in metric fixed point theory which has experienced rapid development in recent times. The concept of coupled fixed point was introduced some time back in 1987 by Guo et al [17]. But only after the publication of the work of Bhaskar et al [15] a large number of papers have been written on this topic and on topics related to it [7, 9, 18, 21].

Our consideration in this paper is a study related to fixed points of some coupled operators on metric spaces equipped with an appropriate binary relation. A contraction condition of Geraghty type [20, 26, 32] is supposed to be satisfied by the coupled operator for those points which are related by the binary relation. As a consequence of it the assumption here is weaker than the usual case in metric fixed point theory where it is assumed that the inequality condition holds for arbitrarily chosen pairs from the space. Such weakening of conditions have substantially occupied recent interests in fixed point theory. Works of this category have come to be known as relation-theoretic fixed point results. Some instances of these works are in [1, 3, 23, 30].

We use Geraghty's approach [16] to define a coupled contraction condition. It is a part of research where the constants of the contractions are replaced by suitable control functions in order to make the contraction inequality more general. Such works occupy important positions in metric fixed point theory. Some instances of these works are [5, 9, 13, 14, 22].

In this paper we combine the above trends in fixed point theory to define a new problem and then investigate its several aspects and show one application of the result.

Firstly, we show that such problem has a solution, that is, a coupled fixed point of the concerned operator exists. The uniqueness of the coupled fixed point is established under some additional conditions.

Well-posedness has been considered for many fixed point problems in recent times [24, 25, 27, 28]. In the present paper we deal with the well-posedness of the problem mentioned above.

Next we discuss some consequences of our main result. Precisely we obtain some results for α-dominated mappings and results in metric spaces having a graph defined on it. The main result is supported with an example. In the last section we include an application of the main theorem to a problem of nonlinear integral equations.

2 Mathematical background

In the following we discuss the necessary mathematics for the discussion on the topics in the following sections. Let X and Y be two nonempty sets and R be a relation from X to Y, that is, $R \subseteq X \times Y$. We write $(x, y) \in R$ or xRy to mean x is R related to y. The set $P = \{x \in X : (x, y) \in$

R for some $y \in Y\}$ is called the domain of R and the set $Q = \{y \in Y : (x,\ y) \in R$ for some $x \in X\}$ is called the range of R. By R^{-1} we mean the set $\{(y,\ x) : (x,\ y) \in R\}$ which is called the inverse of R.

A relation R from X to X is called a relation on X. Let R be a relation on X. The relation R is said to be directed if for given $x, y \in X$, there exists $z \in X$ such that $(x,\ z) \in R$ and $(y,\ z) \in R$. The relation R is said to be a partial order relation on X if it is reflexive, anti-symmetric and transitive.

Let X be a nonempty set. An element $(x, y) \in X \times X$ is called a coupled fixed point of a function $F : X \times X \to X$ if $x = F(x, y)$ and $y = F(y, x)$.

Problem (P): Let (X, d) be a metric space and $F : X \times X \to X$ be a mapping. We consider the problem of finding a coupled fixed point of F, that is, the problem of finding $(x,\ y) \in X \times X$ such that

$$x = F(x,\ y) \ \text{ and } \ y = F(y,\ x). \tag{2.1}$$

Definition 2.1 ([6])**.** The problem (P) is called well-posed if (i) F has a unique coupled fixed point $(x^*,\ y^*)$, (ii) $x_n \to x^*$ and $y_n \to y^*$ as $n \to \infty$, whenever $\{(x_n,\ y_n)\}$ is any sequence in $X \times X$ for which $\limsup_{n\to\infty}[d(x_n,\ x^*) + d(y_n,\ y^*)]$ is finite and $\lim_{n\to\infty} d(x_n,\ F(x_n,\ y_n)) = \lim_{n\to\infty} d(y_n,\ F(y_n,\ x_n)) = 0$.

We define here the R-dominated mapping.

Definition 2.2. Let X be a nonempty set with a binary relation R on it. A mapping $F : X \times X \to X$ is said to be R-dominated if $(x,\ F(x, y)) \in R$ and $(F(y, x),\ y) \in R$, for any $(x,\ y) \in X \times X$.

Example 2.3. Let $X = [0,\ 1]$ be equipped with usual metric. Let $F : X \times X \to X$ be defined as $F(x,\ y) = \frac{x+y}{16+x+y}$, for $x,\ y \in X$. Let a binary relation R on X be defined as $R = \{(x,\ y) : 0 \leq x \leq 1;\ 0 \leq y \leq \frac{1}{8}$ or $0 \leq x \leq \frac{1}{8};\ 0 \leq y \leq 1\}$. Then $F(x,\ y) = F(y,\ x) \in [0,\ \frac{1}{8}]$, for $x,\ y \in [0,\ 1]$. It follows that $(x,\ F(x, y)) \in R$ and $(F(y, x),\ y) \in R$, for any $(x,\ y) \in X \times X$. Therefore, F is a R-dominated mapping.

We introduce R-regularity condition in metric spaces.

Definition 2.4. Let (X, d) be a metric space with a binary relation R on it. Then X is said to have regular property with respect to R (or R-regular property) if for every sequence $\{x_n\}$ in X converging to $x \in X,\ (x_n,\ x_{n+1}) \in R,$ for all n implies $(x_n,\ x) \in R,$ for all n [or $(x_{n+1},\ x_n) \in R,$ for all n implies $(x,\ x_n) \in R,$ for all n].

The following class of functions has appeared in several recent works related to fixed point theory.

Let $\gamma : [0,\ \infty) \to [0,\ 1)$ be such that for any sequence $\{t_n\}$ in $[0,\ \infty)$, $\lim_{n\to\infty} \gamma(t_n) = 1$ implies $\lim_{n\to\infty} t_n = 0$. We denote the collection all such functions γ by $\boldsymbol{B}$. Such functions have appeared in several papers as for instances in [20, 32, 33].

In our theorems, we use following class of functions:

Let $\beta : [0,\ \infty) \to [0,\ 1)$ be such that for any sequence $\{t_n\}$ in $[0,\ \infty)$, $\limsup_{n\to\infty} \beta(t_n) = 1$ implies $\lim_{n\to\infty} t_n = 0$. We denote the collection all such functions β by $\boldsymbol{B}^*$.

We have the following observation about the class $\boldsymbol{B}^*$. Our class $\boldsymbol{B}^*$ is more generalized than $\boldsymbol{B}$. From the definition of $\boldsymbol{B}$ and $\boldsymbol{B}^*$ it is clear that our class $\boldsymbol{B}^*$ contains $\boldsymbol{B}$ and this containment is proper. The following example makes the fact clear:

Example 2.5. Now consider the function $\beta : [0,\ \infty) \to [0,\ 1)$ defined by

$$\beta(t) = \begin{cases} \left| \frac{\sin t}{t} \right| & , \text{ if } t \text{ is irrational}, \\ \frac{1}{2} & , \text{ if } t \text{ is rational}. \end{cases}$$

Clearly $\beta \in \boldsymbol{B}^*$ but $\beta \notin \boldsymbol{B}$.

3 Main results

In this section we establish a coupled fixed point result. We discuss its uniqueness under some additional conditions. We illustrate it with an example.

Theorem 3.1. *Let (X, d) be a complete metric space with a transitive relation R on it such that X has R-regular property. Suppose that $F : X \times X \to X$ is a R-dominated mapping and there exists $\beta \in \boldsymbol{B}^*$ such that for $(x,\ y),\ (u,\ v) \in X \times X$ with $[(x,\ u) \in R$ and $(v,\ y) \in R]$ or $[(u,\ x) \in R$ and $(y,\ v) \in R]$,*

$$d(F(x,\ y),\ F(u,\ v)) \leq \beta(M(x,\ y,\ u,\ v))\ M(x,\ y,\ u,\ v), \tag{3.1}$$

where

$$M(x,\ y,\ u,\ v) = max \left\{ \frac{d(x,\ u) + d(y,\ v)}{2},\ \frac{d(x, F(x,\ y)) + d(y,\ F(y,\ x))}{2}, \right.$$
$$\left. \frac{d(u, F(u,\ v)) + d(v, F(v,\ u))}{2},\ \frac{d(u, F(x,\ y)) + d(v, F(y,\ x))}{2} \right\}.$$

Then F has a coupled fixed point.

Proof. Let $(x_0,\ y_0) \in X \times X$ be arbitrary. We construct two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$x_{n+1} = F(x_n,\ y_n) \text{ and } y_{n+1} = F(y_n,\ x_n), \text{ for all } n \geq 0. \tag{3.2}$$

As F is R-dominated, we have

$$(x_n,\ F(x_n,\ y_n)) = (x_n, x_{n+1}) \in R \quad \text{and} \quad (F(y_n,\ x_n),\ y_n) = (y_{n+1},\ y_n) \in R, \text{ for all } n \geq 0. \tag{3.3}$$

Let

$$r_n = d(x_n,\ x_{n+1}) + d(y_n,\ y_{n+1}), \text{ for all } \ n \geq 0. \tag{3.4}$$

By (3.1), (3.2), (3.3) and (3.4), we have

$$\begin{aligned} d(x_{n+1}, x_{n+2}) &= d(F(x_n, y_n),\ F(x_{n+1}, y_{n+1})) \\ &\leq \beta(M(x_n, y_n, x_{n+1}, y_{n+1}))\ M(x_n, y_n, x_{n+1}, y_{n+1}) \end{aligned} \tag{3.5}$$

where

$$\begin{aligned} M(x_n, y_n, x_{n+1}, y_{n+1}) &= \max\Big\{\frac{d(x_n, x_{n+1}) + d(y_n, y_{n+1})}{2},\ \frac{d(x_n, F(x_n,\ y_n)) + d(y_n,\ F(y_n,\ x_n))}{2}, \\ &\quad \frac{d(x_{n+1}, F(x_{n+1}, y_{n+1})) + d(y_{n+1}, F(y_{n+1}, x_{n+1}))}{2}, \\ &\quad \frac{d(x_{n+1}, F(x_n,\ y_n)) + d(y_{n+1}, F(y_n,\ x_n))}{2}\Big\} \\ &= \max\Big\{\frac{d(x_n, x_{n+1}) + d(y_n, y_{n+1})}{2},\ \frac{d(x_n, x_{n+1}) + d(y_n,\ y_{n+1})}{2}, \\ &\quad \frac{d(x_{n+1}, x_{n+2}) + d(y_{n+1}, y_{n+2})}{2},\ \frac{d(x_{n+1}, x_{n+1}) + d(y_{n+1}, y_{n+1})}{2}\Big\} \\ &= \max\Big\{\frac{d(x_n, x_{n+1}) + d(y_n, y_{n+1})}{2},\ \frac{d(x_n, x_{n+1}) + d(y_n,\ y_{n+1})}{2}, \\ &\quad \frac{d(x_{n+1}, x_{n+2}) + d(y_{n+1}, y_{n+2})}{2},\ 0\Big\} \\ &= \max\left\{\frac{r_n}{2},\ \frac{r_n}{2}\ \frac{r_{n+1}}{2},\ 0\right\} \\ &= \max\left\{\frac{r_n}{2},\ \frac{r_{n+1}}{2}\right\}. \end{aligned} \tag{3.6}$$

Therefore, from (3.5) and (3.6), we have

$$d(x_{n+1}, x_{n+2}) \leq \beta\left(\max\left\{\frac{r_n}{2},\ \frac{r_{n+1}}{2}\right\}\right) \quad \max\left\{\frac{r_n}{2},\ \frac{r_{n+1}}{2}\right\}. \tag{3.7}$$

Similarly, we can show that

$$d(y_{n+1}, y_{n+2}) \leq \beta\left(\max\left\{\frac{r_n}{2},\ \frac{r_{n+1}}{2}\right\}\right) \quad \max\left\{\frac{r_n}{2},\ \frac{r_{n+1}}{2}\right\}. \tag{3.8}$$

Combining (3.7) and (3.8), we have

$$\begin{aligned} r_{n+1} &= d(x_{n+1}, x_{n+2}) + d(y_{n+1}, y_{n+2}) \\ &\leq 2\ \beta\left(\max\left\{\frac{r_n}{2},\ \frac{r_{n+1}}{2}\right\}\right)\ \max\left\{\frac{r_n}{2},\ \frac{r_{n+1}}{2}\right\} \\ &= \beta\left(\max\left\{\frac{r_n}{2},\ \frac{r_{n+1}}{2}\right\}\right)\ \max\{r_n,\ r_{n+1}\}. \end{aligned} \tag{3.9}$$

Suppose that $0 \leq r_n < r_{n+1}$. From (3.9), we have

$$r_{n+1} \leq \beta\left(\frac{r_{n+1}}{2}\right) r_{n+1} < r_{n+1}$$

which is a contradiction. Therefore, $r_{n+1} \leq r_n$, for all $n \geq 0$, that is, $\{r_n\}$ is a decreasing sequence of nonnegative real numbers. Hence there exists $r \geq 0$ such that $r_n \to r$ as $n \to \infty$. By (3.9), we have

$$r_{n+1} \leq \beta\left(\frac{r_n}{2}\right)\ r_n, \text{ for all } n \geq 0. \tag{3.10}$$

If possible, suppose that $r > 0$. Taking limit supremum in (3.10), we have

$$r \leq \limsup_{n\to\infty} \beta\left(\frac{r_n}{2}\right)\ r,$$

which implies that $1 \leq \limsup_{n\to\infty} \beta(\frac{r_n}{2}) \leq 1$, that is, $\limsup_{n\to\infty} \beta(\frac{r_n}{2}) = 1$. Then it follows by the property of β that $\lim_{n\to\infty} \frac{r_n}{2} = \frac{r}{2} = 0$, that is, $r = 0$ which contradicts our assumption. Hence $r = 0$. Then we have

$$\lim_{n\to\infty} [d(x_n, x_{n+1}) + d(y_n, y_{n+1})] = \lim_{n\to\infty} d(x_n, x_{n+1}) = \lim_{n\to\infty} d(y_n, y_{n+1}) = 0. \tag{3.11}$$

Now we prove that both $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences. If possible, assume that either $\{x_n\}$ or $\{y_n\}$ fails to be a Cauchy sequence. Then

$$\text{either} \quad \lim_{m,\ n\to\infty} d(x_m,\ x_n) \neq 0 \quad \text{or} \quad \lim_{m,\ n\to\infty} d(y_m,\ y_n) \neq 0.$$

Hence,

$$\lim_{m,\ n\to\infty} [d(x_m,\ x_n) + d(y_m,\ y_n)] \neq 0,$$

that is, there exists $\epsilon > 0$ for which we can find subsequences $\{m(k)\}$ and $\{n(k)\}$ of positive integers with $n(k) > m(k) > k$ such that

$$d(x_{m(k)},\ x_{n(k)}) + d(y_{m(k)},\ y_{n(k)}) \geq \epsilon \ \text{ and } \ d(x_{m(k)},\ x_{n(k)-1}) + d(y_{m(k)},\ y_{n(k)-1}) < \epsilon. \tag{3.12}$$

Now,

$$\begin{aligned}
\epsilon &\leq d(x_{n(k)},\ x_{m(k)}) + d(y_{n(k)},\ y_{m(k)}) \\
&\leq [d(x_{n(k)},\ x_{n(k)-1}) + d(y_{n(k)},\ y_{n(k)-1})] + [d(x_{n(k)-1},\ x_{m(k)}) + d(y_{n(k)-1},\ y_{m(k)})] \\
&< d(x_{n(k)},\ x_{n(k)-1}) + d(y_{n(k)},\ y_{n(k)-1}) + \epsilon.
\end{aligned}$$

Taking limit as $k \to \infty$ in the above inequality and using (3.11), we have

$$\lim_{k\to\infty} [d(x_{m(k)},\ x_{n(k)}) + d(y_{m(k)},\ y_{n(k)})] = \epsilon. \tag{3.13}$$

Again,

$$\begin{aligned}
&d(x_{n(k)-1},\ x_{m(k)-1}) + d(y_{n(k)-1},\ y_{m(k)-1}) \\
&\leq [d(x_{n(k)-1},\ x_{m(k)}) + d(y_{n(k)-1},\ y_{m(k)})] + [d(x_{m(k)},\ x_{m(k)-1}) + d(y_{m(k)},\ y_{m(k)-1})] \\
&< \epsilon + [d(x_{m(k)},\ x_{m(k)-1}) + d(y_{m(k)},\ y_{m(k)-1})].
\end{aligned}$$

Again,

$$d(x_{n(k)},\ x_{m(k)}) + d(y_{n(k)},\ y_{m(k)}) \leq [d(x_{n(k)},\ x_{n(k)-1}) + d(y_{n(k)},\ y_{n(k)-1})] +$$
$$[d(x_{n(k)-1},\ x_{m(k)-1}) + d(y_{n(k)-1},\ y_{m(k)-1})] + [d(x_{m(k)-1},\ x_{m(k)}) + d(y_{m(k)-1},\ y_{m(k)})],$$

that is,

$$d(x_{n(k)-1},\ x_{m(k)-1}) + d(y_{n(k)-1},\ y_{m(k)-1}) \geq d(x_{n(k)},\ x_{m(k)}) + d(y_{n(k)},\ y_{m(k)})$$
$$- d(x_{n(k)},\ x_{n(k)-1}) - d(y_{n(k)},\ y_{n(k)-1}) - d(x_{m(k)-1},\ x_{m(k)}) - d(y_{m(k)-1},\ y_{m(k)}).$$

From the above inequalities we have that

$$d(x_{n(k)},\ x_{m(k)}) + d(y_{n(k)},\ y_{m(k)}) - d(x_{n(k)},\ x_{n(k)-1}) - d(y_{n(k)},\ y_{n(k)-1}) - d(x_{m(k)-1},\ x_{m(k)})$$
$$- d(y_{m(k)-1},\ y_{m(k)}) \leq d(x_{n(k)-1},\ x_{m(k)-1}) + d(y_{n(k)-1},\ y_{m(k)-1})$$
$$< \epsilon + [d(x_{m(k)},\ x_{m(k)-1}) + d(y_{m(k)},\ y_{m(k)-1})].$$

Taking limit as $k \to \infty$ in the above inequality and using (3.11) and (3.13), we have

$$\lim_{k\to\infty} [d(x_{m(k)-1},\ x_{n(k)-1}) + d(y_{m(k)-1},\ y_{n(k)-1})] = \epsilon. \tag{3.14}$$

Now,

$$\begin{aligned}
&d(x_{n(k)},\ x_{m(k)}) + d(y_{n(k)},\ y_{m(k)})\\
&\leq [d(x_{n(k)}, x_{n(k)-1}) + d(y_{n(k)}, y_{n(k)-1})] + [d(x_{n(k)-1}, x_{m(k)}) + d(y_{n(k)-1}, y_{m(k)})]\\
&\leq [d(x_{n(k)}, x_{n(k)-1}) + d(y_{n(k)}, y_{n(k)-1})] + [d(x_{n(k)-1}, x_{m(k)-1}) + d(y_{n(k)-1}, y_{m(k)-1})]\\
&+ [d(x_{m(k)-1}, x_{m(k)}) + d(y_{m(k)-1},\ y_{m(k)})].
\end{aligned}$$

Taking limit as $k \to \infty$ in the above inequality and using (3.11), (3.13) and (3.14), we get

$$\lim_{k\to\infty} [d(x_{n(k)-1}, x_{m(k)}) + d(y_{n(k)-1}, y_{m(k)})] = \epsilon. \tag{3.15}$$

Using (3.3) and the transitivity assumption of R, we have

$$(x_{m(k)-1},\ x_{n(k)-1}) \in R \ \text{ and } \ (y_{n(k)-1},\ y_{m(k)-1}) \in R.$$

Applying (3.1), we have

$$\begin{aligned}
d(x_{m(k)},\ x_{n(k)}) &= d(F(x_{m(k)-1}, y_{m(k)-1}),\ F(x_{n(k)-1}, y_{n(k)-1}))\\
&\leq \beta(M(x_{m(k)-1}, y_{m(k)-1}, x_{n(k)-1}, y_{n(k)-1}))\\
&M(x_{m(k)-1}, y_{m(k)-1}, x_{n(k)-1}, y_{n(k)-1})
\end{aligned} \tag{3.16}$$

where

$$
\begin{aligned}
&M(x_{m(k)-1}, y_{m(k)-1}, x_{n(k)-1}, y_{n(k)-1})\\
&= \max \left\{ \frac{d(x_{m(k)-1}, x_{n(k)-1}) + d(y_{m(k)-1},\ y_{n(k)-1})}{2}, \right.\\
&\frac{d(x_{m(k)-1},\ F(x_{m(k)-1}, y_{m(k)-1})) + d(y_{m(k)-1},\ F(y_{m(k)-1}, x_{m(k)-1}))}{2},\\
&\frac{d(x_{n(k)-1},\ F(x_{n(k)-1}, y_{n(k)-1})) + d(y_{n(k)-1}, F(y_{n(k)-1}, x_{n(k)-1}))}{2},\\
&\left. \frac{d(x_{n(k)-1}, F(x_{m(k)-1},\ y_{m(k)-1})) + d(y_{n(k)-1}, F(y_{m(k)-1}, x_{m(k)-1}))}{2} \right\}\\
&= \max \left\{ \frac{d(x_{m(k)-1}, x_{n(k)-1}) + d(y_{m(k)-1}, y_{n(k)-1})}{2},\ \frac{d(x_{m(k)-1}, x_{m(k)}) + d(y_{m(k)-1}, y_{m(k)})}{2}, \right.\\
&\left. \frac{d(x_{n(k)-1}, x_{n(k)}) + d(y_{n(k)-1}, y_{n(k)})}{2}, \frac{d(x_{n(k)-1}, x_{m(k)}) + d(y_{n(k)-1}, y_{m(k)})}{2} \right\}. \qquad (3.17)
\end{aligned}
$$

Similarly, we show that

$$
\begin{aligned}
d(y_{m(k)},\ y_{n(k)}) &= d(F(y_{m(k)-1}, x_{m(k)-1}),\ F(y_{n(k)-1}, x_{n(k)-1}))\\
&\leq \beta(M(y_{m(k)-1}, x_{m(k)-1}, y_{n(k)-1}, x_{n(k)-1}))\\
&\quad M(y_{m(k)-1}, x_{m(k)-1}, y_{n(k)-1}, x_{n(k)-1})\\
&= \beta(M(x_{m(k)-1}, y_{m(k)-1}, x_{n(k)-1}, y_{n(k)-1}))\\
&\quad M(x_{m(k)-1}, y_{m(k)-1}, x_{n(k)-1}, y_{n(k)-1}). \qquad (3.18)
\end{aligned}
$$

Combining (3.16) and (3.18), we have

$$
\begin{aligned}
d(x_{m(k)},\ x_{n(k)}) + d(y_{m(k)},\ y_{n(k)}) &\leq 2\ \beta(M(x_{m(k)-1}, y_{m(k)-1}, x_{n(k)-1}, y_{n(k)-1}))\\
&\quad M(x_{m(k)-1}, y_{m(k)-1}, x_{n(k)-1}, y_{n(k)-1}). \qquad (3.19)
\end{aligned}
$$

Taking limit as $k \to \infty$ in (3.17) and using (3.11), (3.14) and (3.15), we have

$$\lim_{k\to\infty} M(x_{m(k)-1}, y_{m(k)-1}, x_{n(k)-1}, y_{n(k)-1}) = \max \left\{ \frac{\epsilon}{2},\ 0,\ 0,\ \frac{\epsilon}{2} \right\} = \frac{\epsilon}{2}. \qquad (3.20)$$

Taking limit supremum in (3.19) and using (3.13), (3.20), we have

$$
\begin{aligned}
\epsilon &\leq 2\ \limsup_{k\to\infty} \beta(M(x_{m(k)-1}, y_{m(k)-1}, x_{n(k)-1}, y_{n(k)-1}))\ \frac{\epsilon}{2}\\
&= \epsilon\ \limsup_{k\to\infty} \beta(M(x_{m(k)-1}, y_{m(k)-1}, x_{n(k)-1}, y_{n(k)-1})). \qquad (3.21)
\end{aligned}
$$

Using (3.21) and the property of β, we have

$$1 \leq \limsup_{k\to\infty} \beta(M(x_{m(k)-1}, y_{m(k)-1}, x_{n(k)-1}, y_{n(k)-1})) \leq 1,$$

that is, $\limsup_{k\to\infty} \beta(M(x_{m(k)-1}, y_{m(k)-1}, x_{n(k)-1}, y_{n(k)-1})) = 1$. Then it follows by the property of β that $\lim_{k\to\infty} M(x_{m(k)-1}, y_{m(k)-1}, x_{n(k)-1}, y_{n(k)-1}) = \frac{\epsilon}{2} = 0$, that is, $\epsilon = 0$ which is a contradiction. Therefore, $\{x_n\}$ and $\{y_n\}$ are both Cauchy sequences in X. As X is complete, there exist

$x,\ y \in X$ such that

$$\lim_{n\to\infty} x_n = x \quad \text{and} \quad \lim_{n\to\infty} y_n = y. \tag{3.22}$$

Now we show that $(x,\ y)$ is a coupled fixed point of F. If possible let $(x,\ y)$ be not a coupled fixed point of F. Then either $x \neq F(x,y)$ or $y \neq F(y,x)$, that is, either, $d(x,\ F(x,y)) \neq 0$ or $d(y,\ F(y,x)) \neq 0$, that is, $d(x,\ F(x,y)) + d(y,\ F(y,x)) > 0$. Using (3.3), (3.22) and R-regularity property of the space, we have

$$(x_n,\ x) \in R \quad \text{and} \quad (y,\ y_n) \in R. \tag{3.23}$$

By (3.1) and (3.23), we have

$$\begin{aligned} d(x_{n+1},\ F(x,y)) &= d(F(x_n,y_n),\ F(x,\ y)) \\ &\leq \beta(M(x_n,y_n,x,y))\ M(x_n,y_n,x,y) \end{aligned} \tag{3.24}$$

where

$$\begin{aligned} M(x_n,y_n,x,y) &= \max\left\{\frac{d(x_n,\ x)+d(y_n,\ y)}{2},\ \frac{d(x_n,\ F(x_n,y_n))+d(y_n,\ F(y_n,x_n))}{2},\right. \\ &\qquad \left.\frac{d(x,\ F(x,y))+d(y,\ F(y,x))}{2},\ \frac{d(x,\ F(x_n,y_n))+d(y,\ F(y_n,x_n))}{2}\right\} \\ &= \max\left\{\frac{d(x_n,\ x)+d(y_n,\ y)}{2},\ \frac{d(x_n,\ x_{n+1})+d(y_n,\ y_{n+1})}{2},\right. \\ &\qquad \left.\frac{d(x,\ F(x,y))+d(y,\ F(y,x))}{2},\ \frac{d(x,\ x_{n+1})+d(y,\ y_{n+1})}{2}\right\}. \end{aligned} \tag{3.25}$$

Similarly, we show that

$$\begin{aligned} d(y_{n+1},\ F(y,x)) &= d(F(y_n,x_n),\ F(y,\ x)) \leq \beta(M(y_n,x_n,y,x))\ M(y_n,x_n,y,x)) \\ &= \beta(M(x_n,y_n,x,y))\ M(x_n,y_n,x,y). \end{aligned} \tag{3.26}$$

Combining (3.24) and (3.26), we have

$$d(x_{n+1},\ F(x,y)) + d(y_{n+1},\ F(y,x)) \leq 2\ \beta(M(x_n,y_n,x,y))\ M(x_n,y_n,x,y). \tag{3.27}$$

Taking limit as $n \to \infty$ in (3.25), we have

$$\begin{aligned} \lim_{n\to\infty} M(x_n,y_n,x,y) &= \max\left\{0,\ 0,\ \frac{d(x,\ F(x,y))+d(y,\ F(y,x))}{2},\ 0\right\} \\ &= \frac{d(x,\ F(x,y))+d(y,\ F(y,x))}{2}. \end{aligned} \tag{3.28}$$

Taking limit supremum in (3.27) and using (3.22) and (3.28), we have

$$d(x,F(x,y)+d(y,F(y,x)) \leq [d(x,F(x,y)+d(y,F(y,x))] \limsup_{n\to\infty} \beta(M(x_n,y_n,x,y)). \tag{3.29}$$

As explained earlier, we have from (3.29) that

$$1 \leq \limsup_{n\to\infty} \beta(M(x_n,y_n,x,y)) \leq 1,$$

that is, $\limsup_{n\to\infty} \beta(M(x_n, y_n, x, y)) = 1$. By a property of β we have that

$$\lim_{n\to\infty} M(x_n, y_n, x, y) = \frac{d(x, F(x, y)) + d(y, F(y, x))}{2} = 0,$$

that is, $d(x, F(x, y)) + d(y, F(y, x)) = 0$, which contradicts our assumption. Therefore, $d(x,\ F(x, y)) = d(y,\ F(y, x)) = 0$, that is, $x = F(x, y)$ and $y = F(y, x)$, that is, $(x,\ y)$ is a coupled fixed point of F. ☐

Remark 3.2. Our result is a generalization of the result of Bhaskar and Lakshmikantam (in [15]) and of the result of Choudhury and Kundu (in [8]).

If R is taken to be a partial ordered relation, then we have the following corollary:

Corollary 3.3. *Let (X, d) be a complete metric space with a partial order $\preceq$ on it such that X has regular property [that is, if $\{x_n\}$ is a monotone convergent sequence with limit x, then $x_n \preceq x$ or $x \preceq x_n$, according as the sequence is increasing or decreasing]. Suppose that $F : X \times X \to X$ is a dominated map [that is, $x \preceq F(x,\ y)$ and $F(y,\ x) \preceq y$, for any $(x,\ y) \in X \times X$] and there exists $\beta \in \boldsymbol{B}^*$ such that (3.1) of Theorem 3.1 is satisfied for all $(x,\ y)$, $(u,\ v) \in X \times X$ with $[x \preceq u$ and $v \preceq y]$ or $[u \preceq x$ and $y \preceq v]$. Then F has a coupled fixed point.*

If R is taken to be the universal relation, that is, $R = X \times X$, we have the following corollary:

Corollary 3.4. *Let (X, d) be a complete metric space and $F : X \times X \to X$. Suppose there exists $\beta \in \boldsymbol{B}^*$ such that (3.1) of Theorem 3.1 is satisfied for all $(x,\ y)$, $(u,\ v) \in X \times X$. Then F has a coupled fixed point.*

Theorem 3.5. *In addition to the hypothesis of Theorem 3.1, suppose that both R and R^{-1} are directed. Then F has a unique coupled fixed point.*

Proof. By Theorem 3.1, the set of coupled fixed points of F is nonempty. If possible, let $(x,\ y)$ and $(x^*,\ y^*)$ be two coupled fixed points of F. Then $x = F(x,\ y)$; $y = F(y,\ x)$ and $x^* = F(x^*,\ y^*)$; $y^* = F(y^*,\ x^*)$. Our aim is to show that $x = x^*$ and $y = y^*$. By the directed property of R and R^{-1}, there exist $u \in X$ and $v \in X$ such that $(x,\ u) \in R$; $(x^*,\ u) \in R$ and $(y,\ v) \in R^{-1}$; $(y^*,\ v) \in R^{-1}$, that is $(x,\ u) \in R$; $(x^*,\ u) \in R$ and $(v,\ y) \in R$; $(v,\ y^*) \in R$. Put $u_0 = u$ and $v_0 = v$. Then $(x,\ u_0) \in R$ and $(v_0,\ y) \in R$. Let $u_1 = F(u_0,\ v_0)$ and $v_1 = F(v_0,\ u_0)$. Similarly, as in the proof of Theorem 3.1, we inductively define two sequences $\{u_n\}$ and $\{v_n\}$ such that

$$u_{n+1} = F(u_n, v_n) \text{ and } v_{n+1} = F(v_n, u_n), \text{ for all } n \geq 0. \tag{3.30}$$

As F is R-dominated, we have

$$(u_n,\ u_{n+1}) \in R \text{ and } (v_{n+1},\ v_n) \in R, \text{ for all } n \geq 0. \tag{3.31}$$

Arguing similarly as in proof of Theorem 3.1, we prove that $\{u_n\}$ and $\{v_n\}$ are two Cauchy sequences in X and there exists p and $q \in X$ such that

$$\lim_{n\to\infty} u_n = p \ \text{ and } \ \lim_{n\to\infty} v_n = q. \tag{3.32}$$

Now we show that $x = p$ and $y = q$, that is, $d(x,p) + d(y,q) = 0$.
If possible, suppose that $d(x,p) + d(y,q) \neq 0$. We claim that

$$(x,\ u_n) \in R \ \text{ and } \ (v_n,\ y) \in R, \ \text{ for all } \ n \geq 0. \tag{3.33}$$

As $(x,u_0) \in R,\ (u_0,u_1) \in R$ and $(v_1,v_0) \in R,\ (v_0,y) \in R$, by the transitivity property of R, we have $(x,\ u_1) \in R$ and $(v_1,\ y) \in R$. Therefore, our claim is true for $n = 1$. Assume that (3.33) is true for some $m > 1$, that is, $(x,\ u_m) \in R$ and $(v_m,\ y) \in R$. By (3.31), $(u_m,\ u_{m+1}) \in R$ and $(v_{m+1},\ u_m) \in R$. The transitivity property of R guarantees that $(x,\ u_{m+1}) \in R$ and $(v_{m+1},\ y) \in R$ and this proves our claim. Using (3.1) and (3.33), we have for all $n \geq 0$ that

$$\begin{aligned} d(x,u_{n+1}) &= d(F(x,\ y),\ F(u_n,\ v_n)) \\ &\leq \beta(M(x,\ y,\ u_n,\ v_n))\, M(x,\ y,\ u_n,\ v_n), \end{aligned} \tag{3.34}$$

where

$$\begin{aligned} M(x,\ y,\ u_n,\ v_n) &= \max\left\{\frac{d(x,u_n)+d(y,v_n)}{2},\ \frac{d(x,\ F(x,y))+d(y,\ F(y,x))}{2},\right. \\ &\qquad \left.\frac{d(u_n,\ F(u_n,v_n))+d(v_n,\ F(v_n,u_n))}{2},\ \frac{d(u_n,\ F(x,y))+d(v_n,\ F(y,x))}{2}\right\} \\ &= \max\left\{\frac{d(x,u_n)+d(y,v_n)}{2},\ 0,\ \frac{d(u_n,\ u_{n+1})+d(v_n,\ v_{n+1})}{2},\right. \\ &\qquad \left.\frac{d(u_n,\ x)+d(v_n,\ y)}{2}\right\}. \end{aligned} \tag{3.35}$$

Similarly, we show that

$$\begin{aligned} d(y,v_{n+1}) &= d(F(y,\ x),\ F(v_n,\ u_n)) \leq \beta(M(y,\ x,\ v_n,\ u_n))\, M(y,\ x,\ v_n,\ u_n) \\ &= \beta(M(x,\ y,\ u_n,\ v_n))\, M(x,\ y,\ u_n,\ v_n). \end{aligned} \tag{3.36}$$

Combining (3.34) and (3.36), we have

$$d(x,u_{n+1}) + d(y,v_{n+1}) \leq 2\ \beta(M(x,\ y,\ u_n,\ v_n))\, M(x,\ y,\ u_n,\ v_n). \tag{3.37}$$

Taking limit in (3.35) as $n \to \infty$ and using (3.32), we have

$$\lim_{n\to\infty} M(x,y,u_n,v_n) = \frac{d(x,p)+d(y,q)}{2}. \tag{3.38}$$

Taking limit supremum as $n \to \infty$ in (3.37) and using (3.32), (3.38), we have

$$d(x,p) + d(y,q) \leq [d(x,p)+d(y,q)]\ \limsup_{n\to\infty} \beta(M(x,\ y,\ u_n,\ v_n)) \tag{3.39}$$

which implies that

$$1 \leq \limsup_{n\to\infty} \beta(M(x,y,u_n,v_n)) \leq 1, \tag{3.40}$$

that is, $\limsup_{n\to\infty} \beta(M(x,\ y,\ u_n,\ v_n)) = 1$. By a property of β we have that

$$\lim_{n\to\infty} M(x,y,u_n,v_n) = \frac{d(x,p)+d(y,q)}{2} = 0,$$

that is, $d(x,p)+d(y,q) = 0$, which contradicts our assumption that $d(x,p)+d(y,q) \neq 0$. Therefore, $d(x,p)+d(y,q) = 0$, that is, $d(x,p) = d(y,q) = 0$, that is,

$$x = p \ \text{ and } \ y = q. \tag{3.41}$$

Similarly, we can show that

$$x^* = p \text{ and } \ y^* = q. \tag{3.42}$$

From (3.41) and (3.42), we have $x = x^*$ and $y = y^*$. Therefore, the coupled fixed point of F is unique. $\square$

We present the following illustrative example in support of Theorems 3.1.

Example 3.6. Take the metric space $X = [0,\ 1]$ with usual metric d. Let $\beta : [0,\ \infty) \to [0,\ 1)$ be defined as $\beta(t) = \dfrac{\ln(1+t)}{t}$, if $t > 0$ and $\beta(t) = 0$, if $t = 0$. Define $F : X \times X \to X$ by $F(x,\ y) = \ln\left(1 + \dfrac{x+y}{2}\right)$, for all $(x,\ y) \in X \times X$ and binary relation R by $R = \{(x,\ y) : 0 \leq x \leq 1;\ 0 \leq y \leq \ln 2 \ \text{ or } \ 0 \leq x \leq \ln 2;\ 0 \leq y \leq 1\}$.

Then we see that X is regular with respect to R and T is R-dominated. Let $(x,y),\ (u,\ v) \in X \times X$ with $(x,\ u) \in R$ and $(v,\ y) \in R$. Then $\Big[x \in [0,\ 1] \text{ or } x \in [0,\ \ln 2]\Big]$; $\Big[u \in [0,\ 1] \text{ or } u \in [0,\ \ln 2]\Big]$; $\Big[y \in [0,\ 1] \text{ or } y \in [0,\ \ln 2]\Big]$ and $\Big[v \in [0,\ 1] \text{ or } v \in [0,\ \ln 2]\Big]$. Now for those values of $x,\ y,\ y,\ u$ and v, we obtain

$$\begin{aligned}
& d(F(x,y),\ F(u,v)) = d\left(\ln\left(1+\frac{x+y}{2}\right),\ \ \ln\left(1+\frac{u+v}{2}\right)\right) \\
& = \left|\ \ln\left(1+\frac{x+y}{2}\right) - \ln\left(1+\frac{u+v}{2}\right)\ \right| \ = \left|\ \ln\left(\frac{1+\dfrac{x+y}{2}}{1+\dfrac{u+v}{2}}\right)\right| \\
& = \left|\ \ln\left(1+\frac{\dfrac{x+y}{2}-\dfrac{u+v}{2}}{1+\dfrac{u+v}{2}}\right)\ \right| \ \leq \left|\ \ln\left(1+\frac{\left|\ \dfrac{x+y}{2}-\dfrac{u+v}{2}\ \right|}{1+\dfrac{u+v}{2}}\right)\ \right| \\
& \leq \left|\ \ln\left(1+\left|\ \frac{x+y}{2}-\frac{u+v}{2}\ \right|\right)\ \right| \ \ \leq \left|\ \ln\left(1+\frac{|\ u-x\ |+|\ \ v-y\ |}{2}\right)\ \right| \\
& = \ln\left(1+\frac{|\ u-x\ |+|\ \ v-y\ |}{2}\right) \ \ \leq \ln\ \Big(1+M(x,\ y,\ u,\ v)\ \Big) \\
& = \frac{\ln\Big(1+M(x,\ y,\ u,\ v)\Big)}{M(x,\ y,\ u,\ v)}\ M(x,\ y,\ u,\ v) \ \ = \ \ \beta\Big(M(x,\ y,\ u,\ v)\ \Big)\ M(x,\ y,\ u,\ v).
\end{aligned}$$

It follows that the inequality in Theorem 3.1 is satisfied for all $(x, y),\ (u,\ v) \in X \times X$ with $(x,\ u) \in R$ and $(v,\ y) \in R$. Here all the conditions of Theorem 3.1 are satisfied and $(0,\ 0)$ is a coupled fixed point of F.

4 Well-Posedness

We use the following assumption to assure the well-posedness via R-dominated mapping.

(A) If $(x^*,\ y^*)$ is any solution of the problem (P), that is, of (2.1) and $\{(x_n,\ y_n)\}$ is any sequence in $X \times X$ for which $\lim_{n\to\infty} d(x_n,\ F(x_n,\ y_n)) = \lim_{n\to\infty} d(y_n,\ F(y_n,\ x_n)) = 0$, then $(x^*, x_n) \in R$ and $(y_n,\ y^*) \in R$, for all n.

Theorem 4.1. *In addition to the hypothesis of Theorem 3.5, suppose that the assumption (A) holds. Then the coupled fixed point problem (P) is well-posed.*

Proof. By Theorem 3.5, F has a unique coupled fixed point (x^*, y^*) (say). Then (x^*, y^*) is a solution of (2.1), that is, $x^* = F(x^*, y^*)$ and $y^* = F(y^*, x^*)$. Let $\{(x_n,\ y_n)\}$ be any sequence in $X \times X$ for which $\limsup_{n\to\infty}[d(x_n,\ x^*) + d(y_n,\ y^*)]$ is finite and $\lim_{n\to\infty} d(x_n,\ F(x_n,\ y_n)) = \lim_{n\to\infty} d(y_n,\ F(y_n,\ x_n)) = 0$. Then there exists a nonnegative real number M such that $\limsup_{n\to\infty}[d(x^*, x_n)+d(y^*, y_n)] = M$ and also by the assumption (A), $(x^*, x_n) \in R$ and $(y_n, y^*) \in R$, for all n. Using (3.1), we have

$$\begin{aligned} d(x_n,\ x^*) &= d(x_n,\ F(x^*,\ y^*)) \le d(x_n,\ F(x_n,\ y_n) + d(F(x^*,\ y^*),\ F(x_n,\ y_n)) \\ &\le \beta(M(x^*, y^*, x_n, y_n))\ M(x^*, y^*, x_n, y_n) + d(x_n,\ F(x_n,\ y_n)) \end{aligned} \tag{4.1}$$

where

$$\begin{aligned} M(x^*, y^*, x_n, y_n) &= \max\left\{\frac{d(x^*, x_n) + d(y^*, y_n)}{2},\ \frac{d(x^*,\ F(x^*, y^*)) + d(y^*,\ F(y^*, x^*))}{2},\right. \\ &\quad \left.\frac{d(x_n, F(x_n, y_n)) + d(y_n, F(y_n, x_n))}{2},\ \frac{d(x_n, F(x^*, y^*)) + d(y_n, F(y^*, x^*))}{2}\right\} \\ &= \max\left\{\frac{d(x^*, x_n) + d(y^*, y_n)}{2},\ 0,\ \frac{d(x_n,\ F(x_n, y_n)) + d(y_n,\ F(y_n, x_n))}{2},\right. \\ &\quad \left.\frac{d(x_n,\ x^*) + d(y_n,\ y^*)}{2}\right\}. \end{aligned} \tag{4.2}$$

Similarly, we can show that

$$\begin{aligned} d(y_n,\ y^*) &\le \beta(M(y^*, x^*, y_n, x_n))\ M(y^*, x^*, y_n, x_n) + d(y_n,\ F(y_n, x_n)) \\ &\le \beta(M(x^*, y^*, x_n, y_n))\ M(x^*, y^*, x_n, y_n) + d(y_n,\ F(y_n, x_n)). \end{aligned} \tag{4.3}$$

Combining (4.1) and (4.3), we have

$$\begin{aligned} d(x_n,\ x^*) + d(y_n,\ y^*) &\le 2\ \beta(M(x^*, y^*, x_n, y_n))\ M(x^*, y^*, x_n, y_n) \\ &+ d(x_n,\ F(x_n,\ y_n)) + d(y_n,\ F(y_n, x_n)). \end{aligned} \tag{4.4}$$

Taking limit supremum as $n \to \infty$ in (4.2), we have

$$\limsup_{n\to\infty} M(x^*, y^*, x_n, y_n) = \frac{M}{2}. \tag{4.5}$$

If possible, suppose that $\limsup_{n\to\infty}[d(x^*, x_n) + d(y^*, y_n)] = M \neq 0$. Then $M > 0$. Taking limit supremum as $n \to \infty$ in (4.4) and using (4.5), we have

$$M \leq M \limsup_{n\to\infty} \beta(M(x^*, y^*, x_n, y_n)), \text{ that is, } 1 \leq \limsup_{n\to\infty} \beta(M(x^*, y^*, x_n, y_n)) \leq 1.$$

Then $\limsup_{n\to\infty} \beta(M(x^*, y^*, x_n, y_n)) = 1$. By a property of β, $\lim_{n\to\infty} M(x^*, y^*, x_n, y_n) = 0$, that is, $\lim_{n\to\infty}[d(x_n, x^*) + d(y_n, y^*)] = 0$ which is a contradiction. Hence we have $\limsup_{n\to\infty}[d(x^*, x_n) + d(y^*, y_n)] = 0$. Then we have $0 \leq \liminf_{n\to\infty}[d(x^*, x_n) + d(y^*, y_n)] \leq \limsup_{n\to\infty}[d(x^*, x_n) + d(y^*, y_n)] = 0$ which implies that $\lim_{n\to\infty}[d(x^*, x_n) + d(y^*, y_n)] = 0$. It follows that $\lim_{n\to\infty} d(x_n, x^*) = \lim_{n\to\infty} d(y_n, y^*) = 0$, that is, $x_n \to x^*$ and $y_n \to y^*$ as $n \to \infty$. Hence the coupled fixed point problem (P) is well-posed. □

5 Some results for $\alpha-$dominated mapping

Coupled α-dominated mappings are defined here and are conceptual extensions of mappings with admissibility conditions. Various types of admissibility conditions have been used in fixed point theory in works like [10, 11, 19, 29, 31].

Definition 5.1. Let X be a nonempty set and $\alpha : X \times X \to \mathbb{R}$ be a mapping. A mapping $F : X \times X \to X$ is said to be α-dominated if $\alpha(x,\ F(x,y)) \geq 1$ and $\alpha(y,\ F(y,x)) \geq 1$, for $(x,\ y) \in X \times X$.

Definition 5.2. Let X be a nonempty set and $\alpha : X \times X \to \mathbb{R}$ be a mapping. Then α is said to have triangular property if for $x,\ y,\ z \in X$, $\alpha(x,\ y) \geq 1$ and $\alpha(y,\ z) \geq 1$ imply $\alpha(x,\ z) \geq 1$.

Definition 5.3. Let $(X,\ d)$ be a metric space and $\alpha : X \times X \to \mathbb{R}$ be a mapping. Then X is said to have α-regular property if for every convergent sequence $\{x_n\}$ with limit $x \in X$, $\alpha(x_n,\ x_{n+1}) \geq 1$, for all n implies $\alpha(x_n,\ x) \geq 1$, for all n.

Theorem 5.4. Let $(X,\ d)$ be a complete metric space and $\alpha : X \times X \to \mathbb{R}$ be a mapping such that X has α-regular property and α has triangular property. Suppose that $F : X \times X \to X$ be a α-dominated mapping and there exists $\beta \in \boldsymbol{B}^*$ such that (3.1) of Theorem 3.1 is satisfied for all $(x,\ y),\ (u,\ v) \in X \times X$ with $\alpha(x,\ u) \geq 1$ and $\alpha(y,\ v) \geq 1$. Then F has a coupled fixed point.

Proof. Define a binary relation R on X as $(x,\ y) \in R$ if and only if $\alpha(x,\ y) \geq 1$ or $\alpha(y,\ x) \geq 1$. Then (i) $\alpha(x,\ u) \geq 1$ and $\alpha(y,\ v) \geq 1$ imply $(x, u) \in R$ and $(v, y) \in R$, (ii) $\alpha(x,\ F(x,y)) \geq 1$ and $\alpha(y,\ F(y,x)) \geq 1$ imply $(x,\ F(x,y)) \in R$ and $(F(y,x),\ y) \in R$, for $(x,\ y) \in X \times X$,

(iii) $\alpha(x_n,\ x_{n+1}) \geq 1$, $\alpha(x_n,\ x) \geq 1$ imply $(x_n,\ x_{n+1}) \in R$, $(x_n,\ x) \in R$, whenever $\{x_n\}$ is a convergent sequence with $x_n \to x$ and $\alpha(x_n,\ x_{n+1}) \geq 1$. Therefore, all the assumptions reduce to the assumptions of Theorem 3.1. Then by an application of Theorem 3.1, we conclude that F has a coupled fixed point in $X \times X$. □

6 Some results on graphic contraction

Our present section is on graphic contraction. Fixed point problem on the structures of metric spaces with a graph have appeared in works like [2, 4, 12].

Let X be a nonempty set and $\Delta := \{(x,\ x) : x \in X\}$. Let G be a directed graph such that its vertex set $V(G)$ coincides with X, that is, $V(G) = X$ and the edge set $E(G)$ contains all loops, that is, $\Delta \subseteq E(G)$. Assume that G has no parallel edges. By G^{-1} we denote the conversion of a graph G, that is, the graph obtained from G by reversing the directions of the edges. Thus we have $V(G^{-1}) = V(G)$ and $E(G^{-1}) = \{(x,\ y) \in X \times X :\ (y,\ x) \in E(G)\}$. A nonempty set X is said to be endowed with a directed graph $G(V, E)$ if $V(G) = X$ and $\Delta \subseteq E(G)$.

Definition 6.1. Let X be a nonempty set endowed with a graph $G(V, E)$. A mapping $F : X \times X \to X$ is said to be G-dominated if $(x,\ F(x, y)) \in E$ and $(F(y, x), y) \in E$, for $(x, y) \in X \times X$.

Definition 6.2. Let X be a nonempty set endowed with a graph $G(V, E)$. Then G is said to have transitive property if for $x,\ y,\ z \in X$, $(x,\ y) \in E$ and $(y,\ z) \in E$ imply $(x,\ z) \in E$.

Definition 6.3. Let $(X,\ d)$ be a metric space endowed with a directed graph $G(V, E)$. Then X is said to have G-regular property if for every convergent sequence $\{x_n\}$ with limit $x \in X$, $(x_n,\ x_{n+1}) \in E$, for all n implies $(x_n,\ x) \in E$, for all n [or $(x_{n+1},\ x_n) \in E$, for all n implies $(x,\ x_n) \in E$, for all n].

Theorem 6.4. Let $(X,\ d)$ be a complete metric space endowed with a directed graph $G(V, E)$ such that X has G-regular property and G has transitive property. Suppose that $F : X \times X \to X$ is a G-dominated mapping and there exists $\beta \in \boldsymbol{B}^*$ such that (3.1) of Theorem 3.1 is satisfied for all $(x,\ y),\ (u,\ v) \in X \times X$ with $[(x,\ u) \in E$ and $(v,\ y) \in E]$ or $[(u,\ x) \in E$ and $(y,\ v) \in E]$. Then F has a coupled fixed point.

Proof. Let us define a relation R, by xRy holds if $(x, y) \in E$. As $(x,\ y) \in E$, for $(x, y) \in X \times X$ implies that $(x,\ y) \in R$, it is easy to verify that all the assumptions of the theorem reduce to the assumptions of Theorem 3.1. Then by an application of Theorem 3.1, we conclude that F has a coupled fixed point in $X \times X$. □

7 Application to the solution of system nonlinear integral equations

In this section, we present an application of our coupled fixed point results derived in Section 3 to establish the existence and uniqueness of a solution of a system of integral equations. We consider a coupled system of two nonlinear integral equations as follows:

$$\left.\begin{array}{c} x(t) = f(t) + \int_0^t K(t,\ s)h(t,\ s,\ x(s),\ y(s))ds \\ \text{and} \\ y(t) = f(t) + \int_0^t K(t,\ s)h(t,\ s,\ y(s),\ x(s))ds, \end{array}\right\} \tag{7.1}$$

where $T > 0$ be any number, $t, s \in [0, T]$, $K : [0, T] \times [0, T] \to \mathbb{R}$ be a function, which is the kernel of the integral equations, and the unknown functions $x(t)$ and $y(t)$ take real values.

The reason for the choice of this application is that coupled non-linear equations have their uses in modeling situations of wide variety.

Let $X = C([0,\ T])$ be the space of all real valued continuous functions defined on $[0,\ T]$. Here $C([0,\ T])$ with the metric $d(x,\ y) = \max_{t\in[0,\ T]} \mid x(t) - y(t) \mid$ is a complete metric space. Assume that this metric space is endowed with the universal relation U, that is, $(x, y) \in U$, for all $x,\ y \in X$. Define a mapping $F : X \times X \to X$ by

$$F(x,\ y)(t) = f(t) + \int_0^t K(t,\ s)h(t,\ s,\ x(s),\ y(s))ds, \quad \text{for all} \quad t, s \in [0,\ T]. \tag{7.2}$$

We designate the following assumptions by $A_1,\ A_2$ and A_3:

$A_1 :\quad f \in C([0,\ T]$ and $\ h : [0,\ T] \times [0,\ T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous mapping;

$A_2 : \mid K(t,\ s) \mid\leq q,$ where $q > 0$ is a fixed number;

$A_3 : \mid h(t, s, x, y) - h(t, s, u, v) \mid\leq \mathcal{M}(t,\ s,\ x,\ y,\ u,\ v),$ for all $(x, y),\ (u, v) \in X \times X$ and $t,\ s \in [0,\ T]$, where $\mathcal{M}(t,\ s,\ x,\ y,\ u,\ v) = \dfrac{1}{qT}\ \ln\left(1 + \dfrac{\mid x - u \mid + \mid y - v \mid}{2}\right).$

Theorem 7.1. Let $(X = C([0, T]), d),\ F,\ h,\ K(t, s)$ satisfy the assumptions $A_1,\ A_2$ and A_3. Then system of nonlinear integral equation (7.1) has a solution $(x, y) \in C([0, T]) \times C([0, T])$ and the solution is unique.

Proof. It is trivial to observe that the mapping $F : X \times X \to X$ defined by (7.2) is a U dominated mapping and X has U regular property, where U is the universal relation. From assumptions $A_1,\ A_2$ and A_3, for all $(x, y),\ (u, v) \in C([0,\ T]) \times C([0,\ T])$, that is, for all $x,\ y,\ u,\ v \in C([0,\ T])$ and $t, s \in [0,\ T]$, we have

$$
\begin{aligned}
&\mid F(x,y)(t)-F(u,v)(t)\mid = \left|\int_0^t K(t,\ s)[h(t,\ s,\ x(s),\ y(s))-h(t,\ s,\ u(s),\ v(s))]ds\right| \\
\leq\ & \int_0^t \mid K(t,\ s)\mid\mid [h(t,\ s,\ x(s),\ y(s))-h(t,\ s,\ u(s),\ v(s))]\mid ds \\
\leq\ & q\times\int_0^t \mid [h(t,\ s,\ x(s),\ y(s))-h(t,\ s,\ u(s),\ v(s))]\mid ds \quad [\text{by } A_2] \\
\leq\ & q\times\int_0^T \mid [h(t,\ s,\ x(s),\ y(s))-h(t,\ s,\ u(s),\ v(s))]\mid ds \\
\leq\ & q\times\int_0^T \mathcal{M}(t,\ s,\ x,\ y,\ u,\ v)\ ds = q\times\int_0^T \frac{1}{qT}\ \ln\left(1+\frac{\mid x-u\mid+\mid y-v\mid}{2}\right)\ ds \\
=\ & \int_0^T \frac{1}{T}\ \ln\left(1+\frac{\mid x-u\mid+\mid y-v\mid}{2}\right)\ ds \leq \int_0^T \frac{1}{T}\ \ln\left(1+\frac{d(x,\ u)+d(y,\ v)}{2}\right)\ ds \\
=\ & \ln(1+\frac{d(x,\ u)+d(y,\ v)}{2})\ \int_0^T \frac{1}{T}\ ds = \ln\left(1+\frac{d(x,\ u)+d(y,\ v)}{2}\right) \\
\leq\ & \ln(1+M(x,y,u,v))\ [\text{since } \ln(1+t) \text{ is nondecreasing for } \ t>0] \\
=\ & \frac{\ln(1+M(x,y,u,v))}{M(x,y,u,v)}\ M(x,y,u,v) = \beta(M(x,\ y,\ u,\ v))\ M(x,\ y,\ u,\ v) \quad \text{where}
\end{aligned}
$$

$$
\beta(t)=\frac{\ln(1+t)}{t},\ \text{if } t>0 \text{ and } \ \beta(t)=0,\ \text{if } t=0
$$

and

$$
\begin{aligned}
M(x,\ y,\ u,\ v)=\max\ \Bigg\{&\frac{d(x,\ u)+d(y,\ v)}{2},\ \frac{d(x,F(x,\ y))+d(y,\ F(y,\ x))}{2}, \\
&\frac{d(u,F(u,\ v))+d(v,F(v,\ u))}{2},\ \frac{d(u,F(x,\ y))+d(v,F(y,\ x))}{2}\Bigg\}.
\end{aligned}
$$

Hence

$$
d(F(x,\ y),\ F(u,\ v)) \leq \beta(M(x,\ y,\ u,\ v))\ M(x,\ y,\ u,\ v).
$$

Therefore, all the conditions of Theorems 3.1 and 3.5 are satisfied and hence by Theorem 3.1 there exists a coupled fixed point $(x,\ y)$ in $X\times X$ which, by virtue of Theorem 3.5, is unique. In other words, the system of integral equations (7.1) under the conditions stipulated in the theorem has a solution which is unique. □

Acknowledgement: The suggestions of the learned referee are gratefully acknowledged.

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VOLUME 23 · ISSUE 1 · APRIL 2021

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